DOI 10.1007/s10958-019-04519-z
Journal of Mathematical Sciences, Vol. 242, No. 6, November, 2019

To the theory of semilinear equations in the plane

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Dedicated to the memory of Professor Bogdan Bojarski

Abstract. In two dimensions, we present a new approach to the study of the semilinear equations of the form $\operatorname{div}[A(z)\nabla u] = f(u)$, the diffusion term of which is the divergence uniform elliptic operator with measurable matrix functions $A(z)$, whereas its reaction term $f(u)$ is a continuous non-linear function. Assuming that $f(t)/t \to 0$ as $t \to \infty$, we establish a theorem on existence of weak $C(\overline{D})\cap W^{1,2}_{\text{loc}}(D)$ solutions of the Dirichlet problem with arbitrary continuous boundary data in any bounded domains D without degenerate boundary components. As consequences, we give applications to some concrete model semilinear equations of mathematical physics, arising from modeling processes in anisotropic and inhomogeneous media. With a view to the further development of the theory of boundary-value problems for the semilinear equations, we prove a theorem on the solvability of the Dirichlet problem for the Poisson equation in Jordan domains with arbitrary boundary data that are measurable with respect to the logarithmic capacity.

Keywords. Semilinear elliptic equations, quasilinear Poisson equations, anisotropic and inhomogeneous media, conformal and quasiconformal mappings.

1. Introduction

The study of linear and non-linear elliptic partial differential equations in two dimensions by the methods of complex analysis and quasiconformal mappings with concrete applications to the nonlinear elasticity, gas flow, hydrodynamics, and other branches of the natural science was initiated by M. A. Lavrentiev [54], C. B. Morrey [60], L. Bers [8], L. Bers and L. Nirenberg [9], I. N. Vekua [72], B. Bojarski [11], J. Serrin [68], and others, see the references therein. The history of such equations actually goes back as far as the celebrated works by d'Alembert on the Cauchy–Riemann systems in hydrodynamics, by Gauss on the geometry of surfaces, by Lobachevskii on a non-Euclidean geometry, and by the pioneer paper by Beltrami [7].

A rather comprehensive presentation of the present state of the theory is given in the excellent book of Astala, Iwaniec, and Martin [4]. This book concerns the most modern aspects and the most recent developments in the theory of planar quasiconformal mappings and their application in conformal geometry, partial differential equations, and nonlinear analysis. The book contains also the exhaustive bibliography on the topic. Among the variety of deep results in this trend, we single out the fundamental harmonic factorization theorem, see [4], Theorem 16.2.1, for the solutions to the non-linear uniformly elliptic divergence equations

$$\operatorname{div} A(z, \nabla u) = 0, \;\; z \in D \subset \mathbb{C}, \tag{1.1}$$

and the corresponding regularity results. In particular, the factorization theorem claims that every solution $u \in W^{1,2}_{\text{loc}}(D)$ of Eq. (1.1) can be expressed as $u(z) = h(f(z))$, where $f : D \to G$ is K-quasiconformal, and h is harmonic on G.

In a series of our recent papers [33–40], we have proposed another application of the theory of quasiconformal mappings to the study of semilinear partial differential equations of the form

$$\operatorname{div}\,[A(z)\nabla u(z)] = f(u),\;\; z \in D,\;\; D \subseteq \mathbb{C}, \tag{1.2}$$

Translated from Ukrains'kiĭ Matematychnyĭ Visnyk, Vol. 16, No. 1, pp. 105–140 January–March, 2019.
Original article submitted March 29, 2019

the diffusion term of which is the divergence form elliptic operator $L(u)$, whereas its reaction term $f(u)$ is a non-linear function. Here, the symmetric matrix function $A(z) = \{a_{ij}(z)\}$, $\det A(z) = 1$, with measurable entries satisfies the uniform ellipticity condition

$$\frac{1}{K}|\xi|^2 \le \langle A(z)\xi,\, \xi\rangle \le K|\xi|^2 \ \text{ a.e. in } D,\ \ 1 \le K < \infty, \tag{1.3}$$

for every $\xi \in \mathbb{R}^2$. We denote the set of all such matrix functions by $M_K^{2\times 2}(D)$. In the cited papers, we have studied the composition properties of $L(u)$ first for sufficiently smooth functions and then in the Sobolev spaces making use of the fundamental compositional theorems established in [28, 70]. It was shown that, by the chain rule for the function $u = U \circ \omega$, the following basic formula holds:

$$\operatorname{div}[A(z)\nabla(U(\omega(z)))] = J_\omega(z)\triangle U(\omega(z)), \tag{1.4}$$

where $J_\omega(z)$ stands for the Jacobian of a quasiconformal mapping ω, *agreed* with the matrix function $A(z)$. This formula, which is understood in the sense of distributions, takes place, in particular, if $U \in W^{1,2}_{\text{loc}}(G)$, $A \in M_K^{2\times 2}(D)$, and $\omega : D \to G$ is a quasiconformal homeomorphism satisfying the Beltrami equation

$$\omega_{\bar z}(z) = \mu(z)\omega_z(z) \quad \text{a.e. in } D. \tag{1.5}$$

Here, the complex dilatation

$$\mu(z) = \frac{1}{\det\,(I + A)}(a_{22} - a_{11} - 2ia_{12}), \tag{1.6}$$

satisfies the uniform ellipticity condition

$$|\mu(z)| \le \frac{1+K}{1-K}. \tag{1.7}$$

Vice versa, given a measurable complex–valued function μ, satisfying (1.7), one can invert the algebraic system (1.6) to obtain

$$A(z) = \begin{pmatrix} \frac{|1-\mu|^2}{1-|\mu|^2} & \frac{-2\operatorname{Im}\mu}{1-|\mu|^2} \\ \frac{-2\operatorname{Im}\mu}{1-|\mu|^2} & \frac{|1+\mu|^2}{1-|\mu|^2} \end{pmatrix}. \tag{1.8}$$

The compositional property (1.4) for the operator $L(u)$ can be applied to the study of a wide range of problems arising in the contemporary analysis in the plane. For instance, (1.4) is useful in the study of such semilinear partial differential equations in the anisotropic case such as the heat equation

$$u_t - \operatorname{div}[A(z)\nabla u(z)] = f(z, u) \tag{1.9}$$

(the same equation describes the Brownian motion, diffusion models of population dynamics, and many other phenomena), the wave equation

$$u_{tt} - \operatorname{div}[A(z)\nabla u(z)] = f(z, u), \tag{1.10}$$

and the Schrödinger equation

$$iu_t + \operatorname{div}[A(z)\nabla u(z)] = k|u|^p u, \tag{1.11}$$

as well as their stationary counterparts. Note also a very interesting recent preprint [27], where the authors have developed a method for the study of spectral properties of the operators $L(u)$ with the

Neumann boundary condition in (non)convex domains in the complex plane. The suggested method is based on the composition operators on Sobolev spaces with applications to the Poincaré inequalities.

The composition property (1.4) for the operator $L(u)$ implies that the study of semilinear equations of the form (1.2) is decomposed into the research of the proper quasilinear Poisson equation

$$\triangle U(w) = J(w)f(U), \;\; w \in G = \omega(D), \tag{1.12}$$

where the weight $J(w)$ stands for the Jacobian of the inverse quasiconformal mapping $\omega^{-1} : G \to D$, and the study of the mapping $\omega(z)$ agreed with the matrix function $A(z)$. In other words, every weak solution $u(z)$ to Eq. (1.2) in a domain D is represented in the form $u(z) = U(\omega(z))$, where $\omega : D \to G$ stands for a quasiconformal mapping generated by the matrix function $A(z)$, and $U(w)$ is a weak solution to the quasilinear Poisson equation (1.12) in the domain $G = \omega(D)$, see [36], Theorem 4.1.

On the one hand, this opens up new possibilities for the study of (1.12), because we can apply a wide range of effective methods both of the potential theory and genuinely nonlinear methods which did not belong to the world of classical harmonic analysis, see, e.g., [16, 26, 30, 51, 52, 64, 68], and the exhaustive bibliography therein. On the other hand, a comprehensively developed theory of quasiconformal mappings in the plane, see, e.g., [1, 4, 12, 13, 42, 55], and also [41, 43], allows us to study the regularity properties for solutions to Eqs. (1.2) and (1.12) and the proper representation of such solutions in detail. A rather comprehensive treatment of the present state of the theory concerning the semilinear equations of the form (1.12) is given in the excellent books by M. Marcus and L. Véron [58] and L. Véron [73].

Note an important family of quasilinear Poisson equations that involves an absorption term such that $uf(u) \geq 0$. Such equations are of particular interest, because, in particular, they describe two competing effects observed in a number of applications: the diffusion expressed by the linear differential part and the absorption produced by the nonlinearity of the right-hand side. Among the variety of model semilinear equations in the plane, we recall the Liouville–Bieberbach equation

$$\triangle u = e^{u} \tag{1.13}$$

investigated by Bieberbach in his celebrated work [10] related to the study of automorphic functions in the plane.

The Liouville–Bieberbach semilinear equation is one of the principal model equations in the theory of non-linear partial differential equations and their applications, see, e.g., [58] and the references therein. Note that the equation appears also as a model one in problems of differential geometry in relation to the existence of surfaces with negative Gaussian curvature [72] and in studying the equilibrium of a charged gas.

In order to illustrate our approach to the study of Eq. (1.2), we complete the introduction with several non-trivial model examples. The corresponding proofs together with other examples have been given in [36] and [38].

n 1. Assume that the reaction term in (1.2) is non-negative. Since $u = U(\omega(z))$, and U satisfies Eq. (1.12) in $G = \omega(D)$, we see, having in mind that the Jacobian $J(w)$ of the quasiconformal mapping $\omega^{-1}(w)$ is non-negative almost everywhere in G, that U is a subharmonic function in G. Thus we arrive at the following generalization of the above-mentioned harmonic factorization theorem: Every solution $u \in W^{1,2}_{\text{loc}}(D)$ of the semilinear equation (1.2) with $f(u) \geq 0$ can be expressed as $u(z) = U(\omega(z))$, where $\omega : D \to G$ is a K-quasiconformal mapping agreed with the matrix function $A(z)$, and U is subharmonic in G.

n 2. Let us consider the divergence form of the model Liouville–Bieberbach semilinear equation

$$\operatorname{div}[A(z)\nabla u] = e^{u}, \;\; z = x + iy, \tag{1.14}$$

in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. If the matrix function $A(z)$ has the entries

$$a_{11} = 3 - 2\frac{x^2 - y^2 - 2xy}{(x^2 + y^2)}, \ a_{12} = -2\frac{x^2 - y^2 + 2xy}{(x^2 + y^2)},$$

$$a_{22} = 3 + 2\frac{x^2 - y^2 - 2xy}{(x^2 + y^2)},$$

it is easy to verify that the function

$$u(x, y) = -2\log(1 - x^2 - y^2) + \log 8$$

realizes the blow-up solution to Eq. (1.14) in the disk $\mathbb{D}$. In this case, the matrix function $A(z)$ generates the well-known logarithmic spiral quasiconformal mapping

$$\omega(z) = ze^{2i\log|z|}$$

which plays an important role in the study of different problems of contemporary analysis, see, e.g., [13, §13.2], [22, 32]. This function ω maps the unit disk $\mathbb{D}$ onto itself and transforms radial lines into spirals infinitely winding around the origin. Since the mapping ω is volume-preserving, problem (1.14) is reduced to the well-known solvability result, see [10] and [58], Theorem 5.3.7, for the Liouville–Bieberbach equation (1.13).

n 3. Let $\mathbb{C}$ be the complex plane, and let

$$A(z) = \begin{pmatrix} 1 & \mp\frac{2\nu(x)}{\sqrt{1-\nu^2(x)}} \\ \mp\frac{2\nu(x)}{\sqrt{1-\nu^2(x)}} & \frac{1+3\nu^2(x)}{1-\nu^2(x)} \end{pmatrix}, \tag{1.15}$$

where $\nu(x)$, $x \in \mathbb{R}$, stands for an arbitrary measurable real-valued function such that $|\nu(x)| \le k < 1$. Then the semilinear equation

$$\operatorname{div}[A(z)\nabla u] = u^q, \ \ 0 < q < 1, \ \ z \in \mathbb{C}, \tag{1.16}$$

has the following solution with a "dead zone" in the complex plane:

$$u(x, y) = \begin{cases} \gamma\left(y \pm \int\limits_0^x \frac{2\nu(t)}{\sqrt{1-\nu^2(t)}}dt\right)^{\frac{2}{1-q}}, & if\ y > \varphi(x),\ x \in \mathbb{R}, \\ 0 & if\ x \le \varphi(x). \end{cases} \tag{1.17}$$

Here,

$$\gamma = \left(\frac{(1-q)^2}{2(1+q)}\right)^{\frac{1}{1-q}},$$

and

$$y = \varphi(x) = \pm\int\limits_0^x \frac{2\nu(t)dt}{\sqrt{1-\nu^2(t)}}, \ \ \infty < x < +\infty,$$

stands for the corresponding free boundary parametrization.

In this paper, for the sake of completeness, we will collect some basic facts from our recent research concerning semilinear partial differential equations in the plane and give a number of new results on the topic. The paper is organized as follows. In Section 2, we give basic facts from the potential theory. In Sections 3 and 4, one can find existence theorems for the quasilinear Poisson equation (1.12) as well as for the corresponding semilinear equation (1.2) without boundary conditions. We study the solvability of the Dirichlet problem with arbitrary continuous boundary data for the quasilinear Poisson equations (1.12) in Section 5. Section 6 is devoted to the solvability of the Dirichlet problem with continuous boundary data for the semilinear equation (1.2), and it also contains some applications. In the rest sections, we discuss the boundary-value problem for the Poisson equations with boundary data that are measurable with respect to the logarithmic capacity.

2. Basic facts from the potential theory

For the sake of completeness, we repeat the fundamental results [35, 39] concerning the potential theory in the plane and strengthen some of them.

In what follows, $\mathbb{D}$ denotes the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$, $\mathbb{D}_R(z_0) := \{z \in \mathbb{C} : |z - z_0| < R\}$ for $z_0 \in \mathbb{C}$ and $R \in (0, \infty)$, $\mathbb{D}_R := \mathbb{D}_R(0)$.

For z and $w \in \mathbb{D}$ with $z \neq w$, let

$$G(z,w) := \log\left|\frac{1 - z\bar{w}}{z - w}\right| \quad \text{and} \quad P(z, e^{it}) := \frac{1 - |z|^2}{|1 - ze^{-it}|^2} \tag{2.1}$$

be the *Green function* and the *Poisson kernel* in $\mathbb{D}$. If $\varphi \in C(\partial\mathbb{D})$ and $g \in C(\overline{\mathbb{D}})$, then a solution to the *Poisson equation*

$$\triangle f(z) \;=\; g(z) \tag{2.2}$$

satisfying the boundary condition $f|_{\partial\mathbb{D}} = \varphi$ is given by the formula

$$f(z) \;=\; \mathcal{P}_{\varphi}(z) - \mathcal{G}_g(z), \tag{2.3}$$

where

$$\mathcal{P}_{\varphi}(z) \;=\; \frac{1}{2\pi}\int\limits_0^{2\pi} P(z, e^{it})\,\varphi(e^{-it})\,dt\;, \quad \mathcal{G}_g(z) \;=\; \int\limits_{\mathbb{D}} G(z,w)\,g(w)\,dm(w)\;, \tag{2.4}$$

see, e.g., [45], pp. 118-120. Here, $m(w)$ denotes the Lebesgue measure in $\mathbb{C}$.

Next, we give the representation of solutions to the Poisson equation in the form of the Newtonian (normalized antilogarithmic) potential.

Given a finite Borel measure ν on $\mathbb{C}$ with compact support, its *potential* is the function $p_\nu : \mathbb{C} \to [-\infty, \infty)$ defined by

$$p_\nu(z) \;=\; \int\limits_{\mathbb{C}} \ln|z - w|\,d\nu(w), \tag{2.5}$$

see [64], point 3.1.1.

Remark 1. Note that the function p_ν is subharmonic by Theorem 3.1.2 and, consequently, is locally integrable on $\mathbb{C}$ by Theorem 2.5.1 in [64]. Moreover, p_ν is harmonic outside of the support of ν.

This definition can be extended to finite *charges* ν with compact support (named also *signed measures*), i.e., to real-valued sigma-additive functions on Borel sets in $\mathbb{C}$, because of $\nu = \nu^+ - \nu^-$, where ν^+ and ν^- are Borel measures by the well-known Jordan decomposition, see, e.g., Theorem 0.1 in [52].

The key fact is the following statement, see, e.g., Theorem 3.7.4 in [64].

Proposition 1. *Let ν be a finite charge with compact support in $\mathbb{C}$. Then*

$$\triangle p_\nu \ = \ 2\pi \cdot \nu \tag{2.6}$$

in the distributional sense, i.e.,

$$\int\limits_{\mathbb{C}} p_\nu(z)\, \triangle\psi(z)\, d\,m(z) \ = \ 2\pi \int\limits_{\mathbb{C}} \psi(z)\, d\,\nu(z) \qquad \forall\ \psi \in C_0^\infty(\mathbb{C})\ . \tag{2.7}$$

Here as usual, $C_0^\infty(\mathbb{C})$ denotes the class of all infinitely differentiable functions $\psi:\ \mathbb{C}\to\mathbb{R}$ with compact support in $\mathbb{C}$, $\triangle = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplace operator, and $d\,m(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$.

Corollary 1. *In particular, if, for every Borel set B in $\mathbb{C}$,*

$$\nu(B) \ := \ \int\limits_B g(z)\ d\,m(z), \tag{2.8}$$

where $g:\mathbb{C}\to\mathbb{R}$ is an integrable function with compact support, then

$$\triangle N_g \ = \ g\ , \tag{2.9}$$

where

$$N_g(z) \ := \ \frac{1}{2\pi}\int\limits_{\mathbb{C}} \ln|z-w|\, g(w)\ d\,m(w)\ , \tag{2.10}$$

in the distributional sense, i.e.,

$$\int\limits_{\mathbb{C}} N_g(z)\, \triangle\psi(z)\, d\,m(z) \ = \ \int\limits_{\mathbb{C}} \psi(z)\, g(z)\, d\,m(z) \qquad \forall\ \psi \in C_0^\infty(\mathbb{C})\ . \tag{2.11}$$

Here, the function g is called the *density of the charge* ν and the function N_g is said to be the *Newtonian potential* of g.

The next statement on the continuity in the mean of functions $\psi:\mathbb{C}\to\mathbb{R}$ in $L^q(\mathbb{C})$, $q\in[1,\infty)$, with respect to shifts is useful in the study of the Newtonian potential, see, e.g., Theorem 1.4.3 in [69], cf. also Theorem III(11.2) in [67]. The one-dimensional analog of the statement can be found also in [65], Theorem 9.5.

Lemma 1. *Let $\psi\in L^q(\mathbb{C})$, $q\in[1,\infty)$, have a compact support. Then*

$$\lim_{\Delta z\to 0}\int\limits_{\mathbb{C}} |\psi(z+\Delta z)-\psi(z)|^q\ d\,m(z) \ = \ 0\ . \tag{2.12}$$

Recall that a *shift of a set $E\subset\mathbb{C}$ by a vector $\Delta z\in\mathbb{C}$* is the set

$$E+\Delta z \ := \ \{\ \xi\in\mathbb{C}:\ \xi \ = \ z\ +\ \Delta z\ ,\ z\in E\ \}\ .$$

We prefer to give a direct proof of this important statement that may be of independent interest. The proof is based on arguments by contradiction and the absolute continuity of indefinite integrals.

Proof. Let us assume that there is a sequence $\Delta z_n \in \mathbb{C}$, $n = 1, 2, \ldots$, such that $\Delta z_n \to 0$ as $n \to \infty$ and, for some $\delta > 0$ and $\psi_n(z) := \psi(z + \Delta z_n)$, $n = 1, 2, \ldots$,

$$I_n \ := \ \left[\int\limits_{\mathbb{C}} |\psi_n(z) - \psi(z)|^q \ dm(z) \right]^{\frac{1}{q}} \ \geq \ \delta \qquad \forall \ n = 1, 2, \ldots \ . \tag{2.13}$$

Denote by K the closed disk in $\mathbb{C}$ centered at 0 with the minimal radius R that contains the support of ψ. By the Luzin theorem, see, e.g., Theorem 2.3.5 in [20], for every prescribed $\varepsilon > 0$, there is a compact set $C \subset K$ such that $g|_C$ is continuous and $m(K \setminus C) < \varepsilon$. With no loss of generality, we may assume that $C \subset K_*$, where K_* is a closed disk in $\mathbb{C}$ centered at 0 with a radius $r \in (0, R)$, and, moreover, that $C_n \subset K$, where $C_n := C - \Delta z_n$ for all $n = 1, 2, \ldots$. Note that $m(C_n) = m(C)$. Then $m(K \setminus C_n) < \varepsilon$ and, consequently, $m(K \setminus C_n^*) < 2\varepsilon$, where $C_n^* := C \cap C_n$, because $K \setminus C_n^* = (K \setminus C_n) \cup (K \setminus C)$.

Next, setting $K_n = K - \Delta z_n$, we see that $K \cup K_n = C_n^* \cup (K \setminus C_n^*) \cup (K_n \setminus C_n^*)$ and $K_n \setminus C_n^* + \Delta z_n = K \setminus C_n^*$. Hence, by the triangle inequality for the norm in L^p, the following estimate holds:

$$I_n \leq 4 \cdot \left[\int\limits_{K \setminus C_n^*} |\psi(z)|^q dm(z) \right]^{\frac{1}{q}} + \left[\int\limits_{C_n^*} |\psi_n(z) - \psi(z)|^q dm(z) \right]^{\frac{1}{q}} \ \forall \ n = 1, 2, \ldots$$

By construction, both terms on the right-hand side can be made to be arbitrarily small, respectively, for small enough ε because of absolute continuity of indefinite integrals and for all large enough n after the choice of the set C. Thus, assumption (2.13) is disproved. $\square$

Let (X, d) and (X', d') be metric spaces with distances d and d', respectively. A family $\mathfrak{F}$ of mappings $f : X \to X'$ is called **equicontinuous at a point**$x_0 \in X$, if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathfrak{F}$ and $x \in X$ with $d(x, x_0) < \delta$. The family $\mathfrak{F}$ is said to be **equicontinuous,**if $\mathfrak{F}$ is equicontinuous at every point $x_0 \in X$.

Lemma 2. *Let $g : \mathbb{C} \to \mathbb{R}$ be in $L^p(\mathbb{C})$ for $p > 1$ with compact support. Then N_g is continuous. A collection $\{N_g\}$ is equicontinuous, if the collection $\{g\}$ is bounded by the norm in $L^p(\mathbb{C})$ with supports in a fixed disk K. Moreover, under the latter hypothesis, on each compact set S in $\mathbb{C}$,*

$$\|N_g\|_C \ \leq \ M \cdot \|g\|_p, \tag{2.14}$$

where M is a constant depending, in general, on S, but not on g.

Proof. By the Hölder inequality with $\frac{1}{q} + \frac{1}{p} = 1$, we have

$$|N_g(z) - N_g(\zeta)| \ \leq \ \frac{\|g\|_p}{2\pi} \cdot \left[\int\limits_K | \ln|z - w| - \ln|\zeta - w| \,|^q \ dm(w) \right]^{\frac{1}{q}}$$

$$= \ \frac{\|g\|_p}{2\pi} \cdot \left[\int\limits_{\mathbb{C}} | \, \psi_\zeta(\xi + \Delta z) - \psi_\zeta(\xi) \,|^q \ dm(\xi) \right]^{\frac{1}{q}},$$

where $\xi = \zeta - w$, $\Delta z = z - \zeta$, $\psi_\zeta(\xi) := \chi_{K+\zeta}(\xi) \ln |\xi|$. Thus, the first two conclusions hold true by Lemma 1, because the function $\ln |\xi|$ belongs to the class $L^q_{\rm loc}(\mathbb{C})$ for all $q \in [1, \infty)$.

The third conclusion similarly follows through the direct estimate

$$|N_g(\zeta)| \leq \frac{\|g\|_p}{2\pi} \cdot \left[\int\limits_K |\ln|\zeta - w||^q \, dm(w) \right]^{\frac{1}{q}}$$

$$= \frac{\|g\|_p}{2\pi} \cdot \left[\int\limits_{\mathbb{C}} |\psi_\zeta(\xi)|^q \, dm(\xi) \right]^{\frac{1}{q}} ,$$

because the last integral is continuous in $\zeta \in \mathbb{C}$. Indeed, by the triangle inequality for the norm in $L^q(\mathbb{C})$, we see that

$$|\|\psi_\zeta\|_q \; - \; \|\psi_{\zeta_*}\|_q| \; \leq \; \|\psi_\zeta - \psi_{\zeta_*}\|_q \; = \; \left\{ \int\limits_\Delta |\ln|\xi||^q \, dm(\xi) \right\}^{\frac{1}{q}} ,$$

where Δ denotes the symmetric difference of the disks $K + \zeta$ and $K + \zeta_*$. Thus, the statement follows from the absolute continuity of the indefinite integral. □

The corresponding statement on the continuity of integrals of the potential type in higher dimensions can be found in [69], Theorem 1.6.1.

Proposition 2. *There exist functions $g \in L^1(\mathbb{C})$ with compact support whose potentials N_g are not continuous, furthermore, $N_g \notin L^\infty_{\text{loc}}$.*

Proof. Indeed, let us consider the function

$$\omega(t) := \frac{1}{t^2(1-\ln t)^\alpha} \ , \ t \in (0,1] \ , \ \alpha \in (1,2) \ , \qquad \omega(0) := \infty$$

and, correspondingly,

$$g(z) := \omega(|z|) \ , \ z \in \overline{\mathbb{D}} \ , \qquad g(z) := 0 \ , \ z \in \mathbb{C} \setminus \overline{\mathbb{D}} \ .$$

Then, setting $\Omega(t) = t \cdot \omega(t)$, we see, firstly, that

$$\int\limits_{\mathbb{D}} |g(w)| d\, m(w) = 2\pi \lim_{\rho \to +0} \int\limits_\rho^1 \Omega(t) d\, t = 2\pi \lim_{\rho \to +0} \int\limits_\rho^1 \frac{d \ln t}{(1-\ln t)^\alpha} = \frac{2\pi}{\alpha - 1}$$

and, secondly, the Newtonian potential N_g at the origin is equal to

$$\lim_{\rho \to +0} \int\limits_\rho^1 \Omega(t) \ln t \, dt = \lim_{\rho \to +0} \left\{ \left[\ln \frac{1}{t} \int\limits_t^1 \Omega(\tau) d\tau \right]_\rho^1 + \int\limits_\rho^1 \left(\frac{1}{t} \int\limits_t^1 \Omega(\tau) d\tau \right) dt \right\}$$

$$= \frac{1}{\alpha - 1} \cdot \lim_{\rho \to +0} \left(\left[\frac{\ln t}{(1-\ln t)^{\alpha-1}} \right]_\rho^1 - \int\limits_\rho^1 \frac{d\, t}{t(1-\ln t)^{\alpha-1}} \right)$$

$$= \frac{1}{\alpha - 1} \cdot \lim_{\rho \to +0} \left[\frac{1}{(1-\ln t)^{\alpha-1}} \; + \frac{\alpha - 1}{2 - \alpha} \cdot (1 - \ln t)^{2-\alpha} \right]_\rho^1 \; = \; -\infty \ .$$

□

The following lemma on the Newtonian potentials is important for obtaining the solutions of a higher regularity to the Poisson equations, as well as to the corresponding semilinear equations.

In this connection, recall the definition of the formal complex derivatives:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left\{\frac{\partial}{\partial x} - i\cdot\frac{\partial}{\partial y}\right\}, \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left\{\frac{\partial}{\partial x} + i\cdot\frac{\partial}{\partial y}\right\}, \quad z = x+iy\ . \tag{2.15}$$

The elementary algebraic calculations show their relation to the Laplacian

$$\triangle := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\cdot\frac{\partial^2}{\partial z\partial\overline{z}} = 4\cdot\frac{\partial^2}{\partial\overline{z}\partial z}. \tag{2.16}$$

Further, we apply the theory of the well-known integral operators

$$Tg(z) := \frac{1}{\pi}\int\limits_{\mathbb{C}} g(w)\,\frac{dm(w)}{z-w}\ , \quad \overline{T}g(z) := \frac{1}{\pi}\int\limits_{\mathbb{C}} g(w)\,\frac{dm(w)}{\overline{z}-\overline{w}}$$

defined for integrable functions with a compact support K and studied in detail. Recall the known results on them in Chapter 1 of [71], confining the case $K = \overline{\mathbb{D}}$, that are relevant to the proof of Theorem 2.

First of all, if $g \in L^1(\mathbb{C})$, then, by Theorem 1.13, the integrals Tg and $\overline{T}g$ exist a.e. in $\mathbb{C}$ and belong to $L^q_{\rm loc}(\mathbb{C})$ for all $q \in [1,2)$. By Theorem 1.14, they have generalized derivatives by Sobolev $(Tg)_{\overline{z}} = g = (\overline{T}g)_z$. Furthermore, if $g \in L^p(\mathbb{C})$, $p>1$, then, by Theorem 1.27 and (6.27), Tg and $\overline{T}g$ belong to $L^q_{\rm loc}(\mathbb{C})$ for some $q>2$. Moreover, by Theorems 1.36–1.37, $(Tg)_z$ and $(\overline{T}g)_{\overline{z}}$ also belong to $L^p_{\rm loc}(\mathbb{C})$. Finally, if $g \in L^p(\mathbb{C})$ for $p>2$, then, by Theorem 1.19, Tg and $\overline{T}g$ belong to $C^\alpha_{\rm loc}(\mathbb{C})$ with $\alpha = (p-2)/p$.

Here, given a domain D in $\mathbb{C}$, a function $g : D \to \mathbb{R}$ is assumed to be extended onto $\mathbb{C}$ by zero outside of D.

Lemma 3. *Let D be a bounded domain in $\mathbb{C}$. Suppose that $g \in L^1(D)$. Then $N_g \in W^{1,q}_{\rm loc}(\mathbb{C})$ for all $q \in [1,2)$, and there exist the generalized derivatives by Sobolev $\frac{\partial^2 N_g}{\partial z\partial\overline{z}}$ and $\frac{\partial^2 N_g}{\partial\overline{z}\partial z}$ and*

$$4\cdot\frac{\partial^2 N_g}{\partial z\partial\overline{z}} = \triangle N_g = 4\cdot\frac{\partial^2 N_g}{\partial\overline{z}\partial z} = g \qquad a.e.\ in\ \mathbb{C} \tag{2.17}$$

Moreover, $N_g \in L^s_{\rm loc}(\mathbb{C})$ for all $s \in [1,\infty)$. More precisely,

$$\|N_g\|_s \le \|g\|_1\cdot\|\ln|\xi|\|_s \qquad \forall\ s\in[1,\infty)\ , \tag{2.18}$$

where $\|N_g\|_s$ is in $\mathbb{D}_r$ for all $r\in(0,\infty)$, and $\|\ln|\xi|\|_s$ is in $\mathbb{D}_{R+r}$, if $D \subseteq \mathbb{D}_R$.

If $g \in L^p(D)$ for some $p\in(1,2]$, then $N_g \in W^{2,p}_{\rm loc}(\mathbb{C})$ and

$$N_g \in W^{1,\gamma}_{\rm loc}(\mathbb{C}) \qquad \forall\ \gamma\in(1,q)\ ,\ where\ \ q = 2p/(2-p) > 2\ . \tag{2.19}$$

In addition, the collection $\{N_g\}$ is locally $\beta-$Hölder equicontinuous in $\mathbb{C}$ for all $\beta\in(0,1-2/q)$, and the collection $\{N_g'\}$ of its first partial derivatives is strictly compact in $L^\gamma(D)$ for all $\gamma\in(1,q)$, if the collection $\{g\}$ is bounded in $L^p(D)$.

Finally, if $g\in L^p(D)$ for some $p>2$, then $N_g \in C^{1,\alpha}_{\rm loc}(\mathbb{C})$ with $\alpha=(p-2)/p$. Furthermore, the collection $\{N_g'\}$ is locally Hölder equicontinuous in $\mathbb{C}$ with the given α, if $\{g\}$ is bounded in $L^p(D)$.

Proof. Note that N_g is the convolution $\psi * g$, where $\psi(\xi) = \ln|\xi|$, and, hence, (2.18) follows, e.g., from Corollary 4.5.2 in [46]. Moreover, $\frac{\partial \psi * g}{\partial z} = \frac{\partial \psi}{\partial z} * g$ and $\frac{\partial \psi * g}{\partial \overline{z}} = \frac{\partial \psi}{\partial \overline{z}} * g$, see, e.g., (4.2.5) in [46]. By elementary calculations, we get

$$\frac{\partial}{\partial z} \ln|z-w| \;=\; \frac{1}{2} \cdot \frac{1}{z-w} \;, \qquad \frac{\partial}{\partial \overline{z}} \ln|z-w| \;=\; \frac{1}{2} \cdot \frac{1}{\overline{z}-\overline{w}} \;. \tag{2.20}$$

Consequently,

$$\frac{\partial N_g(z)}{\partial z} \;=\; \frac{1}{4} \cdot Tg(z) \;, \qquad \frac{\partial N_g(z)}{\partial \overline{z}} \;=\; \frac{1}{4} \cdot \overline{T}g(z) \;. \tag{2.21}$$

Thus, the rest conclusions for $g \in L^1(D)$ follow from Theorems 1.13–1.14 in [71].

Next, if $g \in L^p(D)$ with $p \in (1,2]$, then $N_g \in W^{1,\gamma}_{\rm loc}(\mathbb{C})$ for all $\gamma \in (1,q)$, where $q = 2p/(2-p) > 2$, by Theorem 1.27, (1.27) in [71]. Moreover, $N_g \in W^{2,p}_{\rm loc}(\mathbb{C})$ by Theorems 1.36–1.37 in [71]. In addition, a collection $\{N_g\}$ is locally $\beta-$Hölder equicontinuous in $\mathbb{C}$ for all $\beta \in (0, 1-2/q)$, see, e.g., Lemma 2.7 in [15], and the collection $\{N_g'\}$ of its first partial derivatives is strictly compact in $L^\gamma(D)$ for all $\gamma \in (1,q)$, if the collection $\{g\}$ is bounded by the norm in $L^p(D)$, see, e.g., Theorem 1.4.3 in [69] and Theorem 1.27 in [71].

Finally, if $g \in L^p(D)$ for some $p > 2$, then $N_g \in C^{1,\alpha}_{\rm loc}(\mathbb{C})$ with $\alpha = (p-2)/p$ by Theorem 1.19 in [71]. Furthermore, by the last theorem, the collection $\{N_g'\}$ is also locally $\alpha-$Hölder equicontinuous in $\mathbb{C}$ with $\alpha = (p-2)/p$, if the collection $\{g\}$ is bounded by the norm in $L^p(D)$, $p > 2$. □

Remark 2. Note that, generally speaking, $N_g \notin W^{2,1}_{\rm loc}$ in the case $g \in L^1(\mathbb{C})$, see, e.g., example 7.5 in [25], p. 141. Note also that the corresponding Newtonian potentials N_g in $\mathbb{R}^n$, $n \geq 3$, also belong to $W^{2,p}_{\rm loc}$, if $g \in L^p(\mathbb{C})$ for $p > 1$ with compact support, see, e.g., [26], Theorem 9.9.

As above, we assume that $g : D \to \mathbb{R}$ is extended by zero outside of D.

Corollary 2. *Let D be a subdomain of $\mathbb{D}$, and let $g : D \to \mathbb{R}$ be in $L^1(D)$ and in $L^p_{\rm loc}(D)$ for some $p > 1$. Then N_g satisfies (2.17) a.e. in D. Moreover, $N_g \in W^{1,q}_{\rm loc}(D)$ for $q > 2$, and N_g is locally Hölder continuous in D. Furthermore, $N_g \in C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$, if $g \in L^p_{\rm loc}(D)$ for $p > 2$.*

In addition, the collection $\{N_g\}$ is locally $\beta-$Hölder equicontinuous in D for all $\beta \in (0, 1-2/q)$, and the collection $\{N_g'\}$ of its first partial derivatives is strictly compact in $L^\gamma_{\rm loc}(D)$ for all $\gamma \in (1,q)$, if a collection $\{g\}$ is bounded in $L^1(D)$ and in $L^p_{\rm loc}(D)$ for some $p \in (1,2]$, where q is defined in (2.19).

Finally, the collection $\{N_g'\}$ is locally $\alpha-$Hölder equicontinuous in D with the given α, if a collection $\{g\}$ is bounded in $L^1(D)$ and in $L^p_{\rm loc}(D)$ for $p > 2$.

Proof. Given $z_0 \in D$ and $0 < R < \operatorname{dist}(z_0, \partial D)$, $N_g = N_{g_1} + N_{g_2}$ with $g_2 := g - g_1$ and $g_1 := g \cdot \chi$, where χ is the characteristic function of the disk $\mathbb{D}_R(z_0)$. The first summand satisfies all desired properties by Lemma 3, and the second one is a harmonic function in $\mathbb{D}_R(z_0)$, see, e.g., Theorem 3.1.2 in [64]. Thus, the first part holds. In the proof of the rest part, it is applied the same decomposition. However, in the case we need two following explicit estimates for the second summand in a smaller disk $\mathbb{D}_r(z_0)$, $r \in (0,R)$:

$$|N_{g_2}(z_2) - N_{g_2}(z_1)| \leq \left| \int\limits_{z_1}^{z_2} \frac{\partial N_{g_2}}{\partial z} dz \right| + \left| \int\limits_{z_1}^{z_2} \frac{\partial N_{g_2}}{\partial \bar{z}} \, d\bar{z} \right| \leq \frac{1}{2\pi} \cdot \frac{\|g\|_1}{(R-r)} \cdot |z_2 - z_1|.$$

Since the function Tg_2 is analytic in $\mathbb{D}_r(z_0)$, and the function $\overline{T}g_2 = \overline{Tg_2}$ (for the real-valued function g_2) is antianalytic in $\mathbb{D}_r(z_0)$, we get similarly

$$|N'_{g_2}(z_2) - N'_{g_2}(z_1)| \;\leq\; \frac{1}{4} \left| \int\limits_{z_1}^{z_2} \frac{\partial\, Tg_2}{\partial z}\, dz \right| \;\leq\; \frac{1}{4\pi} \cdot \frac{\|g\|_1}{(R-r)^2} \cdot |z_2 - z_1| \;.$$

Here we denote by N'_{g_2} any of the first partial derivatives of N_{g_2}, see (2.15):

$$\frac{\partial}{\partial x} \;=\; \frac{\partial}{\partial z} \;+\; \frac{\partial}{\partial \bar{z}}\;, \qquad \frac{\partial}{\partial y} \;=\; i \cdot \left(\frac{\partial}{\partial z} \;-\; \frac{\partial}{\partial \bar{z}} \right)\;, \qquad z = x + iy\;,$$

take into account relation (2.20) and calculate the given integrals over the segment $[z_1, z_2] \subset D_r(z_0)$ of straight line going through z_1, $z_2 \in \mathbb{D}_r(z_0)$. $\Box$

3. On the solvability of quasilinear Poisson equations

In this section, we study the solvability problem for quasilinear Poisson equations of the form $\triangle U = h(z) f(U)$. The well-known Leray–Schauder approach allows us to reduce the problem to the study of the corresponding linear Poisson equation from the previous section.

For the sake of completeness, we recall some definitions and basic facts of the celebrated paper [56].

First of all, Leray and Schauder define a *completely continuous* mapping from a metric space M_1 into a metric space M_2 as a continuous mapping on M_1 which takes bounded subsets of M_1 into *relatively compact* ones of M_2, i.e., with compact closures in M_2. When a continuous mapping takes M_1 into a relatively compact subset of M_1, it is nowadays said to be *compact* on M_1.

Then Leray and Schauder extend the Brouwer degree to compact perturbations of the identity I in a Banach space B, i.e., a complete normed linear space. Namely, given an open bounded set $\Omega \subset B$, a compact mapping $F : B \to B$, and $z \notin \Phi(\partial\Omega)$, $\Phi := I - F$, the *(Leray–Schauder) topological degree* $\deg\,[\Phi, \Omega, z]$ of Φ in Ω over z is constructed from the Brouwer degree, by approximating the mapping F over Ω by mappings F_ε with range in a finite-dimensional subspace B_ε (containing z) of B. It is shown that the Brouwer degrees $\deg\,[\Phi_\varepsilon, \Omega_\varepsilon, z]$ of $\Phi_\varepsilon := I_\varepsilon - F_\varepsilon$, $I_\varepsilon := I|_{B_\varepsilon}$, in $\Omega_\varepsilon := \Omega \cap B_\varepsilon$ over z stabilize for sufficiently small positive ε to a common value defining $\deg\,[\Phi, \Omega, z]$ of Φ in Ω over z.

This topological degree "algebraically counts" the number of fixed points of $F(\cdot) - z$ in Ω and conserves the basic properties of the Brouwer degree such as the additivity and homotopy invariance. Now, let a be an isolated fixed point of F. Then the *local (Leray–Schauder) index* of a is defined by $\mathrm{ind}\,[\Phi, a] := \deg[\Phi, B(a, r), 0]$ for small enough r > 0. If $a = 0$, then we say on the *index* of F. In particular, if $F \equiv 0$, correspondingly, $\Phi \equiv I$, then the index of F is equal to 1.

The fundamental Theorem 1 in [56] can be formulated in the following way: *Let B be a Banach space, and let $F(\cdot, \tau) : B \to B$ be a family of operators with $\tau \in [0, 1]$. Suppose that the following hypotheses hold:*

(H1) *$F(\cdot, \tau)$ is completely continuous on B for every $\tau \in [0, 1]$ and uniformly continuous in the parameter $\tau \in [0, 1]$ on every bounded set in B;*

(H2) *the operator $F := F(\cdot, 0)$ has a finite collection of fixed points whose total index is not equal to zero;*

(H3) *the collection of all fixed points of the operators $F(\cdot, \tau)$, $\tau \in [0, 1]$, is bounded in B.*

Then the collection of all fixed points of the family of operators $F(\cdot, \tau)$ contains a continuum along which τ takes all values in $[0, 1]$.

In the proof of the next theorem, the initial operator $F(\cdot) := F(\cdot, 0) \equiv 0$. Hence, F has the single fixed point (at the origin), and its index is equal to 1. Thus, hypothesis (H2) will be automatically satisfied.

Theorem 1. *Let $h : \mathbb{C} \to \mathbb{R}$ be a function in the class $L^p(\mathbb{C})$ for $p > 1$ with compact support. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous, and*

$$\lim_{t\to\infty} \frac{f(t)}{t} \;=\; 0\ . \tag{3.1}$$

Then there is a continuous function $U : \mathbb{C} \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc}(\mathbb{C})$ such that

$$\triangle U(z) \;=\; h(z)\cdot f(U(z)) \qquad a.e., \tag{3.2}$$

and $U = N_g$, where $g : \mathbb{C} \to \mathbb{R}$ is a function in L^p whose support is in the support of h. The upper bound of $\|g\|_p$ depends only on $\|h\|_p$ and on the function f. Moreover, $U \in W^{1,q}_{\rm loc}(\mathbb{C})$ for some $q > 2$, and U is locally Hölder continuous. Furthermore, $U \in C^{1,\alpha}_{\rm loc}(\mathbb{C})$ with $\alpha = (p-2)/p$, if $p > 2$.

In particular, $U \in C^{1,\alpha}_{\rm loc}(\mathbb{C})$ for all $\alpha = (0, 1)$, if h in Theorem 1 is bounded.

Proof. If $\|h\|_p = 0$ or $\|f\|_C = 0$, then any constant function U in $\mathbb{C}$ gives the desired solution of (3.2). Thus, we may assume that $\|h\|_p \neq 0$ and $\|f\|_C \neq 0$. Set $f_*(s) = \max\limits_{|t|\le s} |f(t)|$, $s \in \mathbb{R}^+ := [0, \infty)$. Then the function $f_* : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing. Moreover, $f_*(s)/s \to 0$ as $s \to \infty$ by (3.1).

By Lemma 2, we obtain the family of operators $F(g;\tau) : L^p_h(\mathbb{C}) \to L^p_h(\mathbb{C})$, where $L^p_h(\mathbb{C})$ consists of functions $g \in L^p(\mathbb{C})$ with supports in the support of h,

$$F(g;\tau) \;:=\; \tau h\cdot f(N_g) \qquad \forall\, \tau\in[0,1] \tag{3.3}$$

which satisfies all groups of hypotheses H1-H3 of Theorem 1 in [56]. Indeed:

H1). First of all, $F(g;\tau) \in L^p_h(\mathbb{C})$ for all $\tau \in [0, 1]$ and $g \in L^p_h(\mathbb{C})$, because, by Lemma 2, the function $f(N_g)$ is continuous, and

$$\|F(g;\tau)\|_p \;\le\; \|h\|_p\, f_*\,(\,M\,\|g\|_p) \;<\; \infty \qquad \forall\,\tau\in[0,1],$$

where M is the constant from estimate (2.14). Thus, by Lemma 2 in combination with the Arzela–Ascoli theorem, see, e.g., Theorem IV.6.7 in [17], the operators $F(g;\tau)$ are completely continuous for every $\tau \in [0, 1]$ and even uniformly continuous in the parameter $\tau \in [0, 1]$.

H2). The index of the operator $F(g; 0)$ is obviously equal to 1.

H3). For solutions to the equations $g = F(g;\tau)$, we have, by Lemma 2:

$$\|g\|_p \;\le\; \|h\|_p\, f_*\,(\,M\,\|g\|_p)$$

i.e.,

$$\frac{f_*(\,M\,\|g\|_p)}{M\,\|g\|_p} \;\ge\; \frac{1}{M\,\|h\|_p}, \tag{3.4}$$

and, hence, $\|g\|_p$ should be bounded in view of condition (3.1).

Thus, by Theorem 1 in [56], there is a function $g \in L^p_h(D)$ with $F(g; 1) = g$, and, by Lemma 3, the function $U := N_g$ gives the desired solution of (3.2). □

Corollary 3. *Let D be a subdomain of $\mathbb{D}$, $h : D \to \mathbb{R}$ be in $L^1(D)$ and in $L^p_{\rm loc}(D)$ for some $p > 1$. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the hypothesis of Theorem 1. Then there is a weak solution $u : D \to \mathbb{R}$ of the quasilinear Poisson equation (3.2) which is locally Hölder continuous in D.*

Proof. Let D_k be an expanding sequence of domains in $\mathbb{C}$ with $\overline{D_k} \subset D$, $k = 1, 2, \ldots$, exhausting D, i.e., $\bigcup\limits_{k=1}^{\infty} D_k = D$. Let us extend h by zero outside of D. Set $h_k = h\chi_k$, where χ_k is a characteristic function of D_k in $\mathbb{C}$, and $U_k = N_{g_k}$, where g_k corresponds to h_k by Theorem 1. Note that the sequence $\|g_k\|_p$, $k = 1, 2, \ldots$, is bounded on each D_m, $m = 1, 2, \ldots$, by Theorem 1. Hence, by Lemma 2, the sequence $|N_{g_k}|_C$ is also bounded on each D_m, $m = 1, 2, \ldots$. Now, by Corollary 2, the family of functions $\{N_{g_k}\}$ is Hölder equicontinuous on each D_m, $m = 1, 2, \ldots$. Thus, by the Arzela–Ascoli theorem, see, e.g., Theorem IV.6.7 in [17], the family of functions $\{N_{g_k}\}$ is compact on each D_m, $m = 1, 2, \ldots$.

Without loss of generality, we may assume that $p \in (1, 2]$. Then, by Corollary 2, the Newtonian potential $\{N_{g_k}\}$, $m = 1, 2, \ldots$, is in the class $W^{1,q}_{\rm loc}$ for some $q > 2$, and the family $\{N'_{g_k}\}$ is also compact on each D_m, $m = 1, 2, \ldots$, by the norm of L^q. Consequently, the sequence $\{N_{g_k}\}$ is compact on each D_m, $m = 1, 2, \ldots$, by any norm $\|\cdot\|$ of $W^{1,q}$, too, see, e.g., Theorem 2.5.1 in [59].

Next, let us apply the so-called diagonal process. Namely, let $u_k^{(1)}$, $k = 1, 2, \ldots$, be a subsequence of $\{N_{g_k}\}$ that converges uniformly and, by the norm $\|\cdot\|$, on the domain D_1 to a function $u : D_1 \to \mathbb{R}$. Of course, we may assume that $\|u_k^{(1)} - u\|_C < 1/k$, as well as $\|u_k^{(1)} - u\| < 1/k$ for all $k = 1, 2, \ldots$. Similarly, a subsequence $u_k^{(2)}$ of $u_k^{(1)}$ with respect to the domain D_2 can be defined. Let us continue the process by induction and, finally, consider the diagonal subsequence $u_m := u_m^{(m)}$, $m = 1, 2, \ldots$, of the sequence N_{g_k}.

It is clear by construction that $u_m|_D$ converges to a function $u : D \to \mathbb{R}$ locally uniformly, and, in $W^{1,q}_{\rm loc}(D)$, $q > 2$. Thus, $u \in C(D) \cap W^{1,q}_{\rm loc}(D)$, and, consequently, u is locally Hölder continuous in D. Moreover, u is a weak solution to Eq. (3.2) in the domain D. Indeed, by Corollary 1 and the definition of generalized derivatives, we have that u_m satisfy the relations

$$\int\limits_D \langle \nabla u_m(z), \nabla \psi(z) \rangle \; dm(z) \; + \; \int\limits_D h_m(z) f(u_m(z))\, \psi(z) \; dm(z)$$

$$= \; 0 \quad \forall \psi \in C_0^\infty(D).$$

Passing to the limit as $m \to \infty$, we obtain the desired conclusion. □

4. On the solvability of semilinear equations

In this section, we study the solvability problem for semilinear equations of the form $\operatorname{div}[A(z)\nabla u] = f(u)$. Following work [36], as *a weak solution* to this equation, we understand a function $u \in C(D) \cap W^{1,2}_{\rm loc}(D)$ such that

$$\int_D \langle A(z)\nabla u(z),\, \nabla\varphi(z)\rangle \, dm(z) + \int_D f(u(z))\varphi(z)\, dm(z) = 0 \tag{4.1}$$

for all $\varphi \in C(D) \cap W_0^{1,2}(D)$.

Theorem 2. *Let D be a domain in $\mathbb{C}$ with a finite area that is not dense in $\mathbb{C}$. Suppose that $A \in M_K^{2\times 2}(D)$, and a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies condition* (3.1)*. Then a weak solution $u : D \to \mathbb{R}$ to the equation*

$$\operatorname{div}[A(z)\nabla u(z)] = f(u) \tag{4.2}$$

exists and is locally Hölder-continuous in D.

Proof. Let us extend, by definition, $A \equiv I$ (the unit matrix) outside of D. By Theorem 4.1 in [36], if u is a weak solution of (4.2), then $u = U \circ \omega$, where $\omega := \Omega|_D$, and Ω is a quasiconformal mapping of $\overline{\mathbb{C}}$ onto itself, $\Omega(\infty) = \infty$ agreed with the extended A, and U is a weak solution to (3.2) with $h = J$, where J is the Jacobian of the mapping $\omega^{-1} : D_* \to D$, $D_* := \omega(D)$.

Note that $\overline{\mathbb{C}} \setminus D$ contains a nondegenerate (connected) component C, because D is not dense in $\mathbb{C}$, see, e.g., Corollary IV.2 and the point II.4.D in [47], see also Lemma 5.1 in [48] or Lemma 6.3 in [57]. Hence, $\overline{\mathbb{C}} \setminus D_*$ contains a component $C_* := \Omega(C)$ whose boundary is a nondegenerate continuum, see again Lemma 5.1 in [48] or Lemma 6.3 in [57]. By the Riemann theorem, there is a conformal mapping H of $\overline{\mathbb{C}} \setminus C_*$ onto $\mathbb{D}$.

Setting $H_* = H|_{D_*}$, we see that H_* maps D_* into $\mathbb{D}$. Moreover, the quasiconformal mapping $\omega_* := H_* \circ \omega : D \to \mathbb{D}_* := H_*(D_*)$ is also agreed with A in D. Thus, by Theorem 4.1 in [36], $u = U_* \circ \omega_*$, where U_* is a weak solution to (3.2) with $h = J_*$ in $\mathbb{D}_* \subseteq \mathbb{D}$. Here, J_* is the Jacobian of the mapping $\omega_*^{-1} : \mathbb{D}_* \to D$.

By Remark 4.1 in [36], inversely, if U_* is a weak solution to (3.2) with $h = J_*$ in $\mathbb{D}_*$, then $u := U_* \circ \omega_*$ is a weak solution to (4.2) in D. The latter implication allows us to reduce the proof of Theorem 2 to that of Corollary 3 with the special $h = J_*$.

Indeed, $J_* \in L^1(\mathbb{D}_*)$, because its integral is equal to the area of the domain D, see, e.g., Theorem 3.2 in [12] and Theorem II.B.3 in [1]. Moreover, $J_* \in L^p_{\rm loc}(\mathbb{D}_*)$ for some $p > 1$, because, by the Bojarski result (see [11] and [12]), the first partial derivatives of the quasiconformal mapping $\omega^* := \omega_*^{-1} : \mathbb{D}_* \to D$ are locally integrable with a power $q > 2$, and $J_* = |\omega^*_w|^2 - |\omega^*_{\bar{w}}|^2$, see, e.g., I.A(9) in [1]. $\square$

Remark 3. Note that it is easy to construct a set C in $\mathbb{C}$ of the Cantor type which is dense in the plane $\mathbb{C}$ whose completion has a finite area, furthermore, an arbitrarily small area.

Indeed, let us cover the plane by a collection $\mathcal{S}$ consisting of closed squares with unit sides oriented along the coordinate axes x and y, $z = x + iy \in \mathbb{C}$, that can intersect each other only by their common sides. Let S_n, $n = 1, 2, \ldots$, be some enumeration of the squares in $\mathcal{S}$, and let $\varepsilon \in (0, 1)$ be arbitrary.

First, let us remove narrow symmetric strips of the same width in S_1 along its sides whose total area is less than $\varepsilon/4$. We have a central square in the rest. Then we cut out narrow centralized horizontal and vertical corridors of the same width in the last square whose total area is less than $\varepsilon/8$. These corridors form a cross that splits the last square into 4 squares. In turn, from these squares, we remove the similar crosses of the total area $\varepsilon/16$ that split them on the whole into 4^2 squares. Repeating the procedure by induction, we remove from S_1 corridors with the total area $\varepsilon/2$, and the intersection of all mentioned squares gives a totally disconnected compactum $C_1 \neq \emptyset$ of the Cantor type, see, e.g., 4.41($2'$) in [50].

Similarly, we are able to construct such the set $C_n \subset S_n$ with its completion in S_n whose area is less than $\varepsilon/2^{1+n}$ for each $n = 1, 2, \ldots$. Then the set $C := \bigcup\limits_{n=1}^{\infty} C_n$ has the completion in $\mathbb{C}$ whose area is less than ε. Note that, by our construction, the set C is totally disconnected. Thus, its topological dimension is equal to 0, see, e.g., Proposition II.4.D in [47]. It is clear that $D := \mathbb{C} \setminus C$ is a domain, see, e.g., Theorem IV.4 in [47].

Finally, note that our example of a set C of the Cantor type in the plane with its topological dimension 0 is essentially different from the well-known *Sierpinski cover* whose topological dimension is equal to 1.

5. Dirichlet problem with continuous data for quasilinear Poisson equations

Let D be a bounded domain in $\mathbb{C}$ without degenerate boundary components, i.e., any connected component of the boundary of D is not degenerated to a single point. Given a continuous boundary function $\varphi : \partial D \to \mathbb{R}$, we denote, by $\mathcal{D}_\varphi$, the harmonic function in D that has the continuous extension to $\overline{D}$ with φ as its boundary data. Such a function exists, and it is unique, see, e.g., Corollary 4.1.8 and Theorem 4.2.2 in [64]. Thus, the *Dirichlet operator* $\mathcal{D}_\varphi$ is well defined in the given domains. We do not need its explicit description for our goals.

By Lemma 3, we come to the following result on the existence, regularity, and representation of solutions to the Poisson equation for the Dirichlet problem in arbitrary bounded domains D in $\mathbb{C}$ without degenerate boundary components, where we assume that the charge density g is extended by zero outside of D.

Theorem 3. *Let D be a bounded domain in $\mathbb{C}$ without degenerate boundary components, let $\varphi : \partial D \to \mathbb{R}$ be a continuous function, and let $g : D \to \mathbb{R}$ belong to the class $L^p(D)$ for $p > 1$. Then the function*

$$U \ := \ N_g \ - \ \mathcal{D}_{N_g^*} \ + \ \mathcal{D}_\varphi \ , \qquad N_g^* \ := \ N_g|_{\partial D} \ , \tag{5.1}$$

is continuous in $\overline{D}$ with $U|_{\partial D} = \varphi$, belongs to the class $W^{2,p}_{\rm loc}(D)$, and satisfies the Poisson equation $\triangle U = g$ a.e. in D. Moreover, $U \in W^{1,q}_{\rm loc}(D)$ for some $q > 2$, and U is locally Hölder-continuous in D. Furthermore, $U \in C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$, if $g \in L^p(D)$ for $p > 2$.

Remark 4. Note that a generalized solution to the Poisson equation for the Dirichlet problem in the class $C(\overline{D}) \cap W^{1,2}_{\rm loc}(D)$ is unique at all, see, e.g., Theorem 8.30 in [26], and (5.1) gives the effective representation of this unique solution.

The case of quasilinear Poisson equations is reduced to the case of the linear Poisson equations again by the Leray–Schauder approach, as in the last section.

Theorem 4. *Let D be a bounded domain in $\mathbb{C}$ without degenerate boundary components, let $\varphi : \partial D \to \mathbb{R}$ be a continuous function, and let $h : D \to \mathbb{R}$ be a function in the class $L^p(D)$ for $p > 1$. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous and*

$$\lim_{t\to\infty} \frac{f(t)}{t} \ = \ 0 \ . \tag{5.2}$$

Then there is a continuous function $U : \overline{D} \to \mathbb{R}$ with $U|_{\partial D} = \varphi$ and $U|_D \in W^{2,p}_{\rm loc}$ such that

$$\triangle U(z) \ = \ h(z) \cdot f(U(z)) \qquad \textit{for a.e. } z \in D \ . \tag{5.3}$$

Moreover, $U \in W^{1,q}_{\rm loc}(D)$ for some $q > 2$, and U is locally Hölder-continuous. Furthermore, $U \in C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ if $p > 2$.

In particular, the last statement in Theorem 4 implies that $U \in C^{1,\alpha}_{\rm loc}(D)$ for all $\alpha = (0,1)$, if h is bounded.

Proof. If $\|h\|_p = 0$ or $\|f\|_C = 0$, then the Dirichlet operator $\mathcal{D}_\varphi$ gives the desired solution to Eq. (5.3) for the Dirichlet problem, see, e.g., I.D.2 in [49]. Hence, we may assume further that $\|h\|_p \neq 0$ and $\|f\|_C \neq 0$. Set $f_*(s) = \max\limits_{|t|\leq s} |f(t)|$, $s \in \mathbb{R}^+$. Then the function $f_* : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing, and, moreover, $f_*(s)/s \to 0$ as $s \to \infty$ by (5.2).

By Lemma 2 and the maximum principle for harmonic functions, we obtain the family of operators $F(g;\tau) : L^p(D) \to L^p(D)$, $\tau \in [0,1]$:

$$F(g;\tau) \ := \ \tau h \cdot f(N_g - \mathcal{D}_{N_g^*} + \mathcal{D}_\varphi) \ , \ N_g^* := N_g|_{\partial D} \ , \qquad \forall\, \tau \in [0,1] \tag{5.4}$$

which satisfies all groups of hypotheses H1-H3 of Theorem 1 in [56]. Indeed:

H1). First of all, $F(g;\tau) \in L^p(D)$ for all $\tau \in [0,1]$ and $g \in L^p(D)$, because,, by Lemma 2, $f(N_g - \mathcal{D}_{N_g^*} + \mathcal{D}_\varphi)$ is a continuous function, and, moreover,

$$\|F(g;\tau)\|_p \ \le \ \|h\|_p \ f_*\,(\,2M\,\|g\|_p + \|\varphi\|_C) \ < \ \infty \qquad \forall\, \tau \in [0,1] \ .$$

Thus, by Lemma 2 in combination with the Arzela–Ascoli theorem, see, e.g., Theorem IV.6.7 in [17], the operators $F(g;\tau)$ are completely continuous for each $\tau \in [0,1]$ and even uniformly continuous in the parameter $\tau \in [0,1]$.

H2). The index of the operator $F(g;0)$ is obviously equal to 1.

H3). By Lemma 2 and the maximum principle for harmonic functions, we have the estimate for solutions $g \in L^p$ to the equations $g = F(g;\tau)$:

$$\|g\|_p \ \le \ \|h\|_p \ f_*\,(\,2M\,\|g\|_p + \|\varphi\|_C) \ \le \ \|h\|_p \ f_*(\,3M\,\|g\|_p)$$

whenever $\|g\|_p \ge \|\varphi\|_C/M$, i.e., then it should be

$$\frac{f_*(\,3M\,\|g\|_p)}{3M\,\|g\|_p} \ \ge \ \frac{1}{3M\,\|h\|_p}. \tag{5.5}$$

Hence, $\|g\|_p$ should be bounded in view of condition (5.2).

Thus, by Theorem 1 in [56], there is a function $g \in L^p(D)$ such that $g = F(g;1)$, and, consequently, by Lemma 3, the function $U := N_g - \mathcal{D}_{N_g^*} + \mathcal{D}_\varphi$ gives the desired solution to the quasilinear Poisson equation (5.3) for the Dirichlet problem. □

Remark 5. As is clear from the proof, condition (5.2) can be replaced by the following weaker condition with M from the estimate in Lemma 2:

$$\limsup_{s\to\infty} \ \frac{f_*(s)}{s} \ < \ \frac{1}{3M\|h\|_p} \ . \tag{5.6}$$

Theorem 4 can be applied to some physical problems. The first circle of such applications is relevant to reaction-diffusion problems. Problems of this type are discussed in [16], p. 4, and in [3] in details. A nonlinear system is obtained for the density u and the temperature T of the reactant. By eliminating T, the system can be reduced to the equation

$$\triangle u \ = \ \lambda \cdot f(u) \tag{5.7}$$

with $h(z) \equiv \lambda > 0$. For isothermal reactions, $f(u) = u^q$, where $q > 0$ is called the order of the reaction. It turns out that the density of the reactant u may be zero in a subdomain called a *dead core.* A particularization of results in Chapter 1 of [16] shows that a dead core may exist just iff $0 < q < 1$ and λ is large enough (see the corresponding examples in [36]). In this connection, the following statements may be of independent interest.

Corollary 4. *Let D be a bounded domain in $\mathbb{C}$ without degenerate boundary components, let $\varphi : \partial D \to \mathbb{R}$ be a continuous function, and let $h : D \to \mathbb{R}$ be a function in the class $L^p(D)$, $p > 1$. Then there exists a continuous function $u : \overline{D} \to \mathbb{R}$ with $u|_{\partial D} = \varphi$ such that $u \in W^{2,p}_{\rm loc}(D)$, and*

$$\triangle u(z) \ = \ h(z) \cdot u^q(z) \ , \quad 0 \ < \ q \ < \ 1 \tag{5.8}$$

a.e. in D. Moreover, $u \in W^{1,\beta}_{\rm loc}(D)$ for some $\beta > 2$, and u is locally Hölder-continuous in D. Furthermore, $u \in C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ if $p > 2$.

Corollary 5. *Let D be a bounded domain in $\mathbb{C}$ without degenerate boundary components, and let $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a continuous function $u : \overline{D} \to \mathbb{R}$ with $u|_{\partial D} = \varphi$ such that $u \in C^{1,\alpha}_{\rm loc}(D)$ for all $\alpha \in (0, 1)$, $u \in W^{2,p}_{\rm loc}(D)$ for all $p \in [1, \infty)$, and*

$$\triangle u(z) \ = \ u^q(z) \ , \quad 0 \ < \ q \ < \ 1 \ , \quad a.e. \ in \ D \, . \tag{5.9}$$

Note also that certain mathematical models of a thermal evolution of a heated plasma lead to nonlinear equations of the type (5.7). Indeed, it is known that some of them have the form $\triangle\psi(u) = f(u)$ with $\psi'(0) = \infty$ and $\psi'(u) > 0$, if $u \neq 0$, as, for instance, $\psi(u) = |u|^{q-1}u$ for $0 < q < 1$, see, e.g., [16]. With the replacement of the function $U = \psi(u) = |u|^q \cdot {\rm sign}\, u$, we have that $u = |U|^Q \cdot {\rm sign}\, U$, $Q = 1/q$, and, with the choice $f(u) = |u|^{q^2} \cdot {\rm sign}\, u$, we come to the equation $\triangle U = |U|^q \cdot {\rm sign}\, U = \psi(U)$.

Corollary 6. *Let D be a bounded domain in $\mathbb{C}$ without degenerate boundary components, and let $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a continuous function $U : \overline{D} \to \mathbb{R}$ with $U|_{\partial D} = \varphi$ such that $U \in C^{1,\alpha}_{\rm loc}(D)$ for all $\alpha \in (0, 1)$, $u \in W^{2,p}_{\rm loc}(D)$ for all $p \in [1, \infty)$, and*

$$\triangle U(z) \ = \ |U(z)|^{q-1} U(z) \ , \quad 0 \ < \ q \ < \ 1 \ , \quad a.e. \ in \ D \, . \tag{5.10}$$

Finally, we recall that, in the combustion theory, see, e.g., [5], [62], and the references therein, the model equation

$$\frac{\partial u(z,t)}{\partial t} \ = \ \frac{1}{\delta} \cdot \triangle u \ + \ e^u \ , \quad t \geq 0, \ z \in D, \tag{5.11}$$

takes a special place. Here, $u \geq 0$ is the temperature of the medium, and δ is a certain positive parameter. We restrict ourselves here by the stationary case, although our approach makes it possible to study the parabolic equation (5.11), see [36]. Namely, Eq. (5.3) is appeared here with $h \equiv \delta > 0$ and the function $f(u) = e^{-u}$ that is bounded.

Corollary 7. *Let D be a bounded domain in $\mathbb{C}$ without degenerate boundary components, and let $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a continuous function $U : \overline{D} \to \mathbb{R}$ with $U|_{\partial D} = \varphi$ such that $U \in C^{1,\alpha}_{\rm loc}(D)$ for all $\alpha \in (0, 1)$, $u \in W^{2,p}_{\rm loc}(D)$ for all $p \in [1, \infty)$, and*

$$\triangle U(z) \ = \ \delta \cdot e^{U(z)} \ , \quad a.e. \ in \ D \, . \tag{5.12}$$

Due to the factorization theorem in [36], we extend these results to semilinear equations describing the corresponding physical phenomena in anisotropic and inhomogeneous media in any bounded domain without degenerate boundary components, see the next section.

6. Dirichlet problem with continuous data for semilinear equations

By the factorization theorem from [36] mentioned in Introduction, the study of semilinear equations (4.2) in bounded domains without degenerate boundary componentss D is reduced, by means of a suitable quasiconformal change of variables, to the study of the corresponding quasilinear Poisson equations (5.3).

Theorem 5. *Let D be a bounded domain in $\mathbb{C}$ without degenerate boundary components, let $A \in M^{2\times 2}_K(D)$, $\varphi : \partial D \to \mathbb{R}$ be an arbitrary continuous function, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\lim_{t\to\infty} \frac{f(t)}{t} \;=\; 0\;. \tag{6.1}$$

Then there is a weak solution $u : D \to \mathbb{R}$ of the class $C(D) \cap W^{1,2}_{\rm loc}(D)$ to the equation

$$\operatorname{div}[A(z)\nabla u] = f(u)$$

which is locally Hölder-continuous in D and continuous in $\overline{D}$ with $u|_{\partial D} = \varphi$.

Proof. Let us extend, by definition, $A \equiv I$ outside of D. By Theorem 4.1 in [36], if u is a weak solution to the equation, then $u = U \circ \omega$, where $\omega := \Omega|_D$, Ω is a quasiconformal mapping of $\mathbb{C}$ onto itself agreed with the extended A, and U is a weak solution to Eq. (5.3) with $h = J$, where J is the restriction of the Jacobian of the mapping $\Omega^{-1} : \mathbb{C} \to \mathbb{C}$ to the domain $D_* := \Omega(D)$.

Inversely, by Remark 4.1 in [36], we see that if U is a weak solution to (5.3) with $h = J$, then $u = U \circ \omega$ is a weak solution to our equation. The latter allows us to reduce Theorem 5 to Theorem 4. Indeed, $\overline{D_*} = \Omega(\overline{D})$ is compact, and, by the celebrated Bojarski result, see [11] and [12], the generalized derivatives of the quasiconformal mapping $\Omega^* := \Omega^{-1} : \mathbb{C} \to \mathbb{C}$ are locally integrable with some power $q > 2$. Note also that the Jacobian J of its restriction $\omega^* := \Omega^*|_{D_*}$ is equal to $|\omega^*_w|^2 - |\omega^*_{\overline{w}}|^2$, see, e.g., I.A(9) in [1]. Consequently, $J \in L^p(D_*)$ for some $p > 1$. □

Specifying the reaction term $f(u)$ of the semilinear equation, we arrive at the following statements concerning some concrete problems of mathematical physics in inhomogeneous and anisotropic media.

Corollary 8. *Let D be a bounded domain in $\mathbb{C}$ without degenerate boundary components, $A \in M^{2\times 2}_K(D)$ and $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a continuous function $u : \overline{D} \to \mathbb{R}$ with $u|_{\partial D} = \varphi$ which is locally Hölder-continuous in D, and it is a weak solution in D to the equation*

$$\operatorname{div}[\,A(z)\,\nabla u(z)\,] \;=\; u^q(z)\;, \quad 0 \;<\; q \;<\; 1\;. \tag{6.2}$$

Corollary 9. *Let D be a bounded domain in $\mathbb{C}$ without degenerate boundary components, let $A \in M^{2\times 2}_K(D)$, and let $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a continuous function $u : \overline{D} \to \mathbb{R}$ with $u|_{\partial D} = \varphi$ which is locally Hölder-continuous in D, and it is a weak solution in D to the equation*

$$\operatorname{div}[\,A(z)\,\nabla u(z)\,] \;=\; |u(z)|^{q-1}u(z)\;, \quad 0 \;<\; q \;<\; 1\;. \tag{6.3}$$

Corollary 10. *Let D be a bounded domain in $\mathbb{C}$ without degenerate boundary components, let $A \in M^{2\times 2}_K(D)$, and let $\varphi : \partial D \to \mathbb{R}$ be a continuous function. Then there is a continuous function*

$u : \overline{D} \to \mathbb{R}$ with $u|_{\partial D} = \varphi$ which is locally Hölder-continuous in D, and it is a weak solution in D to the equation

$$\operatorname{div}\left[\, A(z)\, \nabla u(z)\,\right] \;=\; e^{\alpha u(z)}\,, \qquad \alpha \in \mathbb{R}\;. \tag{6.4}$$

Note that the statements given above remain valid, if the reaction terms in Eqs. (6.2)–(6.4) are multiplied by functions $C \in L^{\infty}(D)$.

The rest of the paper is devoted to the study of the Dirichlet problem for the Poisson equation with measurable boundary data. We start with the notion of a logarithmic capacity.

7. Definition and preliminary remarks on a logarithmic capacity

Given a bounded Borel set E in the plane $\mathbb{C}$, the *mass distribution* on E is a nonnegative completely additive function ν of a set defined on its Borel subsets with $\nu(E) = 1$. The function

$$U^{\nu}(z) := \int\limits_{E} \log\left|\frac{1}{z-\zeta}\right| \, d\nu(\zeta) \tag{7.1}$$

is called a *logarithmic potential* of the mass distribution ν at a point $z \in \mathbb{C}$. The *logarithmic capacity* $C(E)$ of the Borel set E is the quantity

$$C(E) = e^{-V}\;, \qquad V \;=\; \inf_{\nu}\; V_{\nu}(E)\;, \qquad V_{\nu}(E) \;=\; \sup_{z}\; U^{\nu}(z)\;. \tag{7.2}$$

The following geometric characterization of the logarithmic capacity is well known, see, e.g., point 110 in [61]:

$$C(E) \;=\; \tau(E) \;:=\; \lim_{n\to\infty}\; V_n^{\frac{2}{n(n-1)}}, \tag{7.3}$$

where V_n denotes the supremum of the product

$$V(z_1, \ldots, z_n) \;=\; \prod_{k<l}^{l=1,\ldots,n} |z_k - z_l| \tag{7.4}$$

taken over all collections of points $z_1, \ldots, z_n$ in the set E. Following Fékete, see [21], the quantity $\tau(E)$ is called the *transfinite diameter* of the set E.

Remark 6. Thus, we see that if $C(E) = 0$, then $C(f(E)) = 0$ for an arbitrary mapping f that is continuous by Hölder and, in particular, for quasiconformal mappings on compact sets, see, e.g., Theorem II.4.3 in [55].

In order to introduce sets that are measurable with respect to the logarithmic capacity, we define, following [18], *inner* C_* and *outer* C^* *capacities*:

$$C_*(E) \;:=\; \sup_{F\subseteq E}\; C(F), \qquad\qquad C^*(E) \;:=\; \inf_{E\subseteq O}\; C(O), \tag{7.5}$$

where the supremum is taken over all compact sets $F \subset \mathbb{C}$, and infimum is taken over all open sets $O \subset \mathbb{C}$. The *set* $E \subset \mathbb{C}$ is called *measurable with respect to the logarithmic capacity*, if $C^*(E) = C_*(E)$. The common value of $C_*(E)$ and $C^*(E)$ is denoted by $C(E)$.

A *function* $\varphi : E \to \mathbb{C}$ defined on a bounded set $E \subset \mathbb{C}$ is called *measurable with respect to the logarithmic capacity*, if, for all open sets $O \subseteq \mathbb{C}$, the sets $\{z \in E : \varphi(z) \in O\}$ are measurable with respect to the logarithmic capacity. It is clear from the definition that the set E is itself measurable with respect to the logarithmic capacity.

Note that the sets with zero logarithmic capacity coincide with sets of the so-called *absolute harmonic measure* equal to zero introduced by Nevanlinna, see Chapter V in [61]. Hence, the set E has the zero (Hausdorff) length, if $C(E) = 0$, see Theorem V.6.2 in [61]. However, there exist sets of length zero having a positive logarithmic capacity, see, e.g., Theorem IV.5 in [18].

Remark 7. It is known that Borel sets and, in particular, compact and open sets are measurable with respect to the logarithmic capacity, see, e.g., Lemma I.1 and Theorem III.7 in [18]. Moreover, by definition, any set $E \subset \mathbb{C}$ with finite logarithmic capacity can be represented as the union of a sigma-compactum (union of a countable collection of compact sets) and a set with zero logarithmic capacity. Thus, the measurability of functions with respect to the logarithmic capacity is invariant under a Hölder-continuous change of variables.

It is also known that the Borel sets and, in particular, compact sets are measurable with respect to all Hausdorff's measures and, in particular, with respect to the measure of length, see, e.g., theorem II(7.4) in [67]. Consequently, any set $E \subset \mathbb{C}$ with finite logarithmic capacity is measurable with respect to the measure of length. Thus, on such a set, any function $\varphi : E \to \mathbb{C}$ measurable with respect to the logarithmic capacity is also measurable with respect to the measure of length on E. However, there exist functions that are measurable with respect to the measure of length, but not measurable with respect to the logarithmic capacity, see, e.g., Theorem IV.5 in [18].

Dealing with measurable boundary functions $\varphi(\zeta)$ with respect to the logarithmic capacity, we will use the *abbreviation q.e. (quasieverywhere)* on a set $E \subset \mathbb{C}$, if a property holds for all $\zeta \in E$ except its subset with zero logarithmic capacity, see [52].

8. Dirichlet problem with measurable data in the unit disk for the Poisson equations

We start with the following analog of the known Luzin theorem on the primitive, see, e.g., Theorem VII(2.3) in [67], in terms of the logarithmic capacity.

Proposition 3. [19]. *Let $\varphi : [a, b] \to \mathbb{R}$ be a measurable function with respect to the logarithmic capacity. Then there is a continuous function $\Phi : [a, b] \to \mathbb{R}$ with $\Phi'(x) = \varphi(x)$ q.e. on (a, b). Furthermore, Φ can be chosen with $\Phi(a) = \Phi(b) = 0$ and $|\Phi(x)| \leq \varepsilon$ under arbitrary prescribed $\varepsilon > 0$ for all $x \in [a, b]$.*

As a consequence of Proposition 3, we obtain the following statement.

Proposition 4. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a measurable function with respect to the logarithmic capacity. Then there is a continuous function $\Phi : \partial\mathbb{D} \to \mathbb{R}$ such that $\Phi'(e^{it}) = \varphi(e^{it})$ q.e. on $\mathbb{R}$.*

The *Poisson–Stieltjes integral*

$$\Lambda_\Phi(z) \ := \ \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} P_r(\vartheta - t) \ d\Phi(e^{it}) \ , \quad z = re^{i\vartheta}, \ r < 1 \ , \ \vartheta \in \mathbb{R} \tag{8.1}$$

is well-defined for arbitrary continuous functions $\Phi : \partial\mathbb{D} \to \mathbb{R}$, see, e.g., Section 3 in [66].

Directly by the definition of the Riemann–Stieltjes integral and the Weierstrass-type theorem for harmonic functions, see, e.g., Theorem I.3.1 in [29], Λ_Φ is a harmonic function in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, because the function $P_r(\vartheta - t)$ is the real part of the analytic function

$$\mathcal{A}_\zeta(z) \ := \ \frac{\zeta + z}{\zeta - z} \ , \quad \zeta = e^{it}, \quad z = re^{i\vartheta} \ , \ r < 1 \ , \ \vartheta \text{ and } t \in \mathbb{R} \ . \tag{8.2}$$

By Theorem 1 in [66] we have the following useful conclusion.

Proposition 5. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a measurable function with respect to the logarithmic capacity, and let $\Phi : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function with $\Phi'(e^{it}) = \varphi(e^{it})$ q.e. on $\mathbb{R}$. Then Λ_Φ has the angular limit*

$$\lim_{z \to \zeta} \Lambda_\Phi(z) \ = \ \varphi(\zeta) \qquad \textit{q.e. on } \partial\mathbb{D} \ . \tag{8.3}$$

Thus, by Lemma 3 and Proposition 5 and the known Poisson formula, see, e.g., I.D.2 in [49], we come to the following result on the existence, regularity, and representation of solutions to the Poisson equation for the Dirichlet problem with measurable data in the unit disk $\mathbb{D}$. We assume that the charge density g is extended by zero outside of $\mathbb{D}$ in the next theorem.

Theorem 6. *Let a function $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to the logarithmic capacity, and let a continuous function Φ correspond to φ by Proposition 4. Suppose that a function $g : \mathbb{D} \to \mathbb{R}$ is in the class $L^p(\mathbb{D})$ for $p > 1$. Then the function*

$$U \ := \ N_g \ - \ \mathcal{P}_{N_g^*} \ + \ \Lambda_\Phi \ , \qquad\qquad N_g^* \ := \ N_g|_{\partial\mathbb{D}} \ , \tag{8.4}$$

belongs to the class $W^{2,p}_{\rm loc}(\mathbb{D})$, satisfies the Poisson equation $\triangle U = g$ a.e. in $\mathbb{D}$, and has the angular limit

$$\lim_{z \to \zeta} U(z) \ = \ \varphi(\zeta) \qquad \textit{q.e. on } \partial\mathbb{D} \ . \tag{8.5}$$

Moreover, $U \in W^{1,q}_{\rm loc}(\mathbb{D})$ for some $q > 2$, and U is locally Hölder-continuous. Furthermore, $U \in C^{1,\alpha}_{\rm loc}(\mathbb{D})$ with $\alpha = (p-2)/p$, if $g \in L^p(\mathbb{D})$ for $p > 2$.

Remark 8. Note that, by the Luzin result, see also Theorem 2 in [66], the statement of Theorem 6 is valid in terms of the length measure, as well as the harmonic measure on $\partial\mathbb{D}$.

9. Dirichlet problem with measurable data in almost smooth domains

We say that a Jordan curve Γ in $\mathbb{C}$ is *almost smooth*, if Γ has a tangent q.e. Here, it is said that a straight line L in $\mathbb{C}$ is *tangent* to Γ at a point $z_0 \in \Gamma$, if

$$\limsup_{z \to z_0, z \in \Gamma} \frac{\text{dist}\,(z, L)}{|z - z_0|} \ = \ 0 \ . \tag{9.1}$$

In particular, Γ is almost smooth, if Γ has a tangent at all its points except a countable set. The nature of such Jordan curves Γ is complicated enough, because the countable set can be everywhere dense in Γ.

Given a domain D in $\mathbb{C}$, $k_D(z, z_0)$ denotes the *quasihyperbolic distance*,

$$k_D(z, z_0) \ := \ \inf_\gamma \int\limits_\gamma \frac{ds}{d(\zeta, \partial D)} \ , \tag{9.2}$$

introduced in paper [24]. Here, $d(\zeta, \partial D)$ denotes the Euclidean distance from the point $\zeta \in D$ to ∂D, and the infimum is taken over all rectifiable curves γ joining the points z and z_0 in D.

Next, it is said that a domain D satisfies the *quasihyperbolic boundary condition*, if

$$k_D(z, z_0) \ \le \ a \ln \frac{d(z_0, \partial D)}{d(z, \partial D)} + b \qquad \forall\, z \in D \tag{9.3}$$

for constants a and b and a point $z_0 \in D$. The last notion was introduced in [23], but, before it, was first applied in [6].

Remark 9. Consider a Jordan domain D in $\mathbb{C}$ with the almost smooth boundary satisfying the quasihyperbolic boundary condition. By the Riemann theorem, see, e.g., Theorem II.2.1 in [29], there is a conformal mapping $f : D \to \mathbb{D}$ that is extended to a homeomorphism $\tilde{f} : \overline{D} \to \overline{\mathbb{D}}$ by the Carathéodory theorem, see, e.g., Theorem II.3.4 in [29]. Moreover, $f_* := \tilde{f}|_{\partial D}$, as well as f_*^{-1}, is Hölder-continuous by Corollary to Theorem 1 in [6]. Thus, by Remark 7, a function $\varphi : \partial D \to \mathbb{R}$ is measurable with respect to the logarithmic capacity, iff the function $\psi := \varphi \circ f_*^{-1} : \partial\mathbb{D} \to \mathbb{R}$ is so. Set $\Phi := \Psi \circ f_*$, where $\Psi : \partial\mathbb{D} \to \mathbb{R}$ is a continuous function corresponding to ψ by Proposition 4.

Proposition 6. *Let D be a Jordan domain in $\mathbb{C}$ with the almost smooth boundary satisfying the quasihyperbolic boundary condition. Suppose that $\varphi : \partial D \to \mathbb{R}$ is measurable with respect to the logarithmic capacity, and $\Phi : \partial D \to \mathbb{R}$ is the continuous function corresponding to φ by Remark 9. Then the harmonic function $\mathcal{L}_\Phi(z) := \Lambda_{\Phi\circ f_*^{-1}}(f(z))$ has the angular limit φ q.e. on ∂D.*

Proof. Indeed, by Remark 9 and Proposition 5, there is the angular limit

$$\lim_{w\to\xi} \Lambda_\Psi(w) \ = \ \psi(\xi) \qquad \text{q.e. on } \partial\mathbb{D}\ . \tag{9.4}$$

By the Lindelöf theorem, see, e.g., Theorem II.C.2 in [49], if ∂D has a tangent at a point ζ, then

$$\arg\, [\tilde{f}(\zeta) - \tilde{f}(z)] - \arg\, [\zeta - z] \to const \qquad \text{as } z \to \zeta\ .$$

After the change of variables $\xi := \tilde{f}(\zeta)$ and $w := \tilde{f}(z)$, we have that

$$\arg\, [\xi - w] - \arg\, [\tilde{f}^{-1}(\xi) - \tilde{f}^{-1}(w)] \to const \qquad \text{as } w \to \xi\ .$$

In other words, the conformal images of sectors in $\mathbb{D}$ with a vertex at ξ are asymptotically the same as sectors in D with a vertex at ζ. Thus, the nontangential paths in $\mathbb{D}$ are transformed under $\tilde{f}^{-1}$ into nontangential paths in D.

Recall that, firstly, the almost smooth Jordan curve ∂D has a tangent q.e. Secondly, by Remark 6, the mappings f_* and f_*^{-1} are Hölder-continuous, and, thirdly, by Remark 7, they transform the sets with zero logarithmic capacity zero into sets with zero logarithmic capacity. Thus, (9.4) implies the desired conclusion. ☐

Finally, by Lemma 3, Proposition 6, and the Poisson formula, we come to the following result on the existence, regularity, and representation of solutions to the Poisson equation for the Dirichlet problem with measurable data in the Jordan domains. We assume here that the charge density g is extended by zero outside of D in the next theorem.

Theorem 7. *Let D be a Jordan domain in $\mathbb{C}$ with the almost smooth boundary satisfying the quasihyperbolic boundary condition, let a function $\varphi : \partial D \to \mathbb{R}$ be measurable with respect to the* *logarithmic capacity, and let a continuous function* Φ *correspond to* φ *by Remark 9. Suppose that a function* $g : D \to \mathbb{R}$ *is in the class* $L^p(D)$ *for* $p > 1$*. Then the function*

$$U \ := \ N_g \ - \ \mathcal{D}_{N_g^*} \ + \ \mathcal{L}_\Phi \ , \qquad\qquad N_g^* \ := \ N_g|_{\partial D} \ , \tag{9.5}$$

belongs to the class $W^{2,p}_{\rm loc}(D)$*, satisfies the Poisson equation* $\triangle U = g$ *a.e. in* D*, and has the angular limit*

$$\lim_{z\to\zeta} U(z) \ = \ \varphi(\zeta) \qquad q.e. \ on \ \partial D \ . \tag{9.6}$$

Moreover, $U \in W^{1,q}_{\rm loc}(D)$ *for some* $q > 2$*, and* U *is locally Hölder-continuous. Furthermore,* $U \in C^{1,\alpha}_{\rm loc}(D)$ *with* $\alpha = (p-2)/p$*, if* $g \in L^p(D)$ *for* $p > 2$*.*

Remark 10. Note that, by the Luzin result, see also Theorem 3 in [66], the statement of Theorem 7 is valid in terms of the length measure on rectifiable ∂D. Indeed, by the Riesz theorem, length $f_*^{-1}(E) = 0$ whenever $E \subset \partial\mathbb{D}$ with $|E| = 0$, see, e.g., Theorem II.C.1 and Theorem II.D.2 in [49]. Conversely, by the Lavrentiev theorem, $|f_*(\mathcal{E})| = 0$ whenever $\mathcal{E} \subset \partial D$ and length $\mathcal{E} = 0$, see [53] and point III.1.5 in [63].

However, by the well-known Ahlfors—Beurling example, see [2], the sets with zero length, as well as with zero harmonic measure, are not invariant with respect to quasiconformal changes of variables. The last circumstance does not make it is possible to apply the result in the future for the extension of the statement to generalizations of the Poisson equation in anisotropic and inhomogeneous media. Hence, we prefer to use the logarithmic capacity.

Acknowledgements

This work was partially supported by a grant of the Ministry of Education and Science of Ukraine, the project number is 0119U100421.

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