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CONFORMAL LIMIT FOR DIMER MODELS ON THE HEXAGONAL LATTICE

D. Keating,* N. Reshetikhin,† and A. Sridhar‡

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In this note, we derive the asymptotical behavior of local correlation functions in dimer models on a domain of the hexagonal lattice in the continuum limit, when the size of the domain goes to infinity and the parameters of the model scale appropriately. Bibliography: 8 titles.

Dedicated to the 70th birthday of M. Semenov-Tian-Shansky

1. Introduction

In this note, we study the asymptotics of local correlation functions for dimer models on special domains of the hexagonal lattice. The main result is a formula for the asymptotics of the inverse to the Kasteleyn operator computed in two different ways: from the integral formula and from the definition. This note is a research report. Missing details will be completed in an extended version which will be also posted on the arXiv.

Asymptotical formulas for local correlation functions of height functions in dimer models were computed in a number of papers for various regions and lattices, see, for example, [1–3].

Here we emphasize the relation to Dirac fermions, rather than to a Gaussian field, as it was done, for example, in [1–3]. Dirac fermions can be written in terms of a Gaussian field due to the Bose–Fermi correspondence in two-dimensional space-time, but the resulting expression is nonlocal. However, in many ways it is preferable to think of Dirac fermions as more fundamental objects.

Here is the plan of the paper. The first section is the introduction. In the second section, we recall basic facts about dimer models on the hexagonal lattice. We compute the asymptotic of correlation functions for special domains using the integral representation in the third section. In Sec. 4, we compute the same asymptotic using the definition of the inverse to the Kasteleyn operator in terms of the difference equation. In the fifth section, we state the asymptotical behavior of Kastelyn fermions in the continuum limit. The details will be given in an extended version of the paper.

2. Dimers on the hexagonal lattice and the Kasteleyn operator

2.1. Dimer models on the hexagonal lattice. Let H be the hexagonal lattice with the bipartite structure shown in Fig. 1 and $\Gamma \subset H$ be a finite subgraph that is a connected, simply connected domain in H without 1-valent vertices. In other words, Γ is a connected, simply connected domain assembled from elementary hexagons.

A dimer configuration on D is a perfect matching on vertices connected by edges. In other words, it is a partition of edges into two groups, occupied by a dimer and not occupied, such that each vertex should be occupied by a dimer and two dimers never share a common vertex.

*University of California, Berkeley, USA, e-mail: dkeating@berkeley.edu.

†University of California, Berkeley, USA; St.Petersburg State University, St.Petersburg, Russia; KdV Institute for Mathematics, University of Amsterdam, Amsterdam, The Netherlands, e-mail: reshetik@math.berkeley.edu.

‡Google LLC, e-mail: asridhar@berkeley.edu.

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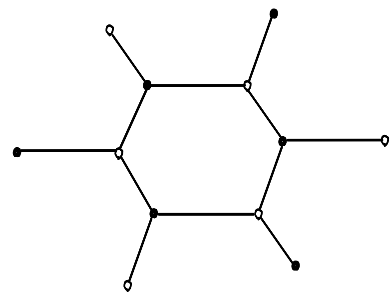

Fig. 1. The hexagonal lattice with bipartite structure.

The Boltzmann weight of a dimer configuration is

$$w(D) = \prod_{e \in D} w(e),$$

where the product is taken over the occupied edges in the dimer configuration D, and $w(e) > 0$ are weights of edges, which should be fixed in order to define the model.

The Boltzmann weights define a probability distribution on dimer configurations on Γ, with

$$\mathrm{Prob}(D) = \frac{w(D)}{Z},$$

where Z is the partition function

$$Z = \sum_{D \subset \Gamma} w(D).$$

The characteristic function of an edge e on the space of dimer configurations is the function σ_e that takes on D the value 1 if e is occupied and 0 if e is not occupied. Local correlation functions for dimer models are expectation values of products of characteristic functions:

$$E(e_1, \dots, e_n) = \sum_{D \subset \Gamma} \mathrm{Prob}(D) \prod_{i=1}^{n} \sigma_{e_i}.$$

It is clear that the dimer probability distribution and, therefore, local correlation functions are invariant with respect to transformations of the form $w(e) \mapsto s(e_+)w(e)s(e_-)$ where s is any function on vertices with positive values and $e_\pm$ are the endpoints of e.

For the hexagonal lattice (our terminology will match Fig. 1), this means that we can choose the weights of the tilted NW-SE edges and of the horizontal edges to be 1. And we will denote the remaining weights of the SW-NE edges by $x(e)$.

2.2. The Kasteleyn operator. As discovered in the 1960s, the partition function and correlation functions of dimer models can be computed in terms of determinants. For details, see the original references [4, 5] and the expository part of [6].

To define such a determinantal solution, we should choose a special orientation of edges, a Kasteleyn orientation. On the hexagonal lattice, it can be chosen as shown in Fig. 2. In order to have determinants, not Pfaffians, one should choose an identification of black and white vertices. We assume that they are identified by horizontal edges.

Choose an embedding of the hexagonal lattice into the square grid as shown in Fig. 2. We will denote the coordinates of centers of horizontal edges by (t, h). Here $h \in \frac{1}{2}\mathbb{Z}$ and $t \in \mathbb{Z}$. Let $\mathcal{D} \subset \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ be a domain in the hexagonal lattice embedded into the square grid.

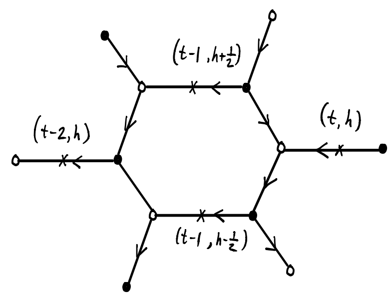

Fig. 2. The hexagonal lattice with the Kasteleyn orientation which we use and with the coordinates of horizontal edges which are identified with adjacent vertices.

The Kasteleyn operator is a linear operator (a difference operator) acting on the vertices of the graph. After the identification of black and white vertices by horizontal dimers, it becomes a difference operator on a domain in the square grid with coordinates $(t, h) \in \mathcal{D}$, acting as

$$(Kf)(h,t) = f(t,h) - f\Big(t-1, h+\frac{1}{2}\Big) + x\Big(t-\frac{1}{2}, h\Big) f\Big(t-1, h-\frac{1}{2}\Big). \tag{1}$$

It is convenient to think of such functions as functions on an extended domain $\widetilde{\mathcal{D}}$, where we add edges with 1-valent vertices to the boundary vertices and define $f(v) = 0$ for each 1-valent vertex v. According to the Kasteleyn theorem, the partition function Z is the absolute value of the determinant of K, and local correlation functions can be computed in terms of the inverse to K.

Let $R(t, h|t', h')$ be the kernel of the inverse to the Kasteleyn operator on $\mathcal{D} \subset \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$. That is, if

$$Kf = g,$$

then

$$f(t,h) = \sum_{(t',h') \in \mathcal{D}} R(t,h|t',h') g(t',h').$$

We have

$$R(t,h|t',h') - R\Big(t-1, h+\frac{1}{2}|t',h'\Big) + x\Big(t-\frac{1}{2}, h\Big) R\Big(t-1, h-\frac{1}{2}|t',h'\Big) = \delta(t,t')\delta(h,h'), \tag{2}$$

with the boundary conditions $R(t,h|t',h') = 0$ when (h,t) corresponds to a 1-valent vertex. If the domain $\mathcal{D}$ is noncompact, one should impose boundary conditions when $(t,h) \to \infty$, but we will not go into details of this here.

Consider horizontal edges with coordinates $x_k = (t_k, h_k)$. Then we have the following formula for the local correlation function:

$$E(x_1, \ldots, x_n) = \det(R(t_k, h_k|t_l, h_l))_{k,l=1}^{n}. \tag{3}$$

Note that Kasteleyn operators can be defined for noncompact domains as well, but this should be supplemented by appropriate boundary conditions.

2.3. Kasteleyn fermions. The Kasteleyn solution of dimer models (the determinant formulas above) can be written in terms of the Grassmann integral. Let $V_{\mathcal{D}}$ be the real vector space whose basis is enumerated by the vertices in the region $\mathcal{D}$. Choose an element $I \in \wedge^N V_{\mathcal{D}}$. It defines the Grassmann integral over $\wedge^\bullet V_{\mathcal{D}}$ as

$$\int f = f_I$$

where $f \in \wedge^\bullet V_{\mathcal{D}}$ and f_I is its component in the basis $I \in \wedge^N V_{\mathcal{D}}$. Let $\psi(t,h)$ be the elements of $\wedge^\bullet V_{\mathcal{D}}$ corresponding to the basis vectors in $V_{\mathcal{D}}$. Typically, I is chosen as a monomial in ψ (a longest ordered product with no repetitions). There are two choices of such an integral, I and $-I$.

Elements ψ are generators of the Grassmann algebra $\wedge^\bullet V_{\mathcal{D}}$. In physics, they are called fermions, since

$$\psi(t',h')\psi(t,h) = -\psi(t,h)\psi(t',h').$$

In terms of generators, we will write

$$\int f = \int f d\psi.$$

Similarly, the Grassmann integral can be defined for the dual vector space $V_{\mathcal{D}}^*$. We will denote the corresponding fermions by $\psi^*(t,h)$.

The Grassmann algebra $\wedge^\bullet(V_{\mathcal{D}} \oplus V_{\mathcal{D}}^*)$ is naturally isomorphic to $\wedge^\bullet V_{\mathcal{D}} \otimes \wedge^\bullet V_{\mathcal{D}}^*$. The integral on this algebra can be identified with the tensor product of integrals. We will write

$$\int F = \int F d\psi^* d\psi$$

for such an integral where F is a polynomial in the anticommuting variables ψ, ψ^*, the generators of $\wedge^\bullet(V_{\mathcal{D}} \oplus V_{\mathcal{D}}^*)$.

Define

$$A = \sum_{(t,h)\in\mathcal{D}} \psi^*(t,h)(K\psi)(t,h),$$

where $K\psi$ is defined as in (1).

The determinant formulas for the partition function and correlation functions can be written in terms of fermions as

$$Z = \left| \int e^A d\psi^* d\psi \right|$$

and

$$E(x_1,\ldots,x_n) = \frac{\int e^A \psi(t_1,h_1)\psi^*(t_1,h_1)\ldots\psi(t_n,h_n)\psi^*(t_n,h_n) d\psi^* d\psi}{\int e^A d\psi^* d\psi}.$$

Note that neither the formula for the partition function, nor the formula for local correlation functions depends on the choice of monomials defining the integrals.

Also, note that the inverse to the Kasteleyn matrix can be written as

$$R(t,h|t',h') = \frac{\int e^A \psi(t,h)\psi^*(t',h') d\psi^* d\psi}{\int e^A d\psi^* d\psi}.$$

We will call ψ, ψ^* Kasteleyn fermions.

3. Continuum limit from the integral representation

3.1. The continuum limit. Denote by $\varphi_\epsilon : \mathbb{Z} \times \frac{1}{2}\mathbb{Z} \to \mathbb{R}^2$ the embedding of the square grid into $\mathbb{R}^2$ such that $(t,h) \mapsto (\epsilon t, \epsilon h)$. We are interested in the asymptotic of local correlation functions in the limit $\epsilon \to 0$ when the lattice domain D expands so that the image $\varphi_\epsilon(\mathcal{D})$ fills an $\mathbb{R}^2$-domain $\mathbb{D}$. Because of the determinantal formulas (3), it suffices to find the asymptotic of the kernel $R((t_1,h_1),(t_2,h_2))$ of the inverse Kasteleyn matrix.

We assume that as $\epsilon \to 0$ and the lattice region expands accordingly to fill the Euclidean domain $\mathbb{D}$, the coordinates t_i and h_i behave as $t_i = \tau_i/\epsilon$, $h_i = \chi_i/\epsilon$ where $(\tau_i, \chi_i) \in \mathbb{D}$.

3.2. The integral formula for the inverse to the Kasteleyn operator. For special lattice domains $\mathcal{D} \in \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$, the kernel of $R = K^{-1}$ has a convenient integral representation. For the semiinfinite domain shown in Fig. 3, such a representation was found in [7]. The boundary conditions at infinity are determined by the asymptotical configuration of dimers as shown in Fig. 3.

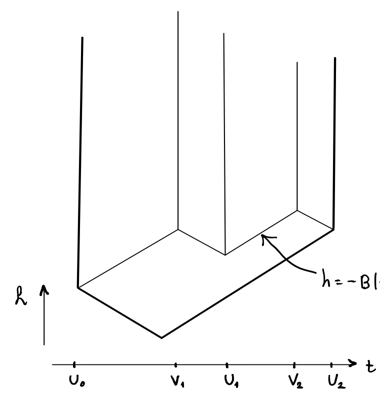

Fig. 3. The lattice domain $\mathcal{D}$ with asymptotical boundary configuration of dimers. The function $B(t)$ is defined in (5). For details, see [7].

Assume that the edge weights $x(t-\frac{1}{2},h)$ in (1) are $x(m,h) = q^m$ when $V_i < m < U_i$ and $x(m,h) = q^{-m}$ when $U_i < m < V_{i+1}$. Define $\mathcal{D}_+$ to be the set of m such that $V_i < m < U_i$ for some i, and $\mathcal{D}_-$ to be the set of m such that $U_i < m < V_{i+1}$ for some i. Then formulas from [7] give the following integral representation of the inverse Kasteleyn operator:

$$R((t_1,h_1),(t_2,h_2)) = \left(\frac{1}{2\pi i}\right)^2 \int_{C_z}\int_{C_w} \frac{\Phi_-(z,t_1)\Phi_+(w,t_2)}{\Phi_+(z,t_1)\Phi_-(w,t_2)} z^{-h_1-B(t_1)} w^{h_2+B(t_2)} \frac{\sqrt{zw}}{z-w}\frac{dz}{z}\frac{dw}{w} \tag{4}$$

where

$$\Phi_+(z,t) = \prod_{\substack{m>t,\\ m\in\mathcal{D}_+}} (1-zq^m), \quad \Phi_-(z,t) = \prod_{\substack{m<t,\\ m\in\mathcal{D}_-}} (1-z^{-1}q^{-m})$$

and

$$B(t) = \frac{1}{2}\sum_{i=1}^{N} |t-V_i| - \frac{1}{2}\sum_{i=1}^{N-1} |t-U_i| \tag{5}$$

for $m \in \mathbb{Z}+\frac{1}{2}$ and $t \in \mathbb{Z}$. We assume that $\sum_{i=1}^{N} V_i = \sum_{i=1}^{N-1} U_i$ and $U_0 + U_N = 0$.

From our setup we see that for the case when $t \in \mathcal{D}_+$, $V_i < t < U_i$, we have

$$\Phi_+(z,t) = \prod_{m=t+\frac{1}{2}}^{U_i-\frac{1}{2}} (1-zq^m) \prod_{m=V_{i+1}+\frac{1}{2}}^{U_{i+1}-\frac{1}{2}} (1-zq^m) \prod_{m=V_{i+2}+\frac{1}{2}}^{U_{i+2}-\frac{1}{2}} \cdots,$$

$$\Phi_-(z,t) = \prod_{m=U_{i-1}+\frac{1}{2}}^{V_i-\frac{1}{2}} (1-z^{-1}q^{-m}) \prod_{m=U_{i-2}+\frac{1}{2}}^{V_{i-1}-\frac{1}{2}} \cdots,$$

and for the case when $t \in \mathcal{D}_-$, $U_i < t < V_{i+1}$,

$$\Phi_+(z,t) = \prod_{m=V_{i+1}+\frac{1}{2}}^{U_{i+1}-\frac{1}{2}} (1-zq^m) \prod_{m=V_{i+2}+\frac{1}{2}}^{U_{i+2}-\frac{1}{2}} \cdots,$$

$$\Phi_-(z,t) = \prod_{m=U_i+\frac{1}{2}}^{t-\frac{1}{2}} (1-z^{-1}q^{-m}) \prod_{m=U_{i-1}+\frac{1}{2}}^{V_i-\frac{1}{2}} (1-z^{-1}q^{-m}) \prod_{m=U_{i-2}+\frac{1}{2}}^{V_{i-1}-\frac{1}{2}} \cdots.$$

3.3. The continuum limit. Now assume that $q = \exp(-\epsilon)$, $\epsilon \to 0$, and that $u_i = U_i\epsilon$, $v_i = V_i\epsilon$, $\tau_a = t_a\epsilon$, $\chi_a = h_a\epsilon$ are kept finite in this limit.

3.3.1. *The lemma on q-dilogarithms.* The following lemma is known. We present it anyway for completeness.

Lemma 1.

$$\prod_{m=t_1+\frac{1}{2}}^{t_2-\frac{1}{2}} (1-zq^m) = e^{\frac{1}{\epsilon}\int_{ze^{-\tau_2}}^{ze^{-\tau_1}} \frac{\ln(1-t)}{t}dt} (1+O(\epsilon)).$$

Proof. Recall the q-Pochhammer symbol (q-dilogarithm) defined by $(z;q)_\infty = \prod_{k=0}^{\infty}(1-zq^k)$. Assume that $(z;q)_\infty$ can be expanded as

$$(z;q)_\infty = e^{\frac{S(z)}{\epsilon}} f(z)(1+O(\epsilon))$$

as $\epsilon \to 0$. Then we have

$$(zq;q)_\infty = \frac{1}{1-z}(z;q)_\infty = e^{\frac{S(z)}{\epsilon}} \frac{f(z)}{1-z}(1+O(\epsilon)),$$

as well as

$$(zq;q)_\infty = e^{\frac{S(zq)}{\epsilon}} f(zq)(1+O(\epsilon)).$$

We now write $q = e^{-\epsilon}$ and expand the above in orders of ϵ:

$$\begin{aligned} S\Big(z - z\epsilon + z\frac{\epsilon^2}{2}\Big) &= S(z) + z\Big(-\epsilon + \frac{\epsilon^2}{2}\Big)S'(z) + z^2\frac{\epsilon^2}{2}S''(z) + \dots \\ &= S(z) - \epsilon z S'(z) + \frac{\epsilon^2}{2}(zS'(z) + z^2 S''(z)) + O(\epsilon^3). \end{aligned}$$

Equating the two expressions for $(zq;q)_\infty$, we have

$$e^{-zS'(z)}\Big(1 + \frac{\epsilon}{2}\left(zS'(z) + z^2S''(z)\right) + \dots\Big)\left(f(z) - \epsilon z f'(z) + \dots\right) = \frac{1}{1-z}f(z).$$

Now let us look at the terms order-by-order. For the 0-order terms, we have

$$S'(z) = \frac{\ln(1-z)}{z}.$$

If S is chosen so that $S(0) = 0$, then

$$S(z) = \int_0^z \frac{\ln(1-t)}{t} dt.$$

For the ϵ-order terms, we have

$$\frac{1}{2}(zS'(z) + z^2 S''(z))f(z) - zf'(z) = \frac{1}{2}z(zS'(z))' f(z) - zf'(z) = 0.$$

Using what we know about $S(z)$, this becomes

$$f'(z) = -\frac{1}{2}\frac{f(z)}{1-z},$$

giving

$$f(z) = \sqrt{z-1}.$$

Putting this all together, we have

$$(z,q)_\infty = \exp\left(\frac{1}{\epsilon}\int_0^z \frac{\ln(1-t)}{t}dt\right)\sqrt{z-1}(1+O(\epsilon)).$$

Now, we write our finite product as a ratio of infinite products and use the above result:

$$\begin{aligned}
\prod_{m=t_1+\frac{1}{2}}^{t_2-\frac{1}{2}} (1-zq^m) &= \frac{(zq^{t_1+\frac{1}{2}};q)_\infty}{(zq^{t_2+\frac{1}{2}};q)_\infty} = \exp\left(\frac{1}{\epsilon}\int_{zq^{t_2+\frac{1}{2}}}^{zq^{t_1+\frac{1}{2}}} \frac{\ln(1-t)}{t}dt\right)\sqrt{\frac{zq^{-\tau_1}-1}{zq^{-\tau_2}-1}}(1+O(\epsilon)) \\
&= \exp\left(\frac{1}{\epsilon}\int_{z_2(1-\frac{\epsilon}{2})}^{z_1(1-\frac{\epsilon}{2})} \frac{\ln(1-t)}{t}dt\right)\sqrt{\frac{zq^{-\tau_1}-1}{zq^{-\tau_2}-1}}(1+O(\epsilon)) \\
&= \exp\left(\frac{1}{\epsilon}\int_{z_2}^{z_1} \frac{\ln(1-t)}{t}dt - \frac{1}{2}(\ln(1-ze^{-\tau_1}) + \ln(1-ze^{-\tau_2}))\right) \\
&\times \sqrt{\frac{zq^{-\tau_1}-1}{zq^{-\tau_2}-1}}(1+O(\epsilon)) = \exp\left(\frac{1}{\epsilon}\int_{ze^{-\tau_2}}^{ze^{-\tau_1}} \frac{\ln(1-t)}{t}dt\right)(1+O(\epsilon)).
\end{aligned}$$

□

Similarly, we have

$$\prod_{t_1+\frac{1}{2}}^{t_2-\frac{1}{2}} (1-z^{-1}q^{-m}) = (-z)^{-(t_2-t_1)} \prod_{m=t_1+\frac{1}{2}}^{t_2-\frac{1}{2}} q^{-m} \prod_{t_1+\frac{1}{2}}^{t_2-\frac{1}{2}} (1-zq^m).$$

So, as $\epsilon \to 0$ we have

$$\prod_{t_1+\frac{1}{2}}^{t_2-\frac{1}{2}} (1 - z^{-1}q^{-m}) = (-1)^{t_2-t_1} z^{-\frac{\tau_2-\tau_1}{\epsilon}} e^{\frac{\tau_2^2-\tau_1^2}{2\epsilon}} \exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-\tau_2}}^{ze^{-\tau_1}} \frac{\ln(1-t)}{t}dt\right)(1+O(\epsilon)).$$

Note that this asymptotic expansion is a meromorphic function of z on the complex plane with branch cuts along $[e^{\tau_1}, e^{\tau_2}]$.

3.3.2. *The functions $\Phi_\pm$ in the continuum limit.* Now we can use the computations from the previous section to find the asymptotic of $\Phi_\pm(z,t)$.

Indeed, for $t \in \mathcal{D}_+$, i.e, $V_i < t < U_i$, we have[1]

$$\begin{aligned}\Phi_+(z,t) &= \prod_{m=t+\frac{1}{2}}^{U_i-\frac{1}{2}} (1-zq^m) \prod_{m=V_{i+1}+\frac{1}{2}}^{U_{i+1}-\frac{1}{2}} (1-zq^m) \prod_{m=V_{i+2}+\frac{1}{2}}^{U_{i+2}-\frac{1}{2}} \cdots = \exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-u_i}}^{ze^{-\tau}} \frac{\ln(1-t)}{t}dt\right) \\ &\times \exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-u_{i+1}}}^{ze^{-v_{i+1}}} \frac{\ln(1-t)}{t}dt\right) \exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-u_{i+2}}}^{ze^{-v_{i+2}}} \frac{\ln(1-t)}{t}dt\right)\ldots \\ &= \exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-u_i}}^{ze^{-\tau}} + \frac{1}{\epsilon}\int\limits_{ze^{-u_{i+1}}}^{ze^{-v_{i+1}}} + \frac{1}{\epsilon}\int\limits_{ze^{-u_{i+2}}}^{ze^{-v_{i+2}}} \frac{\ln(1-t)}{t}dt + \ldots\right),\end{aligned}$$

$$\begin{aligned}\Phi_-(z,t) &= \prod_{m=U_{i-1}+\frac{1}{2}}^{V_i-\frac{1}{2}} (1-z^{-1}q^{-m}) \prod_{m=U_{i-2}+\frac{1}{2}}^{V_{i-1}-\frac{1}{2}} \cdots = (-z)^{-\frac{v_i-u_{i-1}}{\epsilon}} e^{\frac{v_i^2-u_{i-1}^2}{2\epsilon}} \exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-v_i}}^{ze^{-u_{i-1}}} \frac{\ln(1-t)}{t}dt\right) \\ &\times (-z)^{\frac{v_{i-1}-u_{i-2}}{\epsilon}} e^{\frac{v_{i-1}^2-u_{i-2}^2}{2\epsilon}} \exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-v_{i-1}}}^{ze^{-u_{i-2}}} \frac{\ln(1-t)}{t}dt\right)\cdots = (-z)^{-\frac{1}{\epsilon}\sum\limits_{j\le i}(v_j-u_{j-1})} e^{\frac{1}{2\epsilon}\sum\limits_{j\le i} v_j^2-u_{j-1}^2} \\ &\times \exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-v_i}}^{ze^{-u_{i-1}}} + \frac{1}{\epsilon}\int\limits_{ze^{-v_{i-1}}}^{ze^{-u_{i-2}}} \frac{\ln(1-t)}{t}dt + \ldots\right).\end{aligned}$$

Similarly, for $t \in \mathcal{D}_-$, i.e., $U_i < t < V_{i+1}$, we obtain

$$\Phi_+(z,t) = \exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-u_{i+1}}}^{ze^{-v_{i+1}}} + \frac{1}{\epsilon}\int\limits_{ze^{-u_{i+2}}}^{ze^{-v_{i+2}}} \frac{\ln(1-t)}{t}dt + \ldots\right),$$

$$\begin{aligned}\Phi_-(z,t) = (-z)&^{-\frac{\tau-u_i}{\epsilon}-\frac{1}{\epsilon}\sum\limits_{j<i}(v_j-u_{j-1})} e^{\frac{1}{2\epsilon}(\tau^2-u_i^2+\sum\limits_{j<i} v_j^2-u_{j-1}^2)} \\ &\times \exp\left(\frac{1}{\epsilon}\int\limits_{ze^{-\tau}}^{ze^{-u_i}} + \frac{1}{\epsilon}\int\limits_{ze^{-v_i}}^{ze^{-u_{i-1}}} \frac{\ln(1-t)}{t}dt + \ldots\right).\end{aligned}$$

[1] In the expressions below, we will omit the integrand if it is clear which function is integrated.

Recall that

$$B(t) = \frac{1}{2}\sum_{i=1}^{N}|t-V_i| - \frac{1}{2}\sum_{i=1}^{N-1}|t-U_i|.$$

Now define

$$L(t) = \begin{cases} \sum\limits_{j\le i} V_j - U_{j-1} & \text{for } t\in\mathcal{D}_+, V_i<t<U_i,\\ t-U_i+\sum\limits_{j<i} V_j - U_{j-1} & \text{for } t\in\mathcal{D}_-,\ U_i<t<V_{i+1}.\end{cases}$$

For $t\in\mathcal{D}_+$, $V_i<t<U_i$, we have

$$\begin{aligned} B(t)+L(t) &= \frac{1}{2}\sum_{j=1}^{N}|t-V_j| - \frac{1}{2}\sum_{j=1}^{N-1}|t-U_j| + \sum_{j\le i}(V_j-U_{j-1})\\ &= \frac{1}{2}\left(\sum_{j=1}^{i} t - \sum_{j=i+1}^{N} t + \sum_{j=i}^{N-1} t - \sum_{j=1}^{i-1} t - \sum_{j=1}^{i} V_j + \sum_{j=i+1}^{N} V_j - \sum_{j=i}^{N-1} U_j + \sum_{j=1}^{i-1} U_j\right)\\ &\quad + \sum_{j=1}^{i} V_j - \sum_{j=1}^{i-1} U_j + U_0 = \frac{t}{2} + \frac{1}{2}\left(\sum_{j=1}^{N} V_j - \sum_{j=1}^{N-1} U_j\right) - U_0 = \frac{t}{2} - U_0, \end{aligned}$$

where we use the equality $\sum\limits_{j=1}^{N} V_j = \sum\limits_{j=1}^{N-1} U_j$. A similar calculation can be done for $t\in\mathcal{D}_-$, $U_i<t<V_{i+1}$.

Now, for the ratio of the functions Φ we have

$$\frac{\Phi_-(z,t)}{\Phi_+(z,t)} z^{-h-B(t)} = C_\tau \exp\left(\frac{S(z)}{\epsilon}\right)(1+O(\epsilon))$$

where C_τ is a constant in z. The function $S(z)$ is

$$S(z) = \sum_{i=0}^{N}\mathrm{Li}_2(ze^{-u_i}) - \sum_{i=1}^{N}\mathrm{Li}_2(ze^{-v_i}) - \mathrm{Li}_2(ze^{-\tau}) - \left(\frac{\tau}{2} - u_0 + \chi\right)\ln z,$$

and $\mathrm{Li}_2(z) = \int\limits_0^z \frac{\ln(1-x)}{x}dx$ is the dilogarithm.

Combining the results from above, we have the following asymptotical integral representation for R:

$$R((t_1,h_1),(t_2,h_2)) = \frac{C_{\tau_1}}{C_{\tau_2}}\left(\frac{1}{2\pi i}\right)^2 \int\limits_{C_z}\int\limits_{C_w} e^{\frac{S(z,\tau_1,\chi_1)-S(w,\tau_2,\chi_2)}{\epsilon}} \frac{\sqrt{zw}}{z-w}\frac{dz}{z}\frac{dw}{w}(1+O(\epsilon)) \qquad (6)$$

where the function $S(z)$ is as above. The integration contours are shown in Fig. 4.

3.3.3. *The asymptotic of the integral* (6). We will compute the asymptotic using the method of steepest descent, so first we should study the critical points of the function $S(z)$.

Lemma 2. *The following identity holds:*

$$z_0^2 S''(z_0) = \left(\frac{\partial z_0}{\partial \chi}\right)^{-1}$$

where z_0 is a critical point of $S(z)$.

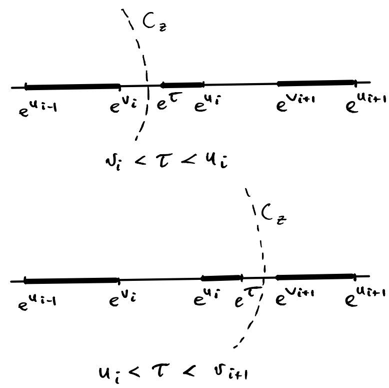

Fig. 4. The integration contours in (6) are circles with $|z| < |w|$ when $\tau_1 < \tau_2$ and $|w| > |z|$ when $\tau_1 > \tau_2$ centered at the origin. The contour C_z intersects the positive part of the real line as shown above with $\tau = \tau_1$. The contour C_w intersects the positive part of the real line similarly with $\tau = \tau_2$.

Proof. For the first derivative of S in z, we have

$$z\frac{\partial S(z)}{\partial z} = \sum_{i=0}^{N} \ln(1 - ze^{-u_i}) - \sum_{i=1}^{N} \ln(1 - ze^{-v_i}) - \ln(1 - ze^{-\tau}) - \left(\frac{\tau}{2} - u_0 + \chi\right)$$

$$= \ln\left(\frac{\prod_{i=0}^{N}(1 - ze^{-u_i})}{\prod_{i=1}^{N}(1 - ze^{-v_i})}\frac{1}{(1 - ze^{-\tau})}\right) - \left(\frac{\tau}{2} - u_0 + \chi\right) = \ln\left(\frac{f(z)}{(1 - ze^{-\tau})}\right) - \left(\frac{\tau}{2} + \chi\right)$$

where $f(z) = \frac{\prod_{i=0}^{N}(1-ze^{-u_i})}{\prod_{i=1}^{N}(1-ze^{-v_i})}e^{u_0}$.

From this we see that if z_0 is a critical point of S, i.e., $S'(z_0) = 0$, then we have

$$e^{\chi+\frac{\tau}{2}} = \frac{f(z_0)}{1 - z_0e^{-\tau}}.$$

This defines z_0 as an implicit function of χ and τ. Taking the derivative, we have

$$1 = \frac{\partial z_0}{\partial \chi}\left(\ln\left(\frac{f(z_0)}{1 - z_0e^{-\tau}}\right)\right)'.$$

For the second derivative of $S(z)$, we have

$$\left(z\frac{\partial}{\partial z}\right)^2 S(z) = z\left(\ln\left(\frac{f(z_0)}{1 - z_0e^{-\tau}}\right)\right)'.$$

Taking into account the equation for the derivative at the critical point in χ and the fact that

$$\left(z\frac{\partial}{\partial z}\right)^2 S(z) = zS'(z) + z^2S''(z),$$

we obtain the value of the second derivative of $S(z)$ at the critical point z_0 and the desired identity. □

Before we compute the asymptotic of (6), we need one more lemma.

Lemma 3. *The following identities hold:*

$$e^{-S_\tau} = \frac{\sqrt{z_0}}{1 - z_0 e^{-\tau}} \quad e^{-\frac{1}{2}S_\chi} = \sqrt{z_0}.$$

Indeed, we have the following identities, which imply the lemma:

$$\begin{aligned} \frac{d}{d\tau}S(z_0) &= \frac{\partial z_0}{\partial \tau}S'(z_0) + \frac{\partial S}{\partial \tau}|_{z_0} = -\frac{1}{2}\ln(z_0) + \ln(1 - z_0 e^{-\tau}), \\ \frac{d}{d\chi}S(z_0) &= \frac{\partial z_0}{\partial \chi}S'(z_0) + \frac{\partial S}{\partial \chi}|_{z_0} = -\ln(z_0). \end{aligned}$$

Theorem 1. *The integral* (6) *has the following asymptotic as* $\epsilon \to 0$ *and all parameters scale as before:*

$$\begin{aligned} R(t,h|t',h') = \epsilon\frac{C_{\tau_1}}{C_{\tau_2}}\Bigg(& e^{\frac{S(z_0)-S(w_0)}{\epsilon}}\frac{\sqrt{\frac{\partial z_0}{\partial \chi_1}\frac{\partial w_0}{\partial \chi_2}}}{z_0 - w_0} + e^{\frac{S(\bar{z}_0)-S(w_0)}{\epsilon}}\frac{\sqrt{\frac{\partial \bar{z}_0}{\partial \chi_1}\frac{\partial w_0}{\partial \chi_2}}}{\bar{z}_0 - w_0} \\ & + e^{\frac{S(z_0)-S(\bar{w}_0)}{\epsilon}}\frac{\sqrt{\frac{\partial z_0}{\partial \chi_1}\frac{\partial \bar{w}_0}{\partial \chi_2}}}{z_0 - \bar{w}_0} + e^{\frac{S(\bar{z}_0)-S(\bar{w}_0)}{\epsilon}}\frac{\sqrt{\frac{\partial \bar{z}_0}{\partial \chi_1}\frac{\partial \bar{w}_0}{\partial \chi_2}}}{\bar{z}_0 - \bar{w}_0}\Bigg)(1 + O(\epsilon)). \end{aligned} \tag{7}$$

Proof. As shown in [7] for (τ, χ), inside the discriminant curve there are two complex conjugate critical points of $S(z)$. The discriminant curve, also known as the arctic circle, is

$$S'(z) = S''(z) = 0.$$

Deforming the integration contours into contours that pass the critical points in the steepest descent direction and computing the corresponding Gaussian integrals, we arrive at (7). ☐

4. Asymptotical solutions to the Kasteleyn difference equation

4.1. Formal asymptotical solutions to the Kasteleyn equations. Here we will study the difference equation

$$f(t,h) - f\Big(t-1, h+\frac{1}{2}\Big) + x\Big(t - \frac{1}{2}, h\Big)f\Big(t-1, h-\frac{1}{2}\Big) = 0 \tag{8}$$

in the continuum limit when $\epsilon \to 0$ and $\tau = \epsilon t$, $\chi = \epsilon h$ are fixed. It is convenient to change the coordinates:

$$\begin{aligned} \xi_+ &= \chi + \frac{\tau}{2}, \quad \xi_- = \chi - \frac{\tau}{2}, \\ \partial_+ &= \partial_\tau + \frac{1}{2}\partial_\chi, \quad \partial_- = -\partial_\tau + \frac{1}{2}\partial_\chi. \end{aligned}$$

Let us look for asymptotical solutions to the difference equation (8) of the form $f(t,h) = e^{\frac{1}{\epsilon}S(\xi_+,\xi_-)}\phi(\xi_+,\xi_-)$. Equation (8) gives

$$\begin{aligned} e^{\frac{1}{\epsilon}S(\xi_+,\xi_-)}\phi(\xi_+,\xi_-) - e^{\frac{1}{\epsilon}S(\xi_+,\xi_- + \epsilon)}\phi(\xi_+,\xi_- + \epsilon) & \\ + v\Big(\xi_+ - \xi_- - \frac{\epsilon}{2}\Big)e^{\frac{1}{\epsilon}S(\xi_+ - \epsilon,\xi_-)}\phi(\xi_+ - \epsilon,\xi_-) &= 0. \end{aligned}$$

Taking the limit $\epsilon \to 0$, we get the following nonlinear differential equation for S at the 0th order in ϵ:

$$1 - e^{\partial_- S} + ve^{-\partial_+ S} = 0. \tag{9}$$

The first-order terms give a linear differential equation for ϕ:

$$-\frac{1}{2}\partial_-^2 Se^{\partial_- S} - \frac{\partial_-\phi}{\phi}e^{\partial_- S} + \frac{1}{2}v\partial_+^2 Se^{-\partial_+ S} - v\frac{\partial_+\phi}{\phi}e^{-\partial_+ S} - \frac{1}{2}v'e^{-\partial_+ S} = 0. \tag{10}$$

4.1.1. *The function S.* Taking into account Eq. (9), we can write

$$e^{\partial_- S} = \frac{1}{1 - z_0 v}, \quad e^{-\partial_+ S} = \frac{z_0}{1 - z_0 v}$$

for some function $z_0(\xi_+, \xi_-)$.

Lemma 4. *The function $z_0(\xi_+, \xi_-)$ satisfies the differential equation*

$$\partial_- z_0(\xi_+, \xi_-) + z_0(\xi_+, \xi_-) v(\xi_+ - \xi_-) \partial_+ z_0(\xi_+, \xi_-) = 0. \tag{11}$$

Proof. Indeed, differentiating (9), we obtain

$$\partial_+(\partial_- S) = -\partial_+ \ln(1 - z_0 v) = \frac{\partial_+(z_0 v)}{1 - z_0 v},$$

$$\partial_-(\partial_+ S) = \partial_-(\ln(1 - z_0 v) - \ln(z_0)) = -\frac{\partial_-(z_0 v)}{(1 - z_0 v) z_0 v} + \frac{\partial_- v}{v}.$$

These two identities imply

$$\partial_-(z_0 v) + (z_0 v)\partial_+(z_0 v) = (1 - z_0 v) z_0 \partial_- v.$$

Sinse $v = v(\xi_+ - \xi_-)$, we can rewrite this as

$$(\partial_- z_0) v - z_0 v' + z_0 v (\partial_+ z_0) v + z_0^2 v v' = -z_0 v' + z_0^2 v v',$$

which gives the desired identity. □

Note that when v is constant, the equation for z_0 is exactly the complex Burgers equation from [8].

4.1.2. *The function ϕ.* Here we will describe the general solution to the differential equation for ϕ.

Theorem 2. *Let $z_0(\xi_+, \xi_-)$ be as above. Then the function*

$$\phi = \psi \frac{\sqrt{(\partial_+ + \partial_-) z_0}}{z_0 - w_0}, \tag{12}$$

where w_0 does not depend on $\xi_\pm$, is a solution to (10) *if and only if ψ satisfies the equation*

$$(\partial_- + z_0 v \partial_+)\psi = 0.$$

Proof. First, let us now look at the terms in Eq. (10) not containing ϕ. We have

$$\begin{aligned}
&\frac{1}{2}\frac{\partial_- \ln(1 - z_0 v)}{1 - z_0 v} + \frac{1}{2} v \partial_+ (\ln(z_0) - \ln(1 - z_0 v)) \frac{z_0}{1 - z_0 v} - \frac{1}{2} v' \frac{z_0}{1 - z_0 v} \\
&= -\frac{1}{2}\frac{\partial_-(z_0 v)}{(1 - z_0 v)^2} - \frac{1}{2} v \left(\frac{\partial_+ z_0}{z_0} + \frac{\partial_+(z_0 v)}{1 - z_0 v} \right) \frac{z_0}{1 - z_0 v} - \frac{1}{2} v' \frac{z_0}{1 - z_0 v} \\
&= -\frac{1}{2}\frac{\partial_-(z_0 v) + v z_0 \partial_+(z_0 v)}{(1 - z_0 v)^2} - \frac{1}{2} v \frac{\partial_+ z_0}{1 - z_0 v} - \frac{1}{2} v' \frac{z_0}{1 - z_0 v} \\
&= -\frac{1}{2}\frac{v(\partial_-(z_0) + v z_0 \partial_+(z_0)) + z_0(\partial_- v + v z_0 \partial_+ v)}{(1 - z_0 v)^2} - \frac{1}{2} v \frac{\partial_+ z_0}{1 - z_0 v} - \frac{1}{2}\frac{v' z_0}{1 - z_0 v} \\
&= \frac{1}{2}\frac{z_0 v'}{1 - z_0 v} - \frac{1}{2}\frac{v \partial_+ z_0}{1 - z_0 v} - \frac{1}{2}\frac{v' z_0}{1 - z_0 v} = -\frac{1}{2}\frac{v \partial_+ z_0}{1 - z_0 v},
\end{aligned}$$

where in the fourth line we use the lemma from above.

Now, the terms containing ϕ after the substitution (12) can be transformed as

$$\frac{1}{1-z_0v}\frac{\partial_-\phi}{\phi}+\frac{vz_0}{1-z_0v}\frac{\partial_+\phi}{\phi}$$
$$=\frac{\partial_-\psi+z_0v\partial_+\psi}{\psi(1-z_0v)}+\frac{1}{2}\frac{(\partial_-+vz_0\partial_+)(\partial_+z_0+\partial_-z_0)}{(\partial_+z_0+\partial_-z_0)(1-z_0v)}-\frac{(\partial_-+z_0v\partial_+)z_0}{(1-z_0v)(z_0-\widetilde{z}_0)}$$
$$=\frac{\partial_-\psi+z_0v\partial_+\psi}{\psi(1-z_0v)}+\frac{1}{2}\frac{(\partial_-+vz_0\partial_+)(\partial_+z_0+\partial_-z_0)}{(\partial_+z_0+\partial_-z_0)(1-z_0v)},$$

where we again use the lemma from above. The denominator of the second term can be written as

$$(\partial_-+z_0v\partial_+)(\partial_+z_0+\partial_-z_0)=(\partial_++\partial_-)(\partial_-+z_0v\partial_+)z_0+(\partial_++\partial_-)(z_0v)\partial_+z_0$$
$$=(\partial_++\partial_-)(z_0v)\partial_+z_0=v(\partial_++\partial_-)(z_0)\partial_+z_0.$$

Here we use the lemma and the fact that $(\partial_++\partial_-)v=0$. Combining the expressions from above, we have the following expression for terms in (10):

$$\frac{1}{1-z_0v}\frac{\partial_-\phi}{\phi}+\frac{vz_0}{1-z_0v}\frac{\partial_+\phi}{\phi}=\frac{\partial_-\psi+z_0v\partial_+\psi}{\psi(1-z_0v)}+\frac{1}{2}\frac{v\partial_+z_0}{1-z_0v}.$$

Putting everything together, the equation for ϕ becomes

$$-\frac{1}{2}\frac{v\partial_+z_0}{1-z_0v}+\frac{\partial_-\psi+z_0v\partial_+\psi}{\psi(1-z_0v)}+\frac{1}{2}\frac{v\partial_+z_0}{1-z_0v}=\frac{\partial_-\psi+z_0v\partial_+\psi}{\psi(1-z_0v)}.$$

The theorem follows. □

4.2. The asymptotical behavior of the inverse to the Kasteleyn operator in the continuum limit. Now, let us find the asymptotic of the inverse to the Kasteleyn operator from the difference equation. Note that the critical points of the function S from the asymptotic of the integral representation satisfy Eq. (11).

Let $z_0(\tau,\chi)$ be the relevant solution to (11), denote $z_0=z_0(\tau,\chi)$ and $w_0=z_0(\tau',\chi')$. Combining the previous results of this section, we arrive at the following asymptotic of $R(t,h|t'h')$:

$$R(t,h|t',h')=\epsilon\frac{C_{\tau_1}}{C_{\tau_2}}\Bigg(e^{\frac{S(z_0)-S(w_0)}{\epsilon}}\frac{\sqrt{\frac{\partial z_0}{\partial\chi_1}\frac{\partial w_0}{\partial\chi_2}}}{z_0-w_0}+e^{\frac{S(\bar z_0)-S(w_0)}{\epsilon}}\frac{\sqrt{\frac{\partial\bar z_0}{\partial\chi_1}\frac{\partial w_0}{\partial\chi_2}}}{\bar z_0-w_0}$$
$$+e^{\frac{S(z_0)-S(\bar w_0)}{\epsilon}}\frac{\sqrt{\frac{\partial z_0}{\partial\chi_1}\frac{\partial\bar w_0}{\partial\chi_2}}}{z_0-\bar w_0}+e^{\frac{S(\bar z_0)-S(\bar w_0)}{\epsilon}}\frac{\sqrt{\frac{\partial\bar z_0}{\partial\chi_1}\frac{\partial\bar w_0}{\partial\chi_2}}}{\bar z_0-\bar w_0}\Bigg)(1+O(\epsilon)). \tag{13}$$

This agrees with (7) when $v(x)=e^{-x}$. We will give a detailed proof in an extended version of this paper.

5. Conformal correlation functions

Note that the asymptotical formula for the inverse to the Kasteleyn operator can be interpreted in terms of Kasteleyn fermions in the following way. In an appropriate sense, one can say that as $\epsilon\to 0$

$$\psi(t,h)=\sqrt{\epsilon}C_\tau\left(a(z_0(\tau,\chi))e^{\frac{S(z_0(\tau,\chi))}{\epsilon}}+a(\overline{z_0(\tau,\chi)})e^{\frac{\overline{S(z_0(\tau,\chi))}}{\epsilon}}\right)(1+O(\epsilon)),$$
$$\psi^*(t,h)=\sqrt{\epsilon}C_\tau^{-1}\left(b(z_0(\tau,\chi))e^{-\frac{S(z_0(\tau,\chi))}{\epsilon}}+b(\overline{z_0(\tau,\chi)})e^{-\frac{\overline{S(z_0(\tau,\chi))}}{\epsilon}}\right)(1+O(\epsilon))$$

where $a(z)$ and $b(z)$ are components of the Dirac fermionic field with correlation functions

$$\langle a(z)b(w)\rangle = \frac{1}{z-w}.$$

We will explain the exact meaning of the convergence and the definition of the Dirac fermionic field in an extended version of this paper. The square roots in (13) appear from the spinor nature of the conformal fields a and b.

The height function of the dimer model is a quadratic combination in a and b, see the extended version of the paper.

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