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VIBRATIONS OF A STRING IN THE CONTEXT OF FINITE FIELDS

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The string wave equation (i.e., the one-dimentional wave equation) is considered in the context of complex functions over finite fields. Analogs of the classical d'Alembert formulas over finite fields are obtained. Bibliography: 9 titles.

To the memory of Oleg Mstislavovich Fomenko

1. Introduction. One can observe that some classical equations and boundary-value problems of mathematical physics can be interpreted in terms of the theory of complex functions over finite fields. In this context, we consider the equation of vibrations of a string (the one-dimensional wave equation) and derive analogs of the d'Alembert formulas. For complex functions over finite fields, see [1–3].

2. Notation and basic functions. Given a prime p, consider the field $\mathbb{F}_q$ with $q = p^l$ elements and the prime subfield $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let $\widehat{\mathbb{F}}_q^\star$ be the group of its multiplicative characters, i.e., the group of all homomorphisms $\chi\colon \mathbb{F}_q^\star \to \mathbb{C}^\star$ of the multiplicative group $\mathbb{F}_q^\star$ of the field $\mathbb{F}_q$ to the multiplicative group $\mathbb{C}^\star$ of the complex field $\mathbb{C}$. Let ϵ be the trivial character, $\epsilon(x) = 1$ for all $x \in \mathbb{F}_q^\star$. We extend every multiplicative character χ to the entire field $\mathbb{F}_q$ by setting $\chi(0) = 0$. In particular, we set $\epsilon(0) = 0$. We set $\delta(0) = \delta(\epsilon) = 1$ and $\delta(x) = \delta(\chi) = 0$ for other $x \in \mathbb{F}_q$ and $\chi \in \widehat{\mathbb{F}}_q^\star$. Note that $\delta(x) + \epsilon(x) = 1$ for all $x \in \mathbb{F}_q$.

Let $e_q\colon \mathbb{F}_q \to \mathbb{C}^\star$ be an additive character of the field $\mathbb{F}_q$. With a certain $h \in \mathbb{F}_q^\star$, one has $e_q(x) = \exp\bigl(2\pi i\,\mathrm{tr}(hx)/p\bigr)$ for all $x \in \mathbb{F}_q$. Here, $\mathrm{tr}\colon \mathbb{F}_q \to \mathbb{F}_p$ denotes the trace, $\mathrm{tr}(z) = z + z^p + \cdots + z^{p^{l-1}}$. This character is fixed throughout the paper. Every additive character of the field $\mathbb{F}_q$ is the function $z \mapsto e_q(kz)$ with a certain $k \in \mathbb{F}_q$. Then to e_q we can attach the real functions $\cos_q$ and $\sin_q$ defined by

$$\cos_q(x) = \frac{e_q(x) + e_q(-x)}{2}, \qquad \sin_q(x) = \frac{e_q(x) - e_q(-x)}{2i}, \tag{1}$$

for which

$$e_q(x) = \cos_q(x) + i\sin_q(x) \quad \text{for all} \quad x \in \mathbb{F}_q.$$

Consider the complex vector space Ω_q of functions $\mathbb{F}_q \to \mathbb{C}$ with the inner product

$$\langle f, g\rangle = \sum_{x\in\mathbb{F}_q} f(x)\overline{g(x)} \quad \text{for all} \quad f, g \in \Omega_q.$$

The dimension of Ω_q equals q. The multiplicative characters and the above-defined function $\delta\colon \mathbb{F}_q \to \mathbb{C}$ form an orthogonal basis of the space Ω_q. The additive characters also form an orthogonal basis of Ω_q. In more detail, for all $a, b, c \in \mathbb{F}_q$ and $\alpha, \beta \in \widehat{\mathbb{F}}_q^\star$, one has

$$\frac{1}{q}\sum_{z\in\mathbb{F}_q} e_q(az)\overline{e_q(bz)} = \delta(a-b), \qquad \frac{1}{q-1}\sum_{x\in\mathbb{F}_q^\star} \alpha(x)\overline{\beta(x)} = \delta(\alpha\overline{\beta}) \tag{2}$$

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and

$$\frac{1}{q-1}\sum_{\chi\in\widehat{\mathbb{F}}_q^\star}\chi(c)=\delta(1-c). \tag{3}$$

For finite fields, see [4–6].

3. Gauss sums. To a character $\chi\in\widehat{\mathbb{F}}_q^\star$ and an element $c\in\mathbb{F}_q$ we attach the Gauss sums

$$G(\chi)=\sum_{x\in\mathbb{F}_q^\star}e_q(x)\,\chi(x),\qquad G(c,\chi)=\sum_{x\in\mathbb{F}_q^\star}e_q(cx)\,\chi(x).$$

As is well known, for $\chi\neq\epsilon$,

$$G(\epsilon)=-1,\quad G(\chi)\,G(\bar\chi)=\chi(-1)\,q,\quad |G(\chi)|^2=q. \tag{4}$$

Also one has $G(0,\epsilon)=q-1$ and $G(c,\chi)=\bar\chi(c)\,G(\chi)$ for all $c\in\mathbb{F}_q^\star$ and $\chi\in\widehat{\mathbb{F}}_q^\star$. For the Gauss sums, see [5] and [6].

4. Fourier transforms. The additive Fourier transform $\hat F\colon\mathbb{F}_q\to\mathbb{C}$ of a function $F\colon\mathbb{F}_q\to\mathbb{C}$ is defined by

$$\hat F(x)=\sum_{y\in\mathbb{F}_q}F(y)e_q(yx)\quad\text{for all }\ x\in\mathbb{F}_q. \tag{5}$$

The inversion formula is as follows:

$$F(z)=\frac{1}{q}\sum_{x\in\mathbb{F}_q}\hat F(x)e_q(-xz)\quad\text{for all }\ z\in\mathbb{F}_q. \tag{6}$$

One can treat (6) as the expansion of the function F in the basis consisting of the additive characters.

The multiplicative Fourier transform $\widehat F\colon\widehat{\mathbb{F}}_q^\star\to\mathbb{C}$ of a function $F\colon\mathbb{F}_q^\star\to\mathbb{C}$ is defined by

$$\widehat F(\chi)=\sum_{x\in\mathbb{F}_q^\star}F(x)\chi(x)\quad\text{for all }\ \chi\in\widehat{\mathbb{F}}_q^\star. \tag{7}$$

The Fourier inversion formula,

$$F(z)=\frac{1}{q-1}\sum_{\chi\in\widehat{\mathbb{F}}_q^\star}\widehat F(\chi)\bar\chi(z)\quad\text{for all }\ z\in\mathbb{F}_q, \tag{8}$$

allows one to recover F from $\widehat F$. Formulas (6) and (8) follow from the orthogonality relations (2) and (3). These facts are well known. For every function $F\colon\mathbb{F}_q\to\mathbb{C}$, there is an expansion

$$F(z)=F(0)\,\delta(z)+\sum_{\chi\in\widehat{\mathbb{F}}_q^\star}C_\chi\,\chi(z),\quad\text{with }\ C_\chi=\frac{1}{q-1}\,\widehat F(\bar\chi)\ \text{ and }\ z\in\mathbb{F}_q,$$

which is similar to (8) but involves an additional term. For instance, for all $z\in\mathbb{F}_q$ one has

$$e_q(-z)=1+\frac{q}{q-1}\sum_{\chi\in\widehat{\mathbb{F}}_q^\star}\frac{\chi(z)}{G(\chi)}.$$

This is an analog of the Taylor expansion for the exponent.

5. Differentiation. Given a multiplicative character χ of the field $\mathbb{F}_q$, consider the linear operator D^χ defined by the formula

$$D^\chi F(x) = \frac{1}{G(\bar\chi)} \sum_{t\in\mathbb{F}_q} \bar\chi(t)\, F(x-t) \quad \text{for all} \quad x \in \mathbb{F}_q, \tag{9}$$

which takes a function $F \in \Omega_q$ to the function $D^\chi F \in \Omega_q$. According to Evans [1], $D^\chi F$ is the derivative of order χ for F. If $\chi \neq \epsilon$, then one can apply (4) and write (9) as

$$\frac{1}{G(\chi)} D^\chi F(x) = \frac{1}{q} \sum_{t\in\mathbb{F}_q} \bar\chi(t)\, F(x+t) \quad \text{for all} \quad x \in \mathbb{F}_q.$$

This relation is similar to the Cauchy integral formula for the derivatives of analytic functions.

The operator D^χ with $\chi \neq \epsilon$ takes constant functions to the zero function. The formulas

$$D^\epsilon F(x) = F(x) - \sum_{t\in\mathbb{F}_q} F(t),$$

$$D^\chi D^{\bar\chi} F(x) = F(x) - \frac{1}{q} \sum_{t\in\mathbb{F}_q} F(t), \tag{10}$$

$$D^\alpha D^\beta = D^{\alpha\beta}$$

hold for all $F \in \Omega_q$, $x \in \mathbb{F}_q$, and characters $\alpha, \beta, \chi \in \widehat{\mathbb{F}}_q^\star$ under the assumptions that $\chi \neq \epsilon$ and $\alpha\beta \neq \epsilon$. For arbitrary $E, F \in \Omega_q$, $x \in \mathbb{F}_q$, and any character $\nu \in \widehat{\mathbb{F}}_q^\star$, one has the formula for integration by parts,

$$\sum_{z\in\mathbb{F}_q} E(z)\, D^\nu F(z) = \nu(-1) \sum_{z\in\mathbb{F}_q} F(z)\, D^\nu E(z),$$

and also the formula

$$D^\nu EF(x) = \frac{1}{q-1} \sum_{\mu\in\widehat{\mathbb{F}}_q^\star} \frac{G(\bar\mu)G(\mu\bar\nu)}{G(\bar\nu)}\, D^\mu E(x)\, D^{\nu\bar\mu} F(x)$$

for the νth derivative of the product EF of E and F, which is similar to the classical Leibniz formula. All these properties of the operators D^χ can be found in [1].

Let $c \in \mathbb{F}_q^\star$, $d \in \mathbb{F}_q$. Consider the composition E of a function $F\colon \mathbb{F}_q \to \mathbb{C}$ and the function $x \mapsto cx + d$. In other words, let $E(x) = F(cx+d)$ for all $x \in \mathbb{F}_q$. Then, for any character $\chi \in \widehat{\mathbb{F}}_q^\star$ and any $x \in \mathbb{F}_q$,

$$D^\chi E(x) = \chi(c) D^\chi F(cx+d). \tag{11}$$

Formula (11) is the simplest analog of the classical formula for the derivative of a composition and immediately follows from the definition (9). Indeed,

$$\begin{aligned} D^\chi E(x) &= \frac{1}{G(\bar\chi)} \sum_{t\in\mathbb{F}_q} \bar\chi(t)\, F\big(c(x-t)+d\big) \\ &= \frac{1}{G(\bar\chi)}\, \chi(c) \sum_{t\in\mathbb{F}_q} \bar\chi(ct)\, F\big((cx+d)-ct\big) \\ &= \frac{1}{G(\bar\chi)}\, \chi(c) \sum_{z\in\mathbb{F}_q} \bar\chi(z)\, F\big((cx+d)-z\big) = \chi(c) D^\chi F(cx+d). \end{aligned}$$

For an odd character χ, the operator D^χ takes even functions to odd ones, and it takes odd functions to even ones. For an even character χ, the operator D^χ takes even functions to even ones and odd functions to odd ones. This can be proved by a simple computation. Indeed, let $F\colon \mathbb{F}_q \to \mathbb{C}$ be, say, an even function and let $x \in \mathbb{F}_q$. Then from (9) it follows that

$$D^\chi F(-x) = \frac{1}{G(\bar\chi)} \sum_{t\in\mathbb{F}_q} \bar\chi(t)\, F(-x-t) = \frac{1}{G(\bar\chi)} \sum_{t\in\mathbb{F}_q} \bar\chi(t)\, F(x+t)$$
$$= \frac{1}{G(\bar\chi)}\, \bar\chi(-1) \sum_{z\in\mathbb{F}_q} \bar\chi(z)\, F(x-z) = \bar\chi(-1)\, D^\chi F(x),$$

as claimed.

Given a character $\chi \in \widehat{\mathbb{F}}_q^\star$, let D_n^χ be the operator that takes a function of several variables $\mathbb{F}_q \times \ldots \times \mathbb{F}_q \to \mathbb{C}$ to its partial derivative of order χ with respect to the nth variable. In this notation, one has

$$D_2^\beta D_1^\alpha = D_1^\alpha D_2^\beta$$

for all characters $\alpha, \beta \in \widehat{\mathbb{F}}_q^\star$. This means that one can evaluate partial derivatives in any order. In more detail, for all $E\colon \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{C}$ and $x, y \in \mathbb{F}_q$,

$$D_2^\beta D_1^\alpha E(x,y) = D_1^\alpha D_2^\beta E(x,y) = \frac{1}{G(\bar\alpha)\, G(\bar\beta)} \sum_{s,t\in\mathbb{F}_q} \bar\alpha(s)\bar\beta(t) E(x-s, y-t).$$

This immediately follows from (9).

Consider some examples. Let $c \in \mathbb{F}_q^\star$ and let $E(x) = e_q(-cx)$ for all $x \in \mathbb{F}_q$. By (11), for all characters $\chi \in \widehat{\mathbb{F}}_q^\star$ and all $x \in \mathbb{F}_q$,

$$D^\chi E(x) = \chi(c)\, e_q(-cx). \tag{12}$$

The definition (1) and formula (12) with $c = 1$ imply that

$$D^\chi \cos_q = -i \sin_q \quad \text{and} \quad D^\chi \sin_q = i \cos_q \tag{13}$$

for all odd characters χ, i.e., for all χ with $\chi(-1) = -1$.

6. The equation of vibrations of a string. Given two functions $u, v\colon \mathbb{R} \to \mathbb{R}$ and a constant $c \in \mathbb{R}$, consider the function $w\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$w(x,t) = u(x-ct) + v(x+ct) \quad \text{for all} \quad x, t \in \mathbb{R}. \tag{14}$$

The function w defined in this way satisfies the differential equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial^2 x}, \tag{15}$$

where u and v are assumed to be twice continuously differentiable. This equation is known in mathematical physics as the equation of vibrations of a string. To be more precise, the equation of free vibrations of an infinite string. A solution of the form (14) is known as its d'Alembert solution. It is the general solution of Eq. (15) in the class of twice continuously differentiable functions and can be used in solving particular problems.

Given functions $a, b\colon \mathbb{R} \to \mathbb{R}$, consider the Cauchy problem of finding solutions of Eq. (15) that satisfy the conditions

$$w\,\Big|_{t=0} = a, \qquad \frac{\partial w}{\partial t}\,\Big|_{t=0} = b. \tag{16}$$

If a is a twice continuously differentiable function and b is a continuously differentiable function, then the problem has a unique solution in the class of twice continuously differentiable functions. The solution is given by d'Alembert's formula

$$w(x,t)=\frac{1}{2}\{a(x-ct)+a(x+ct)\}+\frac{1}{2c}\int_{x-ct}^{x+ct} b(y)\,dy \quad \text{for all} \quad x,t\in\mathbb{R},$$

see, for example, Sobolev [7].

7. Changing the context. Choose a nontrivial multiplicative character ρ of the field $\mathbb{F}_q$. We will write D for the operator D^χ with $\chi=\rho$ and treat D as the differentiation operator of order 1. Also, for a positive integer n, define the operators D^n and D^{-n} as the compositions of n operators D and D^{-1}, respectively. By D^{-1} we mean D^χ with $\chi=\rho^{-1}$. In particular, we treat Df as the first order derivative and $D^{-1}f$ as the primitive function for the function $f\colon \mathbb{F}_q\to\mathbb{C}$; $D^2f=DDf$. In the sequel, given a function of several variables $F\colon \mathbb{F}_q\times\ldots\times\mathbb{F}_q\to\mathbb{C}$, we write D_1^nF, $D_2^nF,\ldots$ for the order n derivatives of F with respect to the first, second, etc. variables. The symbols D_t^nF, D_x^nF, etc. have the same meaning, where the variables are denoted by $t,x,\ldots$ rather than numbered.

We consider the equation of vibrations of a string (15) in the context of functions over finite fields.

Theorem 1. *For arbitrary functions $u,v\colon \mathbb{F}_q\to\mathbb{C}$ and an arbitrary constant $c\in\mathbb{F}_q^\star$, the function $w\colon \mathbb{F}_q\times\mathbb{F}_q\to\mathbb{C}$ defined by*

$$w(x,t)=u(x-ct)+v(x+ct) \quad \textit{for all } x,t\in\mathbb{F}_q \tag{17}$$

satisfies the equation of vibrations of a string

$$D_t^2w=C\,D_x^2w \ \textit{ with the constant } \ C=\rho(c)^2. \tag{18}$$

Conversely, if a function $w\colon \mathbb{F}_q\times\mathbb{F}_q\to\mathbb{C}$ is not a constant and satisfies the equation

$$D_t^2w=C\,D_x^2w \ \textit{ with a certain } \ C\in\mathbb{C},\ \ C\neq 0, \tag{19}$$

then $C=\rho(c)^2$ for a constant $c\in\mathbb{F}_q^\star$, and in the Fourier expansion

$$w(x,t)=\sum_{m,n\in\mathbb{F}_q} r(m,n)e_q(-mx-nt), \qquad x,t\in\mathbb{F}_q, \tag{20}$$

the coefficients $r(m,n)$ are nonzero only if either $m=n=0$ or $\rho(cm/n)=\pm1$, $mn\neq0$. In particular, for a primitive character ρ, these conditions are equivalent to $cm=\pm n$, and the function w can be represented in the form (17)*, with appropriate u and v.*

Proof. Expand the function $u\colon \mathbb{F}_q\to\mathbb{C}$ into the Fourier series,

$$u(x)=\sum_{m\in\mathbb{F}_q} r(m)e_q(-mx) \quad \text{with } x\in\mathbb{F}_q, \tag{21}$$

and consider the function

$$(x,t)\mapsto u(x-ct)=\sum_{m\in\mathbb{F}_q} r(m)e_q(-mx+mct), \qquad x,t\in\mathbb{F}_q. \tag{22}$$

Using formula (12), we find that the operator D_t^2 takes the function (22) to the function

$$(x,t)\mapsto\sum_{m\in\mathbb{F}_q} r(m)\rho(-mc)^2e_q(-mx+mct), \qquad x,t\in\mathbb{F}_q. \tag{23}$$

Similarly, the operator D_x^2 takes (22) to the function

$$(x,t)\mapsto\sum_{m\in\mathbb{F}_q} r(m)\rho(m)^2e_q(-mx+mct) \qquad x,t\in\mathbb{F}_q. \tag{24}$$

Notice that $\rho(-mc)^2 = \rho(c)^2\rho(m)^2$ for all $m \in \mathbb{F}_q$. Comparing (24) with (23), we find that the function (22) is a solution of Eq. (18). Applying similar arguments to v and $-c$ rather than to u and c, we conclude that the function

$$(x,t) \mapsto v(x+ct), \qquad x,t \in \mathbb{F}_q, \tag{25}$$

is a solution of Eq. (18). By the linearity of the differential operators and Eq. (18), the sum of the functions (22) and (25) is a solution of Eq. (18), as claimed. The same result can be obtained in a different way by applying formula (11) with $\chi = \rho^2$ instead of the Fourier expansions.

Now consider the Fourier expansion (20) for a function w satisfying Eq. (19) and apply formula (12) in order to obtain (using term-by-term differentiation) the Fourier expansions of the functions $D_t^2 w$ and $D_x^2 w$. Comparing the Fourier coefficients, we find that (18) is equivalent to

$$C\rho(m)^2 r(m,n) = \rho(n)^2 r(m,n) \qquad \text{for all } m,n \in \mathbb{F}_q. \tag{26}$$

If $m = 0$ and $n \neq 0$, then $\rho(m) = 0$ and $\rho(n) \neq 0$. Thus, from (26) it follows that $r(0,n) = 0$ for all $n \neq 0$. Similarly, we find that $r(m,0) = 0$ for all $m \neq 0$. Further, since the function w is not a constant, there exists a pair (m',n') with $r(m',n') \neq 0$ and $m'n' \neq 0$. For such a pair (m',n'), from (26) it follows that $C = \rho(c)^2$ with $c = n'/m'$. Finally, if $r(m,n) \neq 0$ and $mn \neq 0$, then condition (26) for the pair (m,n) can be written as $\rho(cm)^2 = \rho(n)^2$, which is equivalent to $\rho(cm/n) = \pm 1$.

Assume that ρ is a primitive character. For a field $\mathbb{F}_q$ of odd characteristic and $mn \neq 0$, the condition $\rho(cm/n) = \pm 1$ is equivalent to $n = \pm cm$. Also, in this case, we have $cm \neq -cm$ for all $m \neq 0$. The only terms in the Fourier expansion (20) that can be nonzero are the constant term $r(0,0)$ and the terms attached to the pairs $(m,-cm)$ and (m,cm) with $m \in \mathbb{F}_q^\star$. Thus, (20) implies (17) with

$$u(z) = r(0,0) + \sum_{m\in\mathbb{F}_q^\star} r(m,-cm)e_q(-mz), \qquad v(z) = \sum_{m\in\mathbb{F}_q^\star} r(m,cm)e_q(-mz). \tag{27}$$

For a field $\mathbb{F}_q$ of characteristic 2, the character ρ does not take the value -1. In addition, if $mn \neq 0$, then $cm = -cm$, and the condition $\rho(cm/n) = \pm 1$ is equivalent to $n = cm$ and also to $n = -cm$. In this case, for w we have representation (17) with u from (27) and without the function v at all. □

Theorem 2. *Given a field $\mathbb{F}_q$ of odd characteristic, its primitive character ρ, a pair of functions $a,b\colon \mathbb{F}_q \to \mathbb{C}$, and a constant $C = \rho(c)^2$ with $c \in \mathbb{F}_q^\star$, consider the problem of finding a solution $w\colon \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{C}$ of the differential equation*

$$D_t^2 w = C D_x^2 w \tag{28}$$

satisfying the Cauchy initial conditions

$$w\,\Big|_{t=0} = a, \qquad D_t w\,\Big|_{t=0} = b. \tag{29}$$

This problem has a unique solution. It can be represented as

$$w(x,t) = \frac{a(x+ct)+a(x-ct)}{2} + \bar{\rho}(c)\,\frac{D^{-1}b\,(x+ct) - D^{-1}b\,(x-ct)}{2} \tag{30}$$

for all $x,t \in \mathbb{F}_q$.

Proof. Let w be the function (17) specified by arbitrary functions $u,v\colon \mathbb{F}_q \to \mathbb{C}$. In accordance with Theorem 1, w is a solution of Eq. (28) and, moreover, every solution can be written in such a form.

Show that w, with appropriately chosen u and v, satisfies the Cauchy conditions (29). Clearly, the first condition in (29) is equivalent to

$$u + v = a. \tag{31}$$

The second condition in (29) is equivalent to

$$\rho(-c)Du + \rho(c)Dv = b. \tag{32}$$

This is explained as follows. Apply the operator D_t to the function $(x,t) \mapsto u(x-ct)$ and then take $t = 0$. In this way, we obtain the function $\rho(-c)Du$. Indeed, consider the function

$$(x,t) \mapsto u(x-ct) = \sum_{m \in \mathbb{F}_q} r(m) e_q(-mx + mct), \qquad x, t \in \mathbb{F}_q, \tag{33}$$

where $r(m)$ are the Fourier coefficients of the function u (as in (21)). By (12), we find that the operator D_t takes (33) to the function

$$(x,t) \mapsto \sum_{m \in \mathbb{F}_q} r(m)\rho(-mc) e_q(-mx + mct), \qquad x, t \in \mathbb{F}_q.$$

With $t = 0$, this is exactly the function $\rho(-c)Du$. Similarly, by applying the operator D_t to the function $(x,t) \mapsto v(x+ct)$ and taking $t = 0$, we obtain the function $\rho(c)Dv$. Considering the sum of these functions, we find that condition (32) is equivalent to the second condition in (29). Note that this equivalence can also be established by applying formula (11) instead of the Fourier expansions.

Our assumptions on $\mathbb{F}_q$ and ρ imply that $\rho(-1) = -1$. It follows that (32) is equivalent to

$$-Du + Dv = \bar{\rho}(c)\, b. \tag{34}$$

Thus, in order to determine the functions u and v, we have Eqs. (31) and (34). Apply the operator D^{-1} to the functions in (34). By (10), $D^{-1}Du = u + Q$ and $D^{-1}Dv = v + R$ with some constants $Q, R \in \mathbb{C}$. We see that (34) is equivalent to

$$-u + v = \bar{\rho}(c) D^{-1} b + P \tag{35}$$

with a constant $P \in \mathbb{C}$. Upon solving Eqs. (31) and (35) for u and v and substituting the result into (17), we obtain a solution of our problem and its representation (30).

Now we will prove uniqueness. By the linearity of (28) and (29), it is sufficient to prove uniqueness for the problem with the zero functions a and b only. Let w be a solution of the problem with zero a and b. Express w as in formula (17) in Theorem 1. The functions u and v must satisfy Eqs. (31), (35) with zero functions a and b. Thus, the relations $u + v = 0$ and $-u + v = P$ must hold with a certain constant $P \in \mathbb{C}$. It follows that u and v are constants, and $w = u + v = 0$, as claimed. ☐

8. Boundary-value problems. In connection with the equation of vibrations of a string, a number of different boundary-value problems are considered. Equation (15) and the initial conditions (16) may be supplemented with some boundary conditions.

A typical example is the problem of free vibrations of a string with one end fixed. It consists in finding a function $w\colon [0,\infty] \times [0,\infty) \to \mathbb{R}$ that satisfies Eq. (15), the boundary condition

$$w\big|_{x=0} = g,$$

and the Cauchy conditions (16) with given functions $a, b, g\colon [0,\infty] \to \mathbb{R}$.

Yet another typical example is the problem of free vibrations of a string with fixed ends. Let R denote the distance between the ends of the string. The problem is to find a function $w\colon [0,R] \times [0,\infty) \to \mathbb{R}$ satisfying Eq. (15), the boundary conditions

$$w\big|_{x=0} = 0, \quad w\big|_{x=R} = 0,$$

and the Cauchy conditions (16) with some given functions $a, b\colon [0, R] \to \mathbb{R}$.

In order to change the context and to state similar problems concerning functions over finite fields, it is necessary to give a finite interpretation to intervals. The following two variants look rather natural.

(I) Let the characteristic of $\mathbb{F}_q$ be odd. The squares in the group $\mathbb{F}_q^\star$ form its subgroup $\mathbb{F}_q^{\star 2}$ of index 2. Consider the sets $\mathbb{F}_q^{\star 2}$ and $\mathbb{F}_q^\star \setminus \mathbb{F}_q^{\star 2}$ as analogs of the sets of positive and negative real numbers. Given an $m \in \mathbb{F}_q$, set $(m, \infty) = m + \mathbb{F}_q^{\star 2}$ and $(-\infty, m) = m + \mathbb{F}_q^\star \setminus \mathbb{F}_q^{\star 2}$. We can regard these sets as half-lines in $\mathbb{F}_q$ and as analogs of open half-lines in $\mathbb{R}$. Obviously,

$$\mathbb{F}_q = (-\infty, m) \cup \{m\} \cup (m, \infty).$$

Also, we set $[m, \infty) = \{m\} \cup (m, \infty)$ and $(-\infty, m] = (-\infty, m) \cup \{m\}$. These are half-lines in $\mathbb{F}_q$ and analogs of closed half-lines in $\mathbb{R}$. It is convenient to introduce into consideration a quadratic character κ of the group $\mathbb{F}_q^\star$ extended to a function on $\mathbb{F}_q$ by setting $\kappa(0) = 0$. Given $m, n \in \mathbb{F}_q$, set

$$(m, n) = (-\infty, n) \cap (m, \infty) = \bigl\{z \in \mathbb{F}_q \bigm| \kappa(z - m) = \kappa(n - z) = 1\bigr\}.$$

This is an analog of an open bounded interval in $\mathbb{R}$.

(II) Consider $\mathbb{F}_q$ as a vector space over $\mathbb{F}_p$. We have the linear mapping $\operatorname{tr}\colon \mathbb{F}_q \to \mathbb{F}_p$ with the kernel $\mathbb{S} = \{z^q - z \,|\, z \in \mathbb{F}_q\}$. Given an $m \in \mathbb{F}_q^\star$, consider the space $\mathbb{S}_m = \{mx \,|\, x \in \mathbb{S}\}$, which is the kernel of the additive character $z \mapsto \exp(2\pi i \operatorname{tr}(m^{-1}z)/p)$ of $\mathbb{F}_q$. One can lift functions $\mathbb{F}_q/\mathbb{S}_m \to \mathbb{C}$ to functions $\mathbb{F}_q \to \mathbb{C}$ invariant under translations by vectors in $\mathbb{S}_m$. (This corresponds to periodic complex functions on $\mathbb{R}$ in the classical setting.) Then one can follow the Fourier method of separation of variables and obtain solutions of boundary-value problems represented as linear combinations of the trigonometric functions (1), which are periodic and possess all the necessary properties, see (13) and (12).

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