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MEAN VALUE PROPERTIES OF HARMONIC FUNCTIONS AND RELATED TOPICS (A SURVEY)

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Results involving various mean value properties are reviewed for harmonic, biharmonic and metaharmonic functions. It is also considered how the standard mean value property can be weakened to imply harmonicity and belonging to other classes of functions. Bibliography: 70 *titles.*

To N. N. Uraltseva with deep respect

1 Introduction

It is highly likely that the mean value theorem is the most remarkable and useful fact about harmonic functions. Results on this and other properties of harmonic functions were surveyed by Netuka [1] in the remote 1975, but, unfortunately, there is a number of inaccuracies in this paper. To the best author's knowledge, only one review in this area had appeared since then; namely, the extensive article [2] by Netuka and Veselý which is a substantially extended version of Netuka's survey updated to mid-1993, but still reproducing some of the inaccuracies from the previous paper.

During the past 25 years ([2] was published in 1994), rather many papers on mean value properties and other related topics have appeared. Some of these contain results of significant interest (cf., for example, [3]–[5] to list a few). A number of new as well as some old results deserve to be reviewed, especially, various forms of converse mean value theorem. Mean value theorems are also considered for solutions of equations different from the Laplace equation. Moreover, for several old results, for which only two-dimensional versions were published, proofs are provided for general formulations or just outlined because they were not properly presented in [2]. However, the so-called inverse mean value properties (cf. [2, Sections 7 and 8]) are not treated here, in particular, because these properties were considered in detail in Zaru's thesis [6] available online. Sections 2 and 3 of her thesis deal with inverse mean value properties on balls and annuli and strips, respectively. Of course, many references to the paper [2] are given, since results reviewed here continue research initiated before 1993 and described in Netuka and

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Veselý's article.

In the remaining part of this section, basic classical results about harmonic functions are presented as the basis for considerations in Sections 2–4, and so bibliography used here is restricted to a few monographs, textbooks and pioneering papers (cf. [1, 2] for further references). We begin with the standard formulation of the mean value theorem (see, for example, the monograph [7] by Gilbarg and Trudinger or the textbook [8] by Mikhlin).

Theorem 1.1. *Let D be a domain in $\mathbb{R}^m$, $m \geqslant 2$. If $u \in C^2(D)$ satisfies the Laplace equation*

$$\nabla^2 u = 0 \quad \text{in } D,$$

then we have

$$u(x) = \frac{1}{m\,\omega_m r^{m-1}} \int_{\partial B} u \,\mathrm{d}S \tag{1.1}$$

for every ball $B = B_r(x)$ such that $\overline{B} \subset D$.

Here and below, the following notation is used: $\mathbb{R}^m$ is the Euclidean m-space with points $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$, where x_i, y_i are real numbers, and the norm is $|x| = (x_1^2 + \dots + x_m^2)^{1/2}$.

For a set $G \subset \mathbb{R}^m$ we denote by ∂G its boundary, whereas $\overline{G} = G \cup \partial G$ is its closure; $D \subset \mathbb{R}^m$ is a domain if it is an open and connected set, not necessarily bounded. In particular, $B_r(x) = \{y : |y - x| < r\}$ denotes the open ball of radius r centred at x, the volume of unit ball is $\omega_m = 2\pi^{m/2}/[m\Gamma(m/2)]$ and $\mathrm{d}S$ is the surface area measure.

We denote by $\nabla = (\partial_1, \dots, \partial_m)$ the gradient operator; here, $\partial_i = \partial/\partial x_i$, $\partial_i\partial_j = \partial^2/\partial x_i \partial x_j$ etc. Finally, $C^k(D)$ is the set of continuous functions in D, whose derivatives of order $\leqslant k$ are also continuous there; functions in $C^k(\overline{D})$ are continuous in $\overline{D}$ with all derivatives of order $\leqslant k$.

Since the denominator in (1.1) is equal to the area of sphere of radius r, it is common to refer to this equality as the area version of mean value theorem. Also, it is known as Gauss' theorem of the arithmetic mean (cf. [9, p. 223]). Indeed, one finds this theorem in his paper *Algemeine Lehrsatze in Beziehung auf die im verkehrtem Verhaltnisse des Quadrats der Entfernung Wirkenden Anziehungs- und Abstossungs-Krafte* published in 1840 (cf. also *Gauss Werke*, Bd. 5. S. 197–242); the corresponding quotation from this paper is given in [2, p. 361].

The following consequence of Theorem 1.1 is not widely known, but has important applications in the linear theory of water waves (cf., the monograph [10, Section 4.1]), where a proof of this assertion is given, and the article [11], where further references can be found.

Corollary 1.1. *Zeros of a harmonic function are never isolated.*

Harmonic functions also have a mean value property with respect to volume measure (cf., for example, [12, p. 12]). It follows by integrating (1.1) with respect to the polar radius at x from 0 to some $R > 0$, thus yielding.

Theorem 1.2. *Let D be a bounded domain in $\mathbb{R}^m$, $m \geqslant 2$, and let $B_R(y) \subset D$ be any open ball such that R is less than or equal to the distance from y to ∂D. If u is a Lebesgue integrable harmonic function in D, then we have*

$$u(y) = \frac{1}{\omega_m R^m} \int_{B_R(y)} u \,\mathrm{d}x\,, \tag{1.2}$$

where the ball volume stands in the denominator.

In what follows, we denote by $L(u,x,r)$ and $A(u,x,r)$ the right-hand side terms in (1.1) and (1.2) respectively (the notation used in [2]).

There are many corollaries of Theorem 1.2 and the most important is the maximum principle also referred to as the strong maximum principle (cf., for example, [7, p. 15]).

Theorem 1.3. *Let D be a bounded domain. If a harmonic function u is attained either a global minimum or maximum in D, then u is constant.*

This theorem implies several global estimates (cf. [7, Chapter 2]); first we formulate those known as the weak maximum principle.

Corollary 1.2. *Let $u \in C^2(D) \cap C^0(\overline{D})$, where D is a bounded domain. If u is harmonic in D, then*

$$\min_{x \in \partial D} u \leqslant u(x) \leqslant \max_{x \in \partial D} u \quad \forall\, x \in \overline{D},$$

where equalities hold only for $u \equiv \text{const}$.

Moreover, the derivatives of a harmonic function are estimated in terms of the function itself.

Corollary 1.3. *Let u be harmonic in D. If D' is a compact subset of D, then*

$$\max_{x \in D'} |\partial_i u| \leqslant \frac{m}{d} \sup_{x \in D} |u|, \quad i = 1, \dots, m,$$

where d is the distance between D' and ∂D.

In his note [13] published in 1906, Koebe announced the following converse of Theorem 1.1.

Theorem 1.4. *Let D be a bounded domain in $\mathbb{R}^m$, $m \geqslant 2$. If $u \in C^0(\overline{D})$ satisfies the equality* (1.1) *for every ball $B = B_r(x)$ such that $\overline{B} \subset D$, then u is harmonic in D.*

There is a slightly stronger version of this theorem requiring that (1.1) holds not at every $x \in D$, but only for some sequence $r_k(x) \to 0$ as $k \to \infty$. Of several proofs of this theorem, we mention two. In his classical book [9, pp. 224–226], Kellogg used straightforward but cumbersome calculations for establishing that $u \in C^2(D)$. In [8], these calculations are replaced by application of mollifier technique (cf. the proof of Theorem 11.7.2 in [8]). Moreover, the latter book contains the following important consequence of Corollary 1.2 and Theorem 1.4 (cf. [8, Theorem 11.9.1]).

Corollary 1.4 (the Harnack convergence theorem)**.** *Let a sequence $\{u_k\}$ consist of $C^0(\overline{D})$-functions harmonic in D. If $u_k \to u$ uniformly on ∂D as $k \to \infty$, then $u_k \to u$ uniformly on $\overline{D}$ and u is harmonic in D.*

Various other results about convergence of harmonic functions can be found in Brelot's lectures [14, Appendix, Section 19] along with an elegant proof of Theorem 1.4 based on the Poisson formula and the maximum principle ([14, Appendix, Section 17]).

It is worth mentioning that two early versions of converse to Theorem 1.1 were published by different authors under the same title (cf. [15] and [16]). In the first paper, Levi independently proved the two-dimensional version of Theorem 1.4. (However, this mathematician is more widely known for his paper of 1907, in which a fundamental solution to a general elliptic equation of second order with variable coefficients is constructed (cf. [17]).

In the second paper, Tonelli relaxed the assumptions imposed by Levi; namely, u is required to be Lebesgue integrable on D. Further historical remarks and characterizations of harmonic functions analogous to Theorem 1.4, but expressed in terms of the equality (1.2) instead of (1.1), can be found in [2, pp. 363–364].

In view of Theorem 1.4, the validity of the equality (1.1) for every ball $B = B_r(x)$ such that $\overline{B} \subset D$ can be taken as the definition of harmonicity for functions locally integrable in a domain D. Another illustration (due to Uspenskii [18]) that this definition is reasonable is not so well known. In this two-dimensional consideration, the circumference centred at x is denoted by C instead of ∂B and $L(u, x, C)$ stands for the mean value of u over C.

First, we notice that $L(u, x, C)$ has the following properties:

(i) $L(u, x, C) \geqslant 0$ provided $u \geqslant 0$ on C,

(ii) $L(u_1 + u_2, x, C) = L(u_1, x, C) + L(u_2, x, C)$ for u_1 and u_2 given on C,

(iii) $L(\alpha u, x, C) = \alpha L(u, x, C)$ for all $\alpha \in \mathbb{R}$,

(iv) $L(u, x, C) = 1$ provided that $u \equiv 1$ on C;

(v) $L(u, x, C) = L(u_*, x, C_*)$ when two circumferences C and C_* (u and u_* are given on the respective curve) are congruent in such a way that the functions' values are equal at the corresponding points.

Some of these relations are obvious and the others are straightforward to verify.

It occurs that (i)–(v) provide an axiomatic definition of the mean value over C for the class of integrable functions. Indeed, such a definition is equivalent to (1.1) and all facts that follow from the latter formula can be proved on the basis of (i)–(v). In particular, the main result of [18] is derivation of Corollary 2 from these relations in the case of a disc. Also, an interesting representation in geometric terms is found for the function solving the Dirichlet problem in a disc when a step function is given on its boundary.

Of course, the assertion analogous to Theorem 1.4 with formula (1.2) replacing (1.1) is true as well as some weaker formulations (cf. [19, pp. 17–18] and [2, p. 364]).

Another obvious consequence of harmonicity is the following.

Proposition 1.1 (Zero flux property)**.** *Let u be harmonic in a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$. If D' is a bounded subdomain such that $\overline{D'} \subset D$ and $\partial D'$ is piecewise smooth, then*

$$\int_{\partial D'} \frac{\partial u}{\partial n} \, \mathrm{d}S = 0 \,. \tag{1.3}$$

Here and below n denotes the exterior normal to smooth (of class C^1) parts of domains' boundaries. The name of this assertion has its origin in the hydrodynamic interpretation of harmonic functions as velocity potentials describing irrotational motions of an inviscid fluid in $D \subset \mathbb{R}^m$, $m = 2, 3$, in the absence of sources in which case the influx is equal to outflux for every $D' \subset D$.

Remark 1.1. If u is harmonic in a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$, then u is analytic in D (cf. [7, p. 18]), and so $u \in C^\infty(\overline{B})$ for any closed ball $\overline{B} \subset D$. Moreover, for every $k \geqslant 1$ we have

$$\int\limits_{\partial B} \frac{\partial^k u}{\partial n^k} \, \mathrm{d}S = 0\,. \tag{1.4}$$

This follows from Theorem 1 proved in [20, p. 171], and generalizes (1.3) for $D' = B$.

In 1906, Bôcher [21] and Koebe [13] independently discovered the classical converse to this proposition in two and three dimensions, respectively; its m-dimensional version is as follows.

Theorem 1.5 (Bôcher and Koebe)**.** *Let D be a bounded domain in $\mathbb{R}^m$, $m \geqslant 2$. Then u belonging to $C^0(\overline{D}) \cap C^1(D)$ is harmonic in D provided that it satisfies the equality*

$$\int\limits_{\partial B} \frac{\partial u}{\partial n} \, \mathrm{d}S = 0 \quad \textit{for every ball } B \textit{ such that } \overline{B} \subset D. \tag{1.5}$$

In three dimensions (which does not restrict generality), this theorem is proved in [9, pp. 227–228], by deriving the equality (1.1) from (1.5), which allows us to apply Theorem 1.4. Along with the latter assertion, the Bôcher–Koebe theorem characterizes harmonic functions, but, undeservedly, this fact is not so widely known now. Indeed, the corresponding references are just mentioned in three lines in the extensive survey [2]. Several generalizations of Theorem 1.5 are described below in Section 2.

Another characterization of harmonicity in the two-dimensional case was obtained by Blaschke in 1916. In the brief note [22], he demonstrated that the property, now referred to as the asymptotic behavior of the mean value, is sufficient. The general form of his assertion is as follows.

Theorem 1.6. *Let D be a domain in $\mathbb{R}^m$, $m \geqslant 2$. If for $u \in C^0(D)$ the equality*

$$\lim_{r \to +0} r^{-2} [L(u, x, r) - u(x)] = 0 \tag{1.6}$$

holds for every $x \in D$, then u is harmonic in D.

Now we turn to Kellogg's paper [23] published in 1934 and opening the line of works in which the so-called restricted mean value properties are involved. Much later this notion was defined as follows (cf., for example, [24]).

Definition 1.1. A real-valued function f defined on an open subset $G \subset \mathbb{R}^m$ is said to have the restricted mean value property with respect to balls (spheres) if for each $x \in G$ there exists a ball (sphere) centred at x of radius $r(x)$ such that $B_{r(x)}(x) \subset G$ and the average of f over this ball (its boundary) is equal to $f(x)$.

Then Kellogg's result takes the following form.

Theorem 1.7. *Let D be a bounded domain in $\mathbb{R}^m$, $m \geqslant 2$. If $u \in C^0(\overline{D})$ has the restricted mean value property with respect to spheres, then u is harmonic in D.*

Further applications of restricted mean value properties are considered in Section 3.

An immediate consequence of formulas (1.1) and (1.2) is the following assertion. *Let D be a domain in $\mathbb{R}^m$, $m \geqslant 2$. If u is harmonic in D, then the equality*

$$L(u, x, r) = A(u, x, r) \tag{1.7}$$

holds for all $x \in D$ and all $r > 0$ such that $B_r(x) \subset D$. In the two-dimensional case, its converse was obtained by Beckenbach and Reade [25] in 1943; their simple proof (worth reproducing here) is valid for any $m \geqslant 2$. The general formulation is as follows.

Theorem 1.8. *Let $D \subset \mathbb{R}^m$, $m \geqslant 2$, be a bounded domain. If the equality* (1.7) *holds for $u \in C^0(D)$, all $x \in D$ and all r such that $B_r(x) \subset D$, then u is harmonic in D.*

Proof. If $r \in (0, \rho)$, where ρ is a small positive number, the function $A(u, x, r)$ is defined for x belonging to an open subset of D depending on the smallness of ρ. Moreover, $A(u, x, r)$ is differentiable with respect to r and

$$\partial_r A(u, x, r) = m r^{-1} [L(u, x, r) - A(u, x, r)] = 0, \quad r \in (0, \rho),$$

where the last equality is a consequence of (1.7). Therefore, $A(u, x, r)$ does not depend on $r \in (0, \rho)$. On the other hand, shrinking $B_r(x)$ to its centre by letting $r \to 0$ and taking into account that $u \in C^0(D)$, one obtains that $A(u, x, r) \to u(x)$ as $r \to 0$ for x belonging to an arbitrary closed subset of D. Hence for every $x \in D$ we have that $u(x) = A(u, x, r)$ for all $r \in (0, \rho)$ with some ρ, whose smallness depends on x. Then Theorem 1.7 yields that u is harmonic in D. □

Another proof of this result was published by Freitas and Matos [5], who, presumably, were unaware of the paper [25]. However, their paper contains a generalization of Theorem 1.8 to subharmonic functions (cf. [7, p. 13] for their definition).

Definition 1.2. A function $u \in C^2(D)$ is called *subharmonic* in a domain D if it satisfies the inequality $\nabla^2 u \geqslant 0$ in D.

A characteristic property of a subharmonic function $u \in C^0(D)$ is as follows. *For every ball $B = B_r(x)$ such that $\overline{B} \subset D$ and every harmonic function h in B satisfying $h \geqslant u$ on ∂B the inequality $h \geqslant u$ holds in B as well.* Therefore, for every subharmonic u we have

$$A(u, x, r) \leqslant L(u, x, r) \quad \text{for each } B_r(x) \text{ such that both sides are defined.} \tag{1.8}$$

It is proved in [5] that this inequality characterizes subharmonic functions.

Theorem 1.9. *Let u be continuous in D. Then u is subharmonic in D provided that* (1.8) *holds.*

In conclusion of this section, we mention a property, which is, in some sense, similar to (1.7), and has received much attention (cf. [2, pp. 365–368]). It consists in equating the values of $L(u, x, r)$ corresponding to some u defined on $\mathbb{R}^m$, $m \geqslant 2$, for two different radii $r_1, r_2 > 0$ at every x.

The plan of the main part of the paper is as follows. We begin with generalizations of the Bôcher–Koebe theorem because these results considered in Section 2 are not so widely known. In Section 3, we describe results related to restricted mean value properties. Mean value properties for nonharmonic functions are considered in Section 4.

2 Generalizations of the Bôcher–Koebe Theorem

The Bôcher–Koebe theorem (Theorem 1.5) characterizing harmonic functions in terms of the zero flux property is not so widely known as various results based on mean value properties, in particular, restricted ones. In the survey article [2], two sections are devoted to the latter topic, but the authors just mention a few papers dealing with the zero flux property and its generalizations. The aim of this section is to fill in this gap at least partially.

However, prior to presenting several results of apparent interest it is worth noticing that Theorem 1.5 can be improved. It is mentioned after its formulation that deriving the equality (1.1) from (1.5) and then applying Theorem 1.4 one obtains a proof of Theorem 1.5. The assumption made in Theorem 1.4 that (1.1) holds for all spheres in D is superfluous. It can be weakened by using Theorem 1.7 instead of Theorem 1.4, which leads to the following.

Theorem 2.1. *Let D be a bounded domain in $\mathbb{R}^m$, $m \geqslant 2$. Then u belonging to $C^0(\overline{D}) \cap C^1(D)$ is harmonic in D provided that for every $x \in D$ there exists a radius $r(x)$ such that $\overline{B} \subset D$, where $B = B_{r(x)}(x)$, and the equality* (1.5) *holds for this B.*

Some extensions of the Bôcher–Koebe theorem were obtained by Evans [26] (cf. p. 286 of his paper for the formulation) and Raynor [27] for $m = 2$ and 3, respectively.

2.1. Generalizations of the zero flux property. If $u \in C^2(D)$ is a harmonic function in a bounded domain $D \subset \mathbb{R}^m$, $m \geqslant 2$, and $v \in C^1(D)$, then for any piecewise smooth subdomain D' such that $\overline{D'} \subset D$ the first Green formula for u and v is as follows:

$$\int_{D'} \nabla u \cdot \nabla v \, \mathrm{d}x = \int_{\partial D'} v \, \frac{\partial u}{\partial n} \, \mathrm{d}S \, . \tag{2.1}$$

It occurs that this equality serves itself as a characteristic of harmonic functions and yields several other their characteristic properties.

First, it is well known that (2.1) defines weak solutions of the Laplace equation provided that v is an arbitrary function from $C^1(\overline{D'})$ vanishing on $\partial D'$. It was found long ago that these solutions are harmonic (cf., for example, [17], containing the vast bibliography of classical papers), and in this sense (2.1) characterizes these functions. Second, if $v \equiv 1$, then (2.1) turns into the zero flux property (1.3) discussed in Theorems 1.5 and 2.1.

Furthermore, if v is also harmonic, then (2.1) implies the equality

$$\int_{\partial D'} u \, \frac{\partial v}{\partial n} \, \mathrm{d}S = \int_{\partial D'} v \, \frac{\partial u}{\partial n} \, \mathrm{d}S \, , \tag{2.2}$$

which was used by Gergen [28] for obtaining the following generalization of the Bôcher–Koebe theorem.

Theorem 2.2. *Let $D \subset \mathbb{R}^m$, $m \geqslant 2$, be a bounded domain and let $v \in C^2(\overline{D})$ be a harmonic function in D such that $v > 0$ in $\overline{D}$. Then $u \in C^1(\overline{D})$ is harmonic in D provided that the equality* (2.2) (*with D' changed to B*) *holds for every ball B such that $\overline{B} \subset D$.*

2.2. Local characterizations of harmonicity. In his note [29] published in 1932, Saks improved Theorems 1.5 and 2.2 in the two-dimensional case. The general form of his first assertion (its simple proof is reproduced here) is as follows.

Theorem 2.3. *Let $D \subset \mathbb{R}^m$, $m \geqslant 2$, be a bounded domain. Then $u \in C^1(D)$ is harmonic in D provided that for every $x \in D$*

$$\lim_{r \to +0} \frac{1}{|S_r|\, r^2} \int\limits_{S_r(x)} \frac{\partial u}{\partial n} \, \mathrm{d}S = 0 \,, \tag{2.3}$$

where $S_r(x)$ stands for the sphere of radius r centred at x and $|S_r|$ is its area.

Proof. Let us consider $F(r) = r^{-2}[L(u, x, r) - u(x)]$ for arbitrary $x \in D$ and sufficiently small r. Then

$$\begin{aligned} F(r) &= \frac{1}{|S_r|\, r^2} \int\limits_{S_r(x)} [u(y) - u(x)] \, \mathrm{d}S_y \\ &= \frac{1}{|S_r|\, r^2} \int\limits_{S_r(x)} \frac{\partial u}{\partial n} \left(y + \rho\big([y-x]/r\big) \frac{x-y}{r} \right) \mathrm{d}S_y \,, \end{aligned}$$

where $0 < \rho\big([y-x]/r\big) < r$ for all $x \in D$ and all $[y-x]/r \in S_1$. Since (2.3) implies that $F(r) \to 0$ as $r \to +0$, Theorem 1.6 guarantees that u is harmonic. □

We omit the formulation of the second theorem by Saks because it generalizes Theorem 2.2 in the same way as the last theorem generalizes Theorem 1.5. Instead, we turn to an assertion analogous to the last theorem, but characterizing biharmonic functions, i.e., functions satisfying the equation

$$\Delta^2 u = 0, \quad \Delta = \nabla^2. \tag{2.4}$$

Theorem 2.4 (Cheng [30]). *Let D be a domain in $\mathbb{R}^2$. Then $u \in C^3(D)$ is biharmonic in D provided that*

$$\lim_{r \to +0} \frac{1}{r} \int\limits_0^{2\pi} \frac{\partial^3 u}{\partial r^3}(x_1 + r\cos\theta, x_2 + r\sin\theta) \, \mathrm{d}\theta = 0 \tag{2.5}$$

for every $x \in D$.

One more characterization of harmonic functions based on an asymptotic property was obtained by Beckenbach. The property used in his paper [31] is as follows:

$$\int\limits_{B_r(y)} |\nabla u|^2 \, \mathrm{d}x - \int\limits_{\partial B_r(y)} u \frac{\partial u}{\partial n} \, \mathrm{d}S = o(r^2), \quad r \to 0. \tag{2.6}$$

Here $y \in D \subset \mathbb{R}^2$ and the expression on the right-hand side is obtained from (2.1) by substituting $v = u$ and $D' = B_r(y)$.

Theorem 2.5. *Let $u \in C^1(D)$, and let relation* (2.6) *hold for every $y \in D$. If u does not vanish in D, then u is harmonic there.*

Moreover, it is shown that the assumption $u \neq 0$ in this theorem can be replaced by the requirement that $\partial^2_{x_1} u$ and $\partial^2_{x_2} u$ are Lebesgue integrable in D.

2.3. Harmonicity via the zero flux property for cubes. It occurs that the assertion of Theorem 1.5 remains valid when hyperspheres are replaced by hypersurfaces bounding m-cubes and having edges parallel to the coordinate axes. Let

$$Q_r(x) = \{y \in \mathbb{R}^m : |y_i - x_i| < r,\ i = 1, \dots, m\}, \quad r > 0,$$

denote an open cube centred at $x \in \mathbb{R}^m$; it is the smallest cube of this kind containing $B_r(x)$.

Theorem 2.6 (Beckenbach [32]). *Let D be a domain in $\mathbb{R}^m$, $m \geqslant 2$. If u belongs to $C^1(D)$ and satisfies the equality*

$$\int_{\partial Q} \frac{\partial u}{\partial n}\, \mathrm{d}S = 0 \tag{2.7}$$

for every $Q = Q_r(x) \subset D$, then u is harmonic in D.

Proof. Denoting by $\mathbf{0}$ the origin in $\mathbb{R}^m$, we consider the Steklov type mean function

$$u_r(x) = (2r)^{-m} \int_{Q_r(\mathbf{0})} u(x+y)\, \mathrm{d}y\,.$$

It is defined in some D_r approximating D from inside for small values of r. Then we have

$$\partial^2_{x_i} u_r(x) = \frac{1}{(2r)^m} \int_{-r}^{r} \cdots \int_{-r}^{r} \Big[\partial_{x_i} u(x+y)\big|_{y_i=r} - \partial_{x_i} u(x+y)\big|_{y_i=-r}\Big] \mathrm{d}_i x\,,$$

where $\mathrm{d}_i x = \mathrm{d}x_1 \dots \mathrm{d}x_{i-1}\mathrm{d}x_{i+1} \dots \mathrm{d}x_m$ for $i \neq 1, m$ and $\mathrm{d}_1 x$, $\mathrm{d}_m x$ have an appropriate form. Therefore,

$$\nabla^2 u_r(x) = \frac{1}{(2r)^m} \int_{\partial Q_r(x)} \frac{\partial u}{\partial n}\, \mathrm{d}S$$

vanishes on D_r in view of (2.7). On each compact subset of D, we the function u is the uniform limit of u_r as $r \to +0$. Since u_r is harmonic in D_r and the family of domains approximates D, we conclude that u is harmonic in D. □

3 Results Related to Restricted Mean Value Properties

3.1. The restricted mean value property with respect to spheres combined with solubility of the Dirichlet problem. Neither the proof of Theorem 1.7 nor proofs of related results (cf. [2, Sections 5 and 6] for a review) are trivial. However, there is a class of bounded domains for which the assertion converse to this theorem has a very simple proof. Indeed, this takes place when the restricted mean value property is complemented by the assumption that the Dirichlet problem for the Laplace equation is soluble in the domain.

Theorem 3.1. *Let $u \in C^0(\overline{D})$, where $D \subset \mathbb{R}^m$, $m \geqslant 2$, is a bounded domain in which the Dirichlet problem for the Laplace equation has a solution belonging to $C^0(\overline{D})$ for every continuous function given on ∂D. If u has the restricted mean value property in D with respect to spheres, then u is harmonic in D.*

Proof. First, let us show that the assumptions of the theorem yield that

$$\max_{x \in \overline{D}} u = \max_{x \in \partial D} u \,. \tag{3.1}$$

Denoting the left-hand side by M, we notice that it is sufficient to establish that $u^{-1}(M) \cap \partial D$ is nonempty, where the preimage $u^{-1}(M)$ is a closed subset of $\overline{D}$.

Assuming the contrary, we conclude (in view of boundedness of D) that there is a point $x_0 \in u^{-1}(M)$, whose distance from ∂D is minimal and positive. By the restricted mean value property there exists $r(x_0) > 0$ such that $\partial B_{r(x_0)}(x_0) \subset D$ and the equality (1.1) holds with $r = r(x_0)$. Since $u(x_0) = M$, this maximal value u is attained at every point of $\partial B_{r(x_0)}(x_0)$, but some of these points is closer to ∂D than x_0, which leads to a contradiction proving (3.1).

For $u \in C^0(\overline{D})$ we denote by f its trace on ∂D and by $u_0 \in C^0(\overline{D})$ the solution of the Dirichlet problem in D with f as the boundary data. Hence u_0 has the unrestricted mean value property in D with respect to spheres, and so (3.1) holds for $u - u_0$ as well as for $-(u - u_0)$, which implies that $u \equiv u_0$ in D because $u \equiv u_0$ on ∂D. Thus, u is harmonic in D. □

In the brief note [33] by Burckel, this theorem is proved for two-dimensional domains, whereas simple one-dimensional examples demonstrating that the assumptions about boundedness of D and continuity of u_0 in $\overline{D}$ are essential for validity of the theorem are given in [34, pp. 280–281].

The question how to describe domains in which the Dirichlet problem is soluble has a long history going back to the *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* by George Green (published in 1828), where this problem was posed for the first time. In the 19th century, the well-known results about this problem were obtained by Gauss, Dirichlet, Riemann, Weierstrass, C. Neumann, Poincaré, Lyapunov and Hilbert. The final answer to the above question was given by Wiener [35] in 1924, who introduced the notion of capacity for this purpose. A detailed review of his result as well as of the preceding work accomplished during the first quarter of the 20th century one finds in Kellogg's article [36].

3.2. An example due to Littlewood. It had been found rather long ago that the restricted mean value property with respect to balls does not guarantee harmonicity of a $C^0(D)$-function in a bounded domain D without some extra bounding assumption. There are several examples demonstrating this (cf. [2, p. 369] for references). Here, we reproduce the example proposed by Littlewood and published in Huckemann's paper [37, p. 429] (in [2], this example is mentioned in passing on p. 369, in a quotation from Littlewood's booklet [38]).

Let $D = B_1(\mathbf{0}) \subset \mathbb{R}^2$ and let us define u on D by putting

$$u(x) = a_k \log|x| + b_k, \quad |x| \in [1 - 2^{-k}, 1 - 2^{-k-1}], \quad k = 0, 1, 2, \ldots, \tag{3.2}$$

where a_k and b_k are chosen so that $u \in C^0(D)$ (in particular, this means that $a_0 = 0$) and u satisfies the restricted mean value property with respect to discs for all points on each circumference $|x| = 1 - 2^{-k}$ (it is obvious that this property holds elsewhere). For the latter purpose it is sufficient to require that $a_1, a_2, \ldots$ have alternating signs and $|a_k|$ grows sufficiently fast with k. It is clear that u defined by (3.2) is not harmonic in D.

3.3. On harmonicity of harmonically dominated functions. Presumably, the brief note [39] by Veech was the first publication, in which condition (1.2) was used together with the assumption that the absolute value of the function under consideration is majorized by a positive harmonic function. The result announced in [39] (its improved version was proved in [40]) we formulate keeping the original notation.

Theorem 3.2. *Let Ω be a bounded Lipschitz domain in the plane, and let f be a Lebesgue measurable function on Ω such that $|f(x)| \leqslant g(x)$, $x \in \Omega$, for some positive harmonic function g on Ω. If for each $x \in \Omega$ there is a disc contained in Ω and centered at x over which the average of f is $f(x)$, and if $\delta(x)$, the radius of this disc, as a function of x is bounded away from 0 on compact subsets of Ω, then f is harmonic.*

It is clear straight out of the title of [40] that the proof of this theorem given by Veech relies heavily on probabalistic methods. Purely analytic proof of an analogous result was obtained by Hansen and Nadirashvili in their seminal article [3]. To outline their approach we begin with describing the required notation and definition.

In what follows, h is a fixed harmonic function in a bounded domain $D \subset \mathbb{R}^m$, where the inequality $h \geqslant 1$ holds. A function f in D is called *h-bounded* if there exists a constant $c > 0$ such that $|f| \leqslant ch$ in D.

Definition 3.1. Let a positive function r in D be such that $r(x) \leqslant \rho(x)$, where $\rho(x)$ is the distance from a point $x \in D$ to ∂D. A Lebesgue measurable function f, which is h-bounded in D, is said to be *r-median* if

$$f(y) = \frac{1}{|B_{r(y)}(y)|} \int_{B_{r(y)}(y)} f(x)\,\mathrm{d}x \quad \forall\, x \in D.$$

As in (2.3), by $|\cdot|$ the Lebesgue measure of the corresponding set is denoted.

Now we are in a position to formulate results proved in [3].

Theorem 3.3. *Let D be a bounded domain in $\mathbb{R}^m$, $m \geqslant 2$, in which a harmonic function $h \geqslant 1$ is given, and let r be a function described in Definition 3.1, then the following two assertions are true.*

(i) *If $u \in C^0(D)$ is an h-bounded and r-median function, then u is harmonic in D.*

(ii) *If u is a Lebesgue measurable, h-bounded and r-median function in D, then u is harmonic there provided that r is bounded away from 0 on every compact subset of D.*

Assertion (i) of this theorem establishes, in particular, that the restricted mean value property implies harmonicity for bounded continuous functions, whereas assertion (ii) extends this fact to Lebesgue measurable functions at the expense of an extra assumption imposed on diameters of balls in Definition 3.1. The latter imposes a geometrical restriction on the domain D.

To give an idea how complicated is the proof of Theorem 3.3 it is sufficient to list some of different conceptions used by Hansen and Nadirashvili in their considerations: (1) the Martin compactification; (2) the Schrödinger equation with singularity at the boundary (it is investigated in [3, Section 2], (3) transfinite sweeping of measures (it is studied in [3, Section 3]). In [41], the assertions of Theorem 3.3 are extended to the case of more general mean value properties and to domains which are not necessarily bounded; namely, it is required that $D \neq \mathbb{R}^m$ for $m \geqslant 3$, whereas the complement of D is a nonpolar set when $m = 2$ (cf. [12, Chapter 7, Section 1] for the definition of a polar set).

3.4. On two conjectures related to restricted mean value properties. In his booklet [38] published in 1968, Littlewood posed several questions among which was the following one (it is usually referred to as the one circle problem). Let $D = B_1(\mathbf{0}) \subset \mathbb{R}^2$ and let $u \in C^0(D)$ be bounded on D. Is u harmonic if for every $x \in D$ there exists $r(x) \in (\,0, 1 - |x|\,]$ such that the equality

$$u(x) = \frac{1}{2\pi r(x)} \int\limits_{\partial B_{r(x)}(x)} u \, \mathrm{d}S \quad \text{holds ?} \tag{3.3}$$

Littlewood conjectured that the answer to this question is "No" and this was established by Hansen and Nadirashvili [42], who proved the following assertion.

Theorem 3.4. *Let $D = B_1(\mathbf{0}) \subset \mathbb{R}^2$ and let $\alpha \in (0, 1)$. Then there exists $u \in C^0(D)$ which attains the values in $[0, 1]$ and for every $x \in D$ satisfies the equality (3.3) with some $r(x)$ belonging to $(\,0, \alpha[\,1 - |x|\,]\,)$, but is not harmonic in D.*

In fact, this result is a corollary of another theorem in which u is averaged not over a circumference as it takes place in formula (3.3), but over an annulus centred at x and enclosed in D. It is also worth mentioning that a certain random walk is applied for describing the function, whose existence is asserted in this theorem. The construction "is very delicate" as is emphasized in the subsequent paper [43], where it is substantially simplified. For this purpose a result due to Talagrand [44] is used, it concerns the Lebesgue measure of projections of a certain two-dimensional set on a straight line.

Let $\alpha \in [-\pi/2, \pi/2]$. We denote by $p_\alpha : \mathbb{R}^2 \to \mathbb{R}$ the projection operator mapping onto the x_1-axis parallel to the line going through the origin and forming the angle α with this axis. The following assertion (it is used in [43] to simplify the proof of Theorem 3.4) is a special case of Theorem 1 proved in [44].

Proposition 3.1. *Let $a, b \in \mathbb{R}$, $a < b$. Then there exists a compact set $K \subset [0, 1] \times [a, b]$ such that the orthogonal projection of K on the x_1-axis coincides with $[0, 1]$, whereas the Lebesgue measure of $p_\alpha(K) \subset \mathbb{R}$ is equal to zero for every $\alpha \in (-\pi/2, \pi/2)$.*

The second conjecture was formulated by Veech [45] in 1975. It involves the notion of admissible function on a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$, by which a positive function r is understood such that $B_{r(x)}(x) \subset D$ for every $x \in D$.

Conjecture 3.1. Let D be a bounded domain in $\mathbb{R}^m$ and let r be an admissible function on D which is locally bounded away from zero, i.e., it satisfies the inequality $\inf\limits_{x \in K} r(x) > 0$ for each compact $K \subset D$. Then every nonnegative, r-median function on D is harmonic.

Huckemann [37] demonstrated that an analogous, one-dimensional assertion is true. However, there are measurable and continuous counterexamples to Conjecture 3.1 for $m \geqslant 2$. Like that considered in Theorem 3.4, these examples are based on properties of the random walk given by a certain transition kernel. We restrict ourselves to formulating the following assertion similar to Theorem 3.4 (cf. [46]).

Theorem 3.5. *Let $D = B_1(\mathbf{0}) \subset \mathbb{R}^m$, $m \geqslant 2$, and let $\alpha \in (0, 1]$. Then there exist strictly positive functions u and r belonging to $C^0(D)$ such that $r(x) \leqslant \alpha(1 - |x|)$ for all $x \in D$ and u is r-median, but not harmonic in D.*

3.5. The restricted mean value property for circumferences and the Liouville theorem in two dimensions. In [8, Chapter 12, Section 4], the proof of the general Liouville theorem, asserting that a harmonic function defined in $\mathbb{R}^m$ and bounded from above (or below) is constant, is based on Theorem 1.1 (the mean value property for spheres). However, it occurs that the assumption about harmonicity in the Liouville theorem is superfluous at least in the two-dimensional case. The following assertion shows that it is sufficient to require the restricted mean value property for circumferences.

Theorem 3.6. *Let a real-valued function $u \in C^0(\mathbb{R}^2)$ be bounded. If there exists a strictly positive function r in $\mathbb{R}^2$ and a constant $M > 0$ such that $r(x) \leqslant |x| + M$ if $|x| \geqslant M$, then u is constant provided that*

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} u\big(x + r(x)\,\mathrm{e}^{\mathrm{i}t}\big)\,\mathrm{d}t \tag{3.4}$$

for every $x \in \mathbb{R}^2$.

The original proof of this theorem published by Hansen [47] relies on "a rather technical minimum principle involving the Choquet boundary of a compact set with respect to a function cone" as is pointed out in the subsequent paper [48]. The latter contains a new proof which involves only elementary geometry and basic facts like the inequality $\log(1 + a) \leqslant a$ for $a > 0$ and the mean value property

$$(2\pi)^{-1} \int_0^{2\pi} \log|x + \rho \mathrm{e}^{\mathrm{i}t}|\,\mathrm{d}t = \log|x|, \quad \rho \in (0, |x|),\ x \in \mathbb{R}^2.$$

Some other results concerning the Liouville theorem and the restricted mean volume property in the plane can be found in the brief note [49].

3.6. On the one ball problem in $\mathbb{R}^m$, $m \geqslant 2$. The result presented in this section is an improvement of a theorem obtained by Flatto [50]. It concerns the following question which to some extent is similar to the Littlewood one circle problem considered in Section 3.4. Let $u \in C^0(\mathbb{R}^m)$, $m \geqslant 2$, and for a certain fixed $r > 0$ the mean value equality

$$u(x) = \frac{1}{\omega_m r^m} \int_{|y|<r} u(x + y)\,\mathrm{d}y \tag{3.5}$$

holds for all $x \in \mathbb{R}^m$. What growth condition must be imposed on $u(x)$ as $|x| \to \infty$ to guarantee u be harmonic? The answer involves properties of zeros of the even, entire function

$$\eta(z) = 2^{m/2}\Gamma\left(\frac{m}{2} + 1\right) \frac{J_{m/2}(z)}{z^{m/2}} - 1\,,$$

where J_ν is the νth Bessel function. It occurs (cf. [51, Section 3]) that along with the double zero at the origin this function has a sequence $\{z_k\}_1^\infty$ of simple zeros such that $\operatorname{Re} z_k > 0$ and $|\operatorname{Im} z_k| > 0$ for all $k \geqslant 1$. Moreover, there exists $\mu = \min_{k \geqslant 1} |\operatorname{Im} z_k| > 0$, which allows us to define $h(x, r) = |x|^{(1-m)/2} \exp\{\mu|x|/r\}$, which plays the crucial role in following assertion proved by Volchkov [51] in 1994.

Theorem 3.7. *Let $u \in C^0(\mathbb{R}^m)$, $m \geqslant 2$, satisfy* (3.5) *for all $x \in \mathbb{R}^m$ and some fixed $r > 0$. Then u is harmonic provided that $u(x) = o(h(x,r))$ as $|x| \to \infty$.*

On the other hand, there exists a function $u \in C^\infty(\mathbb{R}^m)$ and $r > 0$ such that (3.5) *holds for all $x \in \mathbb{R}^m$ and $u(x) = O(h(x,r))$ as $|x| \to \infty$, but u is not harmonic.*

Another theorem obtained in [51] deals with the case when the mean value over a second ball is involved through the operator

$$(Bu)(x) = u(x) - \frac{1}{\omega_m r_1^m} \int\limits_{|y|<r_1} u(x+y)\,\mathrm{d}y\,.$$

Let A denote the set of quotients each having some elements of $\{z_k\}_1^\infty$ as the numerator and denominator.

Theorem 3.8. *Let $u \in C^0(\mathbb{R}^m)$, $m \geqslant 2$, satisfy* (3.5) *for all $x \in \mathbb{R}^m$ and some $r > 0$. Then u is harmonic provided that $r/r_1 \notin A$ and $(Bu)(x) = o(h(x,r))$ as $|x| \to \infty$.*

On the other hand, for any $r, r_1 > 0$ there exists a function $u \in C^\infty(\mathbb{R}^m)$ such that (3.5) *holds for all $x \in \mathbb{R}^m$ and $(Bu)(x) = O(h(x,r))$ as $|x| \to \infty$, but u is not harmonic.*

Similar results are true when the volume mean values are changed to the area ones. Furthermore, "local" versions (i.e., for u given in a ball instead of $\mathbb{R}^m$) of these theorems are proved in [52].

4 Mean Value Properties for Nonharmonic Functions

Since Netuka and Veselý [2] had considered exclusively harmonic functions and solutions of the heat equation (cf. also a comprehensive treatment of the heat potential theory in the monograph [53] by Watson), mean value theorems and related results for some other partial differential equations are presented in this section.

4.1. Biharmonic functions. Presumably, the first generalization of the Gauss type mean value formula (1.2) for higher order elliptic equations was obtained by Pizzetti. In 1909, he considered polyharmonic functions, i.e., C^{2k}-functions, $k = 2, 3, \dots$, which satisfy the equation $\Delta^k u = 0$ (cf. the original note [54]), whereas the three-dimensional version of his formula is given in [34, p. 288]. A description of the general form of this formula and certain its generalizations can be found in [55]. To give an idea of Pizzetti's results we restrict ourselves to the case of biharmonic mean formulas valid for functions which satisfy Equation (2.4) in a domain $D \subset \mathbb{R}^m$. The first formula

$$u(y) = \frac{1}{\omega_m R^m} \int\limits_{B_R(y)} u\,\mathrm{d}x - \frac{R^2}{2(m+2)} \nabla^2 u(y)\,, \tag{4.1}$$

involving the mean value over an arbitrary ball $B_R(y) \subset D$ and analogous to (1.2), can be found in the classical book [56] by Nicolescu. The second formula is as follows:

$$u(x) = \frac{1}{|\partial B|} \int\limits_{\partial B} u\,\mathrm{d}S - \frac{r^2}{2m} \nabla^2 u(x)\,. \tag{4.2}$$

Here $B = B_r(x)$ is an arbitrary ball in D and $|\partial B|$ is the area of its boundary. A simple derivation of (4.2) is given in [57]; it uses the Green function for the Laplace equation in a ball and the explicit form of this function is well known in the case of the Dirichlet condition on ∂B.

Let us illustrate how properties of biharmonic functions analogous to those of harmonic ones follow from, say (4.1). A simple example is the Liouville theorem.

Theorem 4.1. *Let u be a bounded biharmonic function on $\mathbb{R}^m$. Then $u \equiv$ const.*

Proof. If $\sup_{x\in\mathbb{R}^m} |u(x)| = M < +\infty$, then (4.1) implies that

$$\sup_{x\in\mathbb{R}^m} |\nabla^2 u(x)| \leqslant \frac{4(m+2)}{R^2} M \,,$$

where $R > 0$ is arbitrary. Hence $\nabla^2 u$ vanishes identically on $\mathbb{R}^m$, and so $u \equiv$ const by the Liouvill theorem for harmonic functions (cf. Section 3.5). □

As another example of similarity between properties of biharmonic and harmonic functions we consider the equality analogous to (1.4). It has the same form

$$\int_{\partial B} \frac{\partial^k u}{\partial n^k} \, \mathrm{d}S = 0$$

with $\overline{B}$ being an arbitrary closed ball in a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$, where u is biharmonic. What distinguishes the last formula from (1.4) is that $k \geqslant 3$ here, whereas $k \geqslant 1$ in (1.4). Again, the last formula follows from Theorem 1 proved in [20, p. 171].

4.2. The restricted mean value property (4.1) and the Liouville theorem. According to Theorem 3.6, it is sufficient to require the restricted mean value property (3.4) instead of harmonicity in order to guarantee that a bounded function is constant. It occurs that the same is true if one imposes condition (4.1) instead of (3.4). This follows from the assertion (cf. [58] for the proof), which differs from Theorem 3.6 in two ways: another restricted mean value property is used and the high-dimensional Euclidean space is considered instead of the plane.

Theorem 4.2. *Let a real-valued, Lebesgue measurable function u be bounded on $\mathbb{R}^m$, $m \geqslant 3$, and let its Laplacian in the distribution sense be a bounded function. If there exists a strictly positive function R on $\mathbb{R}^m$ and a constant $M > 0$ such that $R(y) \leqslant |y| + M$, then u is constant provided that the equality (4.1) with $R = R(y)$ holds for every $y \in \mathbb{R}^m$ and either $u \in C^0(\mathbb{R}^m)$ or R is locally bounded from below by a positive constant.*

There is another result based on the equality (4.1) in the note [58]. It is aimed at proving harmonicity of a locally Lebesgue integrable function which is harmonically dominated in a domain lying in $\mathbb{R}^m$, $m \geqslant 3$.

4.3. Koebe type and Harnack type results for biharmonic functions. Like in the case of harmonic functions, both (4.1) and (4.2) guarantee that a function u is biharmonic provided that these equalities hold for all admissible balls (i.e., lying within a domain D) centred at almost every point of D. Namely, the following assertion was proved in [57] (cf. [2, Theorem 3.1]).

Theorem 4.3. *Let u be a locally Lebesgue integrable function on a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$, and let its Laplacian in the distribution sense be a function. Then the following conditions are equivalent:*

(i) *the function u is biharmonic on D,*

(ii) *for almost every $y \in D$ the equality* (4.1) *holds for all $R > 0$ such that $\overline{B_R(y)} \subset D$,*

(i) *for almost every $y \in D$ the equality* (4.2) *holds for all $R > 0$ such that $\overline{B_R(y)} \subset D$.*

A natural consequence of this theorem is the following assertion analogous to Corollary 1.4.

Corollary 4.1 (The Harnack type convergence theorem)**.** *Let every function of a sequence $\{u_k\}$ be biharmonic in D. If the convergence $u_k \to u$ as $k \to \infty$ is locally uniform in D, then u is biharmonic in D.*

Furthermore, Bramble and Payne [55] obtained direct and converse mean value properties for polyharmonic functions in terms different from those used by Pizzetti. As above, we restrict ourselves to biharmonic functions only in formulations of these properties.

Theorem 4.4. *Let u be a biharmonic function in a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$. If $r_1 < r_2$ are positive numbers and $B_{r_2}(y) \subset D$ for some $y \in D$, then*

$$u(y) = \left[\frac{r_2^2}{\omega_m r_1^m} \int\limits_{B_{r_1}(y)} u \, \mathrm{d}x - \frac{r_1^2}{\omega_m r_2^m} \int\limits_{B_{r_2}(y)} u \, \mathrm{d}x \right] \Big/ \left(r_2^2 - r_1^2\right) . \qquad (4.3)$$

The assertion with volume means changed to spherical ones in the square brackets is also true, whereas the converse is as follows.

Theorem 4.5. *Let u be a locally Lebesgue integrable function on a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$. If u satisfies* (4.3) *for almost every $y \in D$ and all positive $r_1 < r_2$ with sufficiently small r_2, then u is equal almost everywhere in D to a biharmonic function.*

Analogues of Theorems 4.4 and 4.5 for polyharmonic functions involve certain determinants in the numerator and denominator of the fraction on the right-hand side of (4.3).

4.4. Metaharmonic functions in a domain. This term serves as a convenient (though, may be, an out-of-use) abbreviation for "solutions of the Helmholtz equation," like the term "harmonic functions" is a widely used equivalent to "solutions of the Laplace equation." Indeed, the Helmholtz equation

$$\nabla^2 u + \lambda^2 u = 0, \quad \lambda \in \mathbb{C}, \qquad (4.4)$$

has the next level of complexity comparing with the Laplace equation $\nabla^2 u = 0$, and so it is reasonable to use the Greek prefix *meta-* (equivalent to Latin *post-*) in order to denominate solutions of (4.4). Presumably, the term metaharmonic functions was introduced by I. N. Vekua in his still widely cited article [59] (its English translation was published as Appendix 2 in [60] and is available online at: ftp://ftp.math.ethz.ch/hg/EMIS/journals/TICMI/lnt/vol14/vol14.pdf).

Equation (4.4) was briefly considered by Euler and Lagrange in their studies of sound propagation and vibrating membranes as early as 1759, but it was Helmholtz who initiated detailed investigation of this equation now named after him. The aim of his article [61] published in 1860 was to describe sound waves in a tube with one open end (organ pipe), for which purpose he derived a representation of solutions to (4.4) analogues to the Green representation formula for harmonic functions. Thus, the way was opened to obtaining mean value properties for metaharmonic functions, and this was realised by Weber in his papers [62] and [63], in which the following formulas similar to (1.1)

$$u(x) = \frac{\lambda r}{4\pi r^2 \sin \lambda r} \int\limits_{\partial B_r(x)} u \, \mathrm{d}S, \quad u(x) = \frac{1}{2\pi r J_0(\lambda r)} \int\limits_{\partial B_r(x)} u \, \mathrm{d}S \,, \quad \lambda > 0, \tag{4.5}$$

were found in three- and two-dimensional cases respectively (cf. also [34], pp. 288 and 289 respectively); here, J_0 is the Bessel function of order zero. The counterparts of formulas (4.5) for the so-called modified Helmholtz equation (in which $\lambda = \pm \mathrm{i}\varkappa$ with $\varkappa > 0$) are given in [64], where a generalization to solutions of $(\nabla^2 - \varkappa^2)^p u = 0$, $p = 2, 3, \ldots$, is considered. The general mean value property for spheres is derived, for example, in [34, p. 289].

Theorem 4.6. *Let D be a domain in $\mathbb{R}^m$, $m \geqslant 2$. If $u \in C^2(D)$ satisfies Equation (4.4) in D, then*

$$u(x) = \frac{N(m, \lambda)}{m \, \omega_m r^{m-1}} \int\limits_{\partial B} u \, \mathrm{d}S \,, \quad N(m, \lambda) = \frac{(\lambda r/2)^{(m-2)/2}}{\Gamma(m/2) \, J_{(m-2)/2}(\lambda r)} \,, \tag{4.6}$$

for every ball $B = B_r(x)$ such that $\overline{B} \subset D$; J_ν is the νth Bessel function.

It is straightforward to calculate that

$$\frac{N(m, \lambda)}{m \, \omega_m r^{m-1}} = \frac{1}{\lambda J_{(m-2)/2}(\lambda r)} \left(\frac{\lambda}{2\pi r} \right)^{m/2} . \tag{4.7}$$

A new approach to the derivation of (4.6) was developed in the note [65], which also contains the following converse to Theorem 4.6.

Theorem 4.7. *Let D be a bounded domain in $\mathbb{R}^m$, $m \geqslant 2$. If $u \in C^0(D)$ and for every $x \in D$ there exits $r_*(x) > 0$ such that $B_{r_*(x)} \subset D$ and the equality (4.6) holds for every $B = B_r(x)$ with $r < r_*(x)$, then u is metaharmonic in D.*

The proof involves the mollifier technique used in [8] for proving the Koebe theorem.

4.5. Metaharmonic functions on $\mathbb{R}^m$, $m \geqslant 2$. In view of self-similarity, it is sufficient to consider solutions of the equation

$$\nabla^2 u + u = 0, \tag{4.8}$$

in which case an assertion analogous to Theorems 4.6 and 4.7 is as follows.

Theorem 4.8. *A function $u \in C^0(\mathbb{R}^m)$, $m \geqslant 2$, satisfying (4.8) in the distribution sense, is a solution of this equation if and only if the equality*

$$(2\pi r)^{m/2} J_{m/2}(r) \, u(x) = \int\limits_{|y|<r} u(x+y) \, \mathrm{d}y \tag{4.9}$$

holds for every $r > 0$ and every $x \in \mathbb{R}^m$. Hence the integral vanishes when r is equal to any positive zero of $J_{m/2}$.

Remark 4.1. It is worth mentioning that the expressions in (4.7) and (4.9) are used in the estimate for the spectral function of the operator of the Dirichlet problem for the Laplace operator. Dividing $J_{m/2}(\lambda r)$ by $(2\pi r/\lambda)^{m/2}$, we obtain the leading term in the estimate (0.6) proved in [66, p. 268]. Of course, the parameters λ and r in this estimate have meaning different from that in formulas (4.7) and (4.9) (note that $\lambda = 1$ in the latter).

Below, the standard notation is used for the kth positive zero of J_ν, namely, $j_{\nu,k}$, $k = 1, 2, \ldots$. In his note [67] published in 1994, Volchkov proved a converse to the last assertion of Theorem 4.8; it involves the sequence of functions

$$\Phi_k(x) = \int\limits_{|y|<j_{m/2,k}} u(x+y)\,\mathrm{d}y \tag{4.10}$$

defined with the help of zeros of $J_{m/2}$.

Theorem 4.9. *If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, $m \geqslant 2$, is such that Φ_k vanishes identically on $\mathbb{R}^m$ for all $k = 1, 2, \ldots$, then u coincides almost everywhere with a solution of Equation* (4.8)*.*

Furthermore, it occurs that if mean values of u over all balls of one particular radius $j_{m/2,k}$ vanish, then u is metaharmonic provided that all its L^2-type characteristics

$$M_{r,k}(u) = \int\limits_{|x|<r} |\Phi_k(x)|^2\,\mathrm{d}x\,, \quad k = 1, 2, \ldots,$$

grow not not too fast.

Theorem 4.10. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, $m \geqslant 2$, be such that Φ_k vanishes identically on $\mathbb{R}^m$ for some fixed k. Then u coincides almost everywhere with a solution of Equation* (4.8) *provided that $M_{r,k}(u) = o(r)$ as $r \to \infty$ for all $k = 1, 2, \ldots$.*

On the other hand, there exists a function $u \in C^\infty(\mathbb{R}^m)$ such that Φ_k vanishes identically on $\mathbb{R}^m$ for some fixed k and $M_{r,k}(u) = O(r)$ as $r \to \infty$ for all $k = 1, 2, \ldots$, but u is not metaharmonic.

The method used for proving Theorems 4.9 and 4.10 in [67] is applicable to mean values over spheres, thus allowing to obtain similar results in this case.

4.6. Metaharmonic functions on infinite domains in $\mathbb{R}^m$, $m \geqslant 2$. To prove an assertion similar to Theorem 4.9 is much more complicated task in the case when D is an unbounded domain not coinciding with $\mathbb{R}^m$. The reason is that solutions of (4.8) inevitably loose (at least partly) the translation invariance in such a domain. The first result of this kind concerns domains of the form $\mathbb{R}^m \setminus K$, where K is a convex compact set, and its proof requires essentially new methods (cf. [68]).

In the recent paper [69], a wide class of domains, say $\mathscr{O}$, was introduced and each domain $D \in \mathscr{O}$ has the following properties:

(a) it contains the half-space $H = \{x \in \mathbb{R}^m : x_m > 0\}$,

(b) for every $x \in D$ there exists a point x_* such that $x \in \overline{B_{j_{m/2,1}}(x_*)} \subset D$,

(c) the set of all admissible centres x_* is a connected subset of D.

It is clear that any half-space belonging to D takes the form H after an appropriate change of variables involving translation and rotation within D. Furthermore, for every $k = 1, 2, \ldots$ the function Φ_k given by (4.10) is defined on its own subdomain $D_k \subset D$.

Theorem 4.11. *Let a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$, belong to the class $\mathscr{O}$, and let $u \in L^1_{\mathrm{loc}}(D)$ be such that*

$$\int_\alpha^\beta \int_{\mathbb{R}^{m-1}} |u(x)| \, \mathrm{d}x_1 \cdots \mathrm{d}x_{m-1} \mathrm{d}x_m < +\infty \tag{4.11}$$

for any positive $\alpha < \beta$. Then the following two assertions are equivalent:

(1) *Φ_k vanishes identically on D_k for all $k = 1, 2, \ldots$;*

(2) *u coincides almost everywhere in D with a solution of Equation* (4.8).

From the proof given in [69, Section 3] it follows that both conditions (b) and (c) are required to describe the class $\mathscr{O}$ used in this theorem. Indeed, the assertion is not true if either of these conditions is omitted (cf. considerations at the end of Section 3 in [69]). The next theorem demonstrates in what sense (4.11) is essential for guaranteeing that Equation (4.8) holds along with vanishing of all Φ_k, $k = 1, 2, \ldots$.

Theorem 4.12. *For every $\delta > 0$ there exists a sequence $\{u_k\} \subset C^\infty(H)$ such that u_i and u_j are not equal identically for $i \neq j$, and the following properties are fulfilled simultaneously:*

(1) $\int_H |u_k(x)| \, \mathrm{e}^{\delta x_m} \, \mathrm{d}x < +\infty$ *for all $k = 1, 2, \ldots$,*

(2) *each function* $\int_{|y|<j_{m/2,k}} u_k(x+y) \, \mathrm{d}y$ *vanishes identically on the corresponding subset of H,*

(3) *for every $k = 1, 2, \ldots$ the function u_k does not satisfy Equation* (4.8) *in H.*

Thus, these theorems provide a definitive result in the case of a half-space.

4.7. The mean value property over discs in $\mathbb{R}^2$ and metaharmonic functions. According to the approach developed by Chamberland in his note [70], the two-dimensional mean value formula

$$u(x) = \frac{1}{\pi R^2} \int_{B_R(x)} u(y) \, \mathrm{d}y, \quad x \in \mathbb{R}^2, \tag{4.12}$$

where a constant $R > 0$ is fixed, is considered as an equation for the unknown $u \in C^0(\mathbb{R}^2)$. It is clear that the space of these solutions is translation-invariant, rotation-invariant and closed in the usual topology. It occurs that these properties imply the following.

Theorem 4.13. *Every $u \in C^0(\mathbb{R}^2)$ satisfying* (4.12) *belongs to the closed linear span of the solutions of Equation* (4.4), *where $\lambda R = 2J_1(\lambda R)$ and J_1 is the Bessel function of order one.*

This assertion is proved by using the spectral synthesis technique.

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References

1. I. Netuka, “Harmonic functions and mean value theorems” [in Czech], *Čas. Pěst. Mat.* **100**, 391–409 (1975).

2. I. Netuka and J. Veselý, “Mean value property and harmonic functions,” In *Classical and Modern Potential Theory and Applications*, pp. 359–398 Kluwer, Dordrecht (1994).

3. W. Hansen and N. Nadirashvili, “A converse to the mean value theorem for harmonic functions,” *Acta Math.* **171**, 139–163 (1993).

4. W. Hansen and N. Nadirashvili, “On Veech’s conjecture for harmonic functions,” *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.* **22**, No. 1, 137–153 (1995).

5. P. Freitas and J. P. Matos, “On the characterization of harmonic and subharmonic functions via mean-value properties,” *Potential Anal.* **32**, 189–200 (2010).

6. L. Zaru, “On the Mean Value Property of Harmonic Functions,” Master’s Thesis, Department of Mathematics, McGiIl University, Montreal (1992).

7. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin etc. (1983).

8. S. G. Mikhlin, *Mathematical Physics, an Advanced Course*, North-Holland, Amsterdam etc. (1970).

9. O. D. Kellogg, *Foundations of Potential Theory*, Springer, Berlin (1929).

10. N. Kuznetsov, V. Maz’ya, and B. Vainberg, *Linear Water Waves: A Mathematical Approach*, Cambridge Univ. Press, Cambridge (2002).

11. N. Kuznetsov and O. Motygin, “Freely floating structures trapping time-harmonic water waves,” *Quart. J. Mech. Appl. Math.* **68**, 173–193 (2015).

12. L. L. Helms, *Introduction to Potential Theory*, Wiley, New York (1969).

13. P. Koebe, “Herleitung der partiellen Differentialgleichungen der Potentialfunktion aus deren Integraleigenschaft” [in German], *Sitzungsber. Berl. Math. Ges.* **5**, 39–42 (1906).

14. M. Brelot, *Éléments de la Théorie Classique du Potentiel* [in French], CDU, Paris (1961).

15. E. E. Levi, “Sopra una proprietà caratteristica delle funzioni armoniche” [in Italian], *Rom. Acc. L. Rend. (5)*, **18**, No. 1, 10–15 (1909).

16. L. Tonelli, “Sopra una proprietà caratteristica delle funzione armoniche” [in Italian] *Rom. Acc. L. Rend. (5)*, **18**, No. 1, 577–582 (1909).

17. C. Miranda, *Partial Differential Equations of Elliptic Type*, Springer, Berlin etc. (1970).

18. V. A. Uspenskii, “Geometric derivation of basic properties of harmonic functions” [in Russian], *Usp. Mat. Nauk* **4**, 201–205 (1949).

19. S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory*, Springer, New York (1992).

20. V. V. Karachik, “On the mean value property for polyharmonic functions in the ball,” *Sib. Adv. Math.* **24**, No. 3, 169–182 (2014).

21. M. Bôcher, “On harmonic functions in two dimensions,” *American Acad. Proc.* **41**, 557–583 (1906).

22. W. Blaschke, “Ein Mittelwertsatz una eine kennzeichnende Eigenschaft des logarhmischen Potentials” [in German], *Leipz. Ber.* **68**, 3–7 (1916).

23. O. D. Kellogg, “Converses of Gauss’ theorem on the arithmetic mean,” *Trans. Am. Math. Soc.* **36**, 227–242 (1934).

24. D. Heath, “Functions possessing restricted mean value properties,” *Proc. Am. Math. Soc.* **41**, 588–595 (1973).

25. E. F. Beckenbach and M. Reade, “Mean values and harmonic polynomials,” *Trans. Am. Math. Soc.* **53**, 230–238 (1943).

26. G. C. Evans, “Fundamental points of potential theory,” *The Rice Institute Pamphlet* **7**, No. 4, 252–329 (1920).

27. G. E. Raynor, “On the integro-differential equation of the Bôcher type in three space,” *Bull. Am. Math. Soc.* **32**, 654–658 (1926).

28. J. J. Gergen, “Note on a theorem of Bĉher and Koebe,” *Bull. Am. Math. Soc.* **37**, 591–596 (1931).

29. S. Saks, “Note on defining properties of harmonic functions,” *Bull. Am. Math. Soc.* **38**, 380–382 (1932).

30. Min-Teh Cheng, “On a theorem of Nicolesco and generalized Laplace operators,” *Proc. Am. Math. Soc.* **2**, 77–86 (1951).

31. E. F. Beckenbach, “On characteristic properties of harmonic functions,” *Proc. Am. Math. Soc.* **3**, 765–769 (1952).

32. E. F. Beckenbach, “Concerning the definition of harmonic functions,” *Bull. Am. Math. Soc.* **51**, 240–245 (1945).

33. R. B. Burckel, “A strong converse to Gauss’s mean value theorem,” *Am. Math. Mon.* **87**, 819–820 (1980).

34. R. Courant and D. Hilbert, *Methods of Mathematical Physics, vol. 2, Partial Differential Equations*, Wiley, New York (1962).

35. N. Wiener, “The Dirichlet problem,” *J. Math, Phys.* **3**, 127–147 (1924).

36. O. D. Kellogg, “Recent progress with the Dirichlet problem,” *Bull. Am. Math. Soc.* **32**, 601–625 (1926).

37. F. Huckemann, “On the ’one circle’ problem for harmonic functions,” *J. Lond. Math. Soc. (2)* **29**, 491–497 (1954).

38. J. E. Littlewood, *Some Problems in Real and Complex Analysis*, D. C. Heath and Co., Lexington (1968).

39. W. A. Veech, “A converse to Gauss’ theorem,” *Bull. Am. Math. Soc.* **78**, 444–446 (1972).

40. W. A. Veech, “A zero-one law for a class of random walks and a converse to Gauss’ mean value theorem,” *Ann. Math. (2)* **97**, 189–216 (1973).

41. W. Hansen and N. Nadirashvili, “Mean values and harmonic functions,” *Math. Ann.* **297**, 157–170 (1993).

42. W. Hansen and N. Nadirashvili, “Littlewood’s one circle problem,” *J. Lond. Math. Soc.* **50**, 349–360 (1994).

43. W. Hansen, “Littlewood’s one-circle problem, revisited,” *Expo. Math.* **26**, No. 4, 365–374 (2008).

44. M. Talagrand, “Sur la mesure de la projection d’un compact et certaines familles de cercles,” *Bull. Sci. Math.* (2) **104**, 225–231 (1980).

45. W. A. Veech, “A converse to the mean value theorem for harmonic functions,” *Am. J. Math.* **97**, 1007–1027 (1975).

46. W. Hansen and N. Nadirashvili, “On Veech’s conjecture for harmonic functions,” *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV* **22**, No. 1, 137–153 (1995).

47. W. Hansen, “A Liouville property for spherical averages in the plane,” *Math. Ann.* **319**, 539–551 (2001).

48. W. Hansen, “A strong version of Liouville’s theorem,” *Am. Math. Mon.* **115**, 583–595 (2008).

49. W. Hansen, “Liouville’s theorem and the restricted mean volume property in the plane,” *J. Math. Pures Appl.* **77**, 943–947 (1998).

50. L. Flatto, “The converse of Gauss’s theorem for harmonic functions,” *J. Diff. Equations* **1**, 483–490 (1965).

51. V. V. Volchkov, “New two-radii theorems in the theory of harmonic functions,” *Russ. Acad. Sci., Izv., Math.* **44**, No. 1, 181–192 (1995).

52. V. V. Volchkov, “The final version of the mean value theorem for harmonic functions,” *Math. Notes* **59**, No. 3, 247–252 (1996).

53. N. A. Watson, *Introduction to Heat Potential Theory*, Am. Math. Soc., Providence, RI (2012).

54. P. Pizzetti, “Sul significato geometrico del secondo parametro differenziale di una funzione sopra una superficie qualunque” [in Italian] *Rom. Acc. L. Rend. (5)* **18**, 309–316 (1909).

55. J. Bramble and L. E. Payne, “Mean value theorems for polyharmonic functions,” *Am. Math. Mon.* **73**, 124–127 (1966).

56. M. Nicolescu, *Les Fonctions Polyharmoniques* [in French], Hermann, Paris (1936).

57. M. El Kadiri and S. Haddad, “Formules de la moyenne pour les fonctions biharmoniques” [in French], *Rend. Accad. Naz. Sci. XL, Mem. Mat. Appl. (5)* **24**, 243–253 (2000).

58. M. El Kadiri, “Liouville’s theorem and the restricted mean property for biharmonic functions,” *Electron. J. Differ. Equ.* **2004**, Paper No. 66, (2004).

59. I. N. Vekua, “Metaharmonic functions” [in Russian], *Tr. Tbilis. Mat. Inst.* **12**, 105–175 (1943); English transl.: [40, Appendix 2].

60. I. N. Vekua, *New Methods for Solving Elliptic Equations*, North Holland, Amsterdam (1967).

61. H. von Helmholtz, “Theorie der Luftschwingungen in Röhren mit offenen Enden” [in German], *J. Reine Angew. Math.* **57**, 1–72 (1860).

62. H. Weber, “Ueber einige bestimmte Integrale” [in German], *J. Reine Angew. Math.* **69**, 222–237 (1868).

63. H. Weber, “Ueber die Integration der partiellen Differentialgleichung: $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+k^2u=0$” [In German] *Math. Ann.* **1**, 1–36 (1869).

64. H. Poritsky, “Generalizations of the Gauss law of the spherical mean,” *Trans. Am. Math. Soc.* **43**, 199–225 (1938).

65. Yu. M. Grigor’ev and V. V. Naumov, “A converse mean value theorem for the Helmholtz equation” [in Russian], *Vychisl.Tekhnol.* **3**, No. 3, 15–18 (1998).

66. B. M. Levitan, “On decomposition with respect to eigenfunctions of the Laplace operator” [in Russian], *Mat. Sb.* **35**, No. 2, 267–316 (1954).

67. V. V. Volchkov, “New theorems on the mean for solutions of the Helmholtz equation,” *Russ. Acad. Sci., Sb., Math.* **79**, No. 2, 281–286 (1994).

68. V. V. Volchkov, *Integral Geometry and Convolution Equations*, Kluwer, Dordrecht (2003).

69. O. A. Ochakovskaya, “Theorems on ball mean values for solutions of the Helmholtz equation on unbounded domains,” *Izv. Math.* **76**, No. 2, 365–374 (2012).

70. M. Chamberland, “Mean value integral equations and the Helmholtz equation,” *Result. Math.* **30**, No. .1-2, 39–44 (1996).

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