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ALGORITHMS FOR WAVELET DECOMPOSITION OF OF THE SPACE OF HERMITE TYPE SPLINES

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For the space of (not necessarily polynomial) Hermite type splines we develop algorithms for constructing the spline-wavelet decomposition provided that an arbitrary coarsening of a nonuniform spline-grid is a priori given. The construction is based on approximate relations guaranteeing the asymptotically optimal (with respect to the N-diameter of standard compact sets) approximate properties of this decomposition. We study the structure of restriction and extension matrices and prove that each of these matrices is the one-sided inverse of the transposed other. We propose the decomposition and reconstruction algorithms consisting of a small number of arithmetical actions. Bibliography: 5 titles.

Dedicated to Nina Nikolaevna Uraltseva

1 Introduction

As is known, wavelet decompositions are widely used since the end of the last century. The theory of classical wavelets is described in many monographs (cf., for example, [1, 2]). However, multiflow wavelet decompositions are considered relatively recently (cf. [3]–[5]). This paper is devoted to the wavelet decomposition of the space of (not necessarily polynomial) Hermite type splines of the first height. These splines can be used for solving the Hermite interpolation problem (with first order derivatives). In this case, the original flow is combined from the flows of the values of functions and their derivatives.

For a spline-wavelet decomposition one need a pair of spline spaces embedded in each other. The situation becomes much more simpler if we can establish that the embedded spline spaces are obtained on embedded grids. As is known, the use of approximate relations does not guarantee this property (there are simple examples of embedded grids without the expected embedding of spaces). In the case under consideration, such embeddings take place.

Regarding wavelet decompositions of Hermite type splines, the first work in this direction

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was the paper [3], where the elementary case of embedded grids obtaining by eliminating one node was considered. It became clear that, successively removing nodes, one can obtain an embedded Hermite type spline space constructed on an arbitrarily thinned grid. However, the computer implementation of the process of successive removal of nodes is not very effective since it takes a lot of time, leads to significant rounding errors and, eventually, loss of stability of the process.

The goal of this paper is to develop spline-wavelet algorithms for constructing decomposition and reconstruction formulas under the condition that a spline grid coarsening is a priori given (for more details about adapting grid coarsenings we refer, for example, to [5]). Moreover, iteration processes are not used. The proposed algorithms are of explicit character: decomposition and reconstruction consist in computing sums of at most four terms. In the case under consideration, all features of spline-wavelet decompositions (in particular, the asymptotically limit quality of approximations with respect to N-diameter, simplicity of algorithms, the locality property; cf. [4]) are preserved. We consider an open interval (α, β) and an infinite grid on this interval. To pass to a finite grid, it suffices to take restrictions of all functions under consideration on the segment $[a, b]$ contained in the interval (α, β). Moreover, the obtained formulas can be used, as in the previous cases by extending the grid to the exterior of the segment $[a, b]$ with four nodes and assuming that the input flow takes arbitrary values at these nodes.

2 Hermite Type Splines

We consider a four-component vector-valued function $\varphi(t) = ([\varphi]_0(t), [\varphi]_1(t), [\varphi]_2(t), [\varphi]_3(t))^T$ with components $[\varphi]_i(t)$ in the space $C^1(\alpha, \beta)$, $i = 0, 1, 2, 3$. We assume that the following condition holds.

(A) $\quad W(x, y; \varphi) \stackrel{\text{def}}{=} \det(\varphi(x), \varphi'(x), \varphi(y), \varphi'(y)) \neq 0$ for all $x, y \in (\alpha, \beta)$, $x \neq y$.

Denote by X a grid of the form

$$X : \; \ldots < x_{-1} < x_0 < x_1 < \ldots . \tag{2.1}$$

Assume that $\alpha \stackrel{\text{def}}{=} \lim\limits_{j \to -\infty} x_j$ and $\beta \stackrel{\text{def}}{=} \lim\limits_{j \to \infty} x_j$. We introduce the notation

$$G \stackrel{\text{def}}{=} \bigcup_{j \in \mathbb{Z}} (x_j, x_{j+1}), \quad \varphi_j \stackrel{\text{def}}{=} \varphi(x_j), \quad \varphi'_j \stackrel{\text{def}}{=} \varphi'(x_j).$$

We consider functions $\omega_j(t)$, $t \in G$, $j \in \mathbb{Z}$, satisfying the approximation relations

$$\sum_j (\varphi'_{j+1} \omega_{2j-1}(t) + \varphi_{j+1} \omega_{2j}(t)) = \varphi(t), \tag{2.2}$$

under the assumption that

$$\operatorname{supp} \omega_{2j-1} \subset [x_j, x_{j+2}], \quad \operatorname{supp} \omega_{2j} \subset [x_j, x_{j+2}] \quad \forall j \in \mathbb{Z}. \tag{2.3}$$

For fixed $k \in \mathbb{Z}$ from (2.2)–(2.3) with $t \in (x_k, x_{k+1})$ we find

$$\varphi'_k \omega_{2k-3}(t) + \varphi_k \omega_{2k-2}(t) \varphi'_{k+1} \omega_{2k-1}(t) + \varphi_{k+1} \omega_{2k}(t) = \varphi(t). \tag{2.4}$$

By Assumption (A), the system (2.4) is uniquely solvable. For $t \in (x_k, x_{k+1})$ from (2.4) we find

$$\omega_{2k-3}(t) = \frac{\det(\varphi(t), \varphi_k, \varphi'_{k+1}, \varphi_{k+1})}{\det(\varphi'_k, \varphi_k, \varphi'_{k+1}, \varphi_{k+1})},$$

$$\omega_{2k-2}(t) = \frac{\det(\varphi'_k, \varphi(t), \varphi'_{k+1}, \varphi_{k+1})}{\det(\varphi'_k, \varphi_k, \varphi'_{k+1}, \varphi_{k+1})},$$

$$\omega_{2k-1}(t) = \frac{\det(\varphi'_k, \varphi_k, \varphi(t), \varphi_{k+1})}{\det(\varphi'_k, \varphi_k, \varphi'_{k+1}, \varphi_{k+1})},$$

$$\omega_{2k}(t) = \frac{\det(\varphi'_k, \varphi_k, \varphi'_{k+1}, \varphi(t))}{\det(\varphi'_k, \varphi_k, \varphi'_{k+1}. \varphi_{k+1})}.$$

From these formulas we easily derive (subsequently setting $k = q$, $k = q+1$) the equaltieis

$$\omega_{2q-1}(t) = \frac{\det(\varphi'_q, \varphi_q, \varphi(t), \varphi_{q+1})}{\det(\varphi'_q, \varphi_q, \varphi'_{q+1}, \varphi_{q+1})}, \quad t \in (x_q, x_{q+1}), \tag{2.5}$$

$$\omega_{2q-1}(t) = \frac{\det(\varphi(t), \varphi_{q+1}, \varphi'_{q+2}, \varphi_{q+2})}{\det(\varphi'_{q+1}, \varphi_{q+1}, \varphi'_{q+2}, \varphi_{q+2})}, \quad t \in (x_{q+1}, x_{q+2}), \tag{2.6}$$

$$\omega_{2q}(t) = \frac{\det(\varphi'_q, \varphi_q, \varphi'_{q+1}, \varphi(t))}{\det(\varphi'_q, \varphi_q, \varphi'_{q+1}, \varphi_{q+1})}, \quad t \in (x_q, x_{q+1}), \tag{2.7}$$

$$\omega_{2q}(t) = \frac{\det(\varphi'_{q+1}, \varphi(t), \varphi'_{q+2}, \varphi_{q+2})}{\det(\varphi'_{q+1}, \varphi_{q+1}, \varphi'_{q+2}, \varphi_{q+2})}, \quad t \in (x_{q+1}, x_{q+2}), \tag{2.8}$$

for any $q \in \mathbb{Z}$.

Theorem 2.1. *Let $\varphi \in C^1(\alpha, \beta)$, and let Assumption (A) hold. Then for any $q \in \mathbb{Z}$ the functions $\omega_{2q-1}(t)$ and $\omega_{2q}(t)$ defined by formulas (2.3) and (2.5)–(2.8) can be continuously extended to the whole interval (α, β) as functions of class $C^1(\alpha, \beta)$. Furthermore,*

$$\omega_{2q-1}(x_q) = 0, \quad \omega_{2q-1}(x_{q+1}) = 0, \quad \omega_{2q-1}(x_{q+2}) = 0, \tag{2.9}$$

$$\omega'_{2q-1}(x_q) = 0, \quad \omega'_{2q-1}(x_{q+1}) = 1, \quad \omega'_{2q-1}(x_{q+2}) = 0, \tag{2.10}$$

$$\omega_{2q}(x_q) = 0, \quad \omega_{2q}(x_{q+1}) = 1, \quad \omega_{2q}(x_{q+2}) = 0, \tag{2.11}$$

$$\omega'_{2q}(x_q) = 0, \quad \omega'_{2q}(x_{q+1}) = 0, \quad \omega'_{2q}(x_{q+2}) = 0, \tag{2.12}$$

where the same notation is used for the extended functions.

Proof. Computing the corresponding one-sided limits of the functions $\omega_{2q-1}(t)$ and $\omega_{2q}(t)$ and their derivatives at the nodes x_q, x_{q+1}, and x_{q+2} by using the representations (2.3) and (2.5)–(2.8), we conclude that all the assertions of the theorem are valid (cf. also [3]). □

Remark 2.1. If the components $[\varphi(t)]_i$ of the vector $\varphi(t)$ are given by the identities $[\varphi(t)]_i = t^i$, then the functions $\omega_{2q-1}(t)$ and $\omega_{2q}(t)$ form the known interpolation basis for the space of cubic Hermite splines.

The space $\mathbb{S}^1_\varphi(X) \stackrel{\text{def}}{=} \{u \mid u = \sum\limits_j c_j\omega_j \ \forall c_j \in \mathbb{R}^1, j \in \mathbb{Z}\}$ is called the *space of Hermite type splines* (*of the first height*). By Assumption (A), the functions ω_j, $j \in \mathbb{Z}$, are linearly independent. The set $\{\omega_j\}_{j\in\mathbb{Z}}$ is called the *principal basis* for the space $\mathbb{S}^1_\varphi(X)$.

Remark 2.2. The relations (2.9)–(2.12) can be written as

$$\omega_{2s-1}(x_j) = 0, \quad \omega'_{2s-1}(x_j) = \delta_{s+1,j}, \tag{2.13}$$

$$\omega_{2s}(x_j) = \delta_{s+1,j}, \quad \omega'_{2s}(x_j) = 0 \quad \forall s, j \in \mathbb{Z}. \tag{2.14}$$

3 Calibration Relations for Hermite Type Splines

In the set (2.1), we consider the subset $\widehat{X}: \quad \ldots < \widehat{x}_{-2} < \widehat{x}_{-1} < \widehat{x}_0 < \widehat{x}_1 < \widehat{x}_2 < \ldots$, where $\lim\limits_{j\to-\infty} \widehat{x}_j = \alpha$ and $\lim\limits_{j\to\infty} \widehat{x}_j = \beta$. We denote by $\chi(s)$ the monotonically increasing integer-valued function such that

$$\widehat{x}_j = x_{\chi(j)}. \tag{3.1}$$

This function is invertible on $\mathbb{Z}^* = \chi(\mathbb{Z})$ and generates tje mapping $\widehat{X} \mapsto X$ sending a given node $\widehat{x}_j$ (in fact, its number in $\widehat{X}$) to the node $x_{\chi(j)}$ (in fact, its number in X) by formula (3.1). Thus, this mapping determines the embedding of $\widehat{X}$ into X. We note that the inverse mapping χ^{-1} is defined on $\mathbb{Z}^*$.

Repeating the constructions (2.2)–(2.8) with the new grid $\widehat{X}$, we find functions $\widehat{\omega}_j$ such that

$$\operatorname{supp} \widehat{\omega}_{2j-1} \subset [\widehat{x}_j, \widehat{x}_{j+2}], \quad \operatorname{supp} \widehat{\omega}_{2j} \subset [\widehat{x}_j, \widehat{x}_{j+2}] \quad \forall j \in \mathbb{Z}, \tag{3.2}$$

$$\widehat{\varphi}'_i\widehat{\omega}_{2i-3}(t) + \widehat{\varphi}_i\widehat{\omega}_{2i-2}(t)\widehat{\varphi}'_{i+1}\widehat{\omega}_{2i-1}(t) + \widehat{\varphi}_{i+1}\widehat{\omega}_{2i}(t) = \varphi(t) \quad \forall t \in (\widehat{x}_i, \widehat{x}_{i+1}),\ \forall i \in \mathbb{Z}, \tag{3.3}$$

where $\widehat{\varphi}_j = \varphi_{\widehat{x}_j}$ and $\widehat{\varphi}'_j = \varphi'_{\widehat{x}_j}$ for all $j \in \mathbb{Z}$.

From formulas (3.2) and (3.3) we find (with $p \in \mathbb{Z}$)

$$\widehat{\omega}_{2p-1}(t) = \frac{\det(\widehat{\varphi}'_p, \widehat{\varphi}_p, \varphi(t), \widehat{\varphi}_{p+1})}{\det(\widehat{\varphi}'_p, \widehat{\varphi}_p, \widehat{\varphi}'_{p+1}, \widehat{\varphi}_{p+1})}, \quad t \in (\widehat{x}_p, \widehat{x}_{p+1}), \tag{3.4}$$

$$\widehat{\omega}_{2p-1}(t) = \frac{\det(\varphi(t), \widehat{\varphi}_{p+1}, \widehat{\varphi}'_{p+2}, \widehat{\varphi}_{p+2})}{\det(\widehat{\varphi}'_{p+1}, \widehat{\varphi}_{p+1}, \widehat{\varphi}'_{p+2}, \widehat{\varphi}_{p+2})}, \quad t \in (\widehat{x}_{p+1}, \widehat{x}_{p+2}), \tag{3.5}$$

$$\widehat{\omega}_{2p}(t) = \frac{\det(\widehat{\varphi}'_p, \widehat{\varphi}_p, \widehat{\varphi}'_{p+1}, \varphi(t))}{\det(\widehat{\varphi}'_p, \widehat{\varphi}_p, \widehat{\varphi}'_{p+1}, \widehat{\varphi}_{p+1})}, \quad t \in (\widehat{x}_p, \widehat{x}_{p+1}), \tag{3.6}$$

$$\widehat{\omega}_{2p}(t) = \frac{\det(\widehat{\varphi}'_{p+1}, \varphi(t), \widehat{\varphi}'_{p+2}, \widehat{\varphi}_{p+2})}{\det(\widehat{\varphi}'_{p+1}, \widehat{\varphi}_{p+1}, \widehat{\varphi}'_{p+2}, \widehat{\varphi}_{p+2})}, \quad t \in (\widehat{x}_{p+1}, \widehat{x}_{p+2}). \tag{3.7}$$

We can write analogs of formulas (2.13) and (2.14) for the functions (3.4)–(3.7):

$$\widehat{\omega}_{2s-1}(\widehat{x}_j) = 0, \quad \widehat{\omega}'_{2s-1}(\widehat{x}_j) = \delta_{s+1,j}, \tag{3.8}$$

$$\widehat{\omega}_{2s}(\widehat{x}_j) = \delta_{s+1,j}, \quad \widehat{\omega}'_{2s}(\widehat{x}_j) = 0 \quad \forall s, j \in \mathbb{Z}. \tag{3.9}$$

Assume that $q = \chi(i)$ and $q+k = \chi(i+1)$, so that between the nodes $\widehat{x}_i$ and $\widehat{x}_{i+1}$ there are the nodes x_j, $j = q+1, q+2, \ldots, q+k-1$:

$$\widehat{x}_i = x_q < x_{q+1} < x_{q+2} < \ldots < x_{q+k-1} < x_{q+k} = \widehat{x}_{i+1}. \tag{3.10}$$

As is shown in [3], if one node is removed from the original grid, then the coordinate functions $\widehat{\omega}_j$ related to the new grid are linear combinations of the coordinates funcitons connected with the original grid (these linear combinations are called the *calibration relations*). Therefore, while removing a group of nodes, the corresponding coordinate functions also have this property. To find the coefficients of the calibration relations, we use the biorthogonal system of functionals presented by formulas (3.8) and (3.9).

Thus, taking into account the location of supports of the funcitons $\widehat{\omega}_j(t)$, $j \in \{2i-3, 2i-2, 2i-1, 2i\}$ and ω_{2s-3}, ω_{2s-}, ω_{2s-1}, ω_{2s} (cf. formula (2.3)), we have the representations

$$\widehat{\omega}_j(t) = \sum_{(\widehat{x}_i,\widehat{x}_{i+1})\cap(x_s,x_{s+2})\neq\varnothing} (c^{(j)}_{2s-1}\omega_{2s-1}(t) + c^{(j)}_{2s}\omega_{2s}(t)), \tag{3.11}$$

where $t \in (\widehat{x}_i, \widehat{x}_{i+1})$ and $j \in \{2i-3, 2i-2, 2i-1, 2i\}$.

Theorem 3.1. *Let i be a fixed integer, and let $k = \chi(i+1) - \chi(i) + 1$. Under the above assumptions,*

$$\widehat{\omega}_j(t) = \sum_{s=q-1}^{q+k-1} \big(\widehat{\omega}'_j(x_{s+1})\omega_{2s-1}(t) + \widehat{\omega}_j(x_{s+1})\omega_{2s}(t)\big), \tag{3.12}$$

where $t \in (\widehat{x}_i, \widehat{x}_{i+1})$, $j \in \{2i-3, 2i-2, 2i-1, 2i\}$, and $q = \chi(i)$.

Proof. The relation (3.11) can be written as

$$\widehat{\omega}_j(t) = \sum_{s=q-1}^{q+k-1} (c^{(j)}_{2s-1}\omega_{2s-1}(t) + c^{(j)}_{2s}\omega_{2s}(t)), \quad j \in \{2i-3, 2i-2, 2i-1, 2i\}. \tag{3.13}$$

Substituting $t = x_r$, $r \in \{q, q+1, \ldots, q+k\}$ into (3.13), we find

$$\widehat{\omega}_j(x_r) = \sum_{s=q-1}^{q+k-1} (c^{(j)}_{2s-1}\omega_{2s-1}(x_r) + c^{(j)}_{2s}\omega_{2s}(x_r)). \tag{3.14}$$

Using the equalities $\omega_{2s-1}(x_r) = 0$ and $\omega_{2s}(x_r) = \delta_{s+1,r}$ on the right-hand side of (3.14), we can find at most one nonzero term; namely, the term indexed by $s = r-1$:

$$\widehat{\omega}_j(x_r) = c^{(j)}_{2r-2}\omega_{2r-2}(x_r) = c^{(j)}_{2r-2}.$$

Thus,

$$c^{(j)}_{2s} = \widehat{\omega}_j(x_{s+1}) \quad \forall s \in \{q-1, q, \ldots, q+k-1\}. \tag{3.15}$$

Differentiating (3.15) and substituting $t = x_r$ into the obtained identity, we find

$$\widehat{\omega}'_j(x_r) = \sum_{s=q-1}^{q+k-1} (c^{(j)}_{2s-1}\omega'_{2s-1}(x_r) + c^{(j)}_{2s}\omega'_{2s}(x_r)). \tag{3.16}$$

Taking into account the equalities $\omega'_{2s-1}(x_r) = \delta_{s+1,r}$ and $\omega_{2s}(x_r) = 0$, we see that the right-hand side of (3.16) contains at most one nonzero term (in this case, the "first" term); namely, the term indexed by $s = r-1$. Thus, $\widehat{\omega}'_j(x_r) = c^{(j)}_{2r-3}$. Hence

$$c^{(j)}_{2s-1} = \widehat{\omega}'_j(x_{s+1}) \quad \forall s \in \{q-1, q, \dots, q+k-1\}. \tag{3.17}$$

Substituting (3.15) and (3.17) into (3.13), we obtain (3.12). □

Theorem 3.2. *Under the assumptions of Theorem* 2.1 *the relations* (3.12) *can be represented in the form*

$$\widehat{\omega}_{2i-3}(t) = \omega_{2q-3}(t) + \sum_{s'=q+1}^{q+k-1} \big(\widehat{\omega}'_{2i-3}(x_{s'})\omega_{2s'-3}(t)\widehat{\omega}_{2i-3}(x_{s'})\omega_{2s'-2}(t)\big), \tag{3.18}$$

$$\widehat{\omega}_{2i-2}(t) = \omega_{2q-2}(t) + \sum_{s'=q+1}^{q+k-1} \big(\widehat{\omega}'_{2i-2}(x_{s'})\omega_{2s'-3}(t) + \widehat{\omega}_{2i-2}(x_{s'})\omega_{2s'-2}(t)\big), \tag{3.19}$$

$$\widehat{\omega}_{2i-1}(t) = \sum_{s'=q+1}^{q+k-1} \big(\widehat{\omega}'_{2i-1}(x_{s'})\omega_{2s'-3}(t) + \widehat{\omega}_{2i-1}(x_{s'})\omega_{2s'-1}(t)\big) + \omega_{2q+2k-3}(t), \tag{3.20}$$

$$\widehat{\omega}_{2i}(t) = \sum_{s'=q+1}^{q+k-1} \big(\widehat{\omega}'_{2i}(x_{s'})\omega_{2s'-3}(t) + \widehat{\omega}_{2i}(x_{s'})\omega_{2s'-2}(t)\big)\omega_{2q+2k-2}(t). \tag{3.21}$$

Proof. The relation (3.12) can be written as

$$\widehat{\omega}_j(t) = \widehat{\omega}'_j(x_q)\omega_{2q-3}(t) + \widehat{\omega}_j(x_q)\omega_{2q-2}(t) + \sum_{s=q}^{q+k-2} \big(\widehat{\omega}'_j(x_{s+1})\omega_{2s-1}(t) + \widehat{\omega}_j(x_{s+1})\omega_{2s}(t)\big)$$
$$+ \widehat{\omega}'_j(x_{q+k})\omega_{2q+2k-3}(t) + \widehat{\omega}_j(x_{q+k})\omega_{2q+2k-2}(t).$$

Taking into account that $x_q = \widehat{x}_i$ and $x_{q+k} = \widehat{x}_{i+1}$, we find

$$\widehat{\omega}_j(t) = \widehat{\omega}'_j(\widehat{x}_i)\omega_{2q-3}(t) + \widehat{\omega}_j(\widehat{x}_i)\omega_{2q-2}(t) + \sum_{s=q}^{q+k-2} \big(\widehat{\omega}'_j(x_{s+1})\omega_{2s-1}(t) + \widehat{\omega}_j(x_{s+1})\omega_{2s}(t)\big)$$
$$+ \widehat{\omega}'_j(\widehat{x}_{i+1})\omega_{2q+2k-3}(t) + \widehat{\omega}_j(\widehat{x}_{i+1})\omega_{2q+2k-2}(t). \tag{3.22}$$

From formula (3.22) with $j = 2i-3$ we have

$$\widehat{\omega}_{2i-3}(t) = \widehat{\omega}'_{2i-3}(\widehat{x}_i)\omega_{2q-3}(t) + \widehat{\omega}_{2i-3}(\widehat{x}_i)\omega_{2q-2}(t)$$
$$+ \sum_{s=q}^{q+k-2} \big(\widehat{\omega}'_{2i-3}(x_{s+1})\omega_{2s-1}(t) + \widehat{\omega}_{2i-3}(x_{s+1})\omega_{2s}(t)\big)$$
$$+ \widehat{\omega}'_{2i-3}(\widehat{x}_{i+1})\omega_{2q+2k-3}(t) + \widehat{\omega}_{2i-3}(\widehat{x}_{i+1})\omega_{2q+2k-2}(t). \tag{3.23}$$

From (3.8) and (3.9) it follows that

$$\widehat{\omega}'_{2i-3}(\widehat{x}_i) = 1, \widehat{\omega}_{2i-3}(\widehat{x}_i) = \widehat{\omega}'_{2i-3}(\widehat{x}_{i+1}) = \widehat{\omega}_{2i-3}(\widehat{x}_{i+1}) = 0$$

and, consequently, (3.23) can be written in the form (3.18).

For $j = 2i - 2$ formula (3.22) takes the form

$$\widehat{\omega}_{2i-2}(t) = \widehat{\omega}'_{2i-2}(\widehat{x}_i)\omega_{2q-3}(t) + \widehat{\omega}_{2i-2}(\widehat{x}_i)\omega_{2q-2}(t) + \sum_{s=q}^{q+k-2} \big(\widehat{\omega}'_{2i-2}(x_{s+1})\omega_{2s-1}(t) + \widehat{\omega}_{2i-2}(x_{s+1})\omega_{2s}(t)\big) + \widehat{\omega}'_{2i-2}(\widehat{x}_{i+1})\omega_{2q+2k-3}(t) + \widehat{\omega}_{2i-2}(\widehat{x}_{i+1})\omega_{2q+2k-2}(t). \tag{3.24}$$

Taking into account (3.8) and (3.9), we find

$$\widehat{\omega}'_{2i-2}(\widehat{x}_i) = 0, \quad \widehat{\omega}_{2i-2}(\widehat{x}_i) = 1, \quad \widehat{\omega}'_{2i-2}(\widehat{x}_{i+1}) = \widehat{\omega}_{2i-2}(\widehat{x}_{i+1}) = 0,$$

so that (3.24) implies (3.19).

We consider (3.2) with $j = 2i - 1$:

$$\widehat{\omega}_{2i-1}(t) = \widehat{\omega}'_{2i-1}(\widehat{x}_i)\omega_{2q-3}(t) + \widehat{\omega}_{2i-1}(\widehat{x}_i)\omega_{2q-2}(t) + \sum_{s=q}^{q+k-2} \big(\widehat{\omega}'_{2i-1}(x_{s+1})\omega_{2s-1}(t) + \widehat{\omega}_{2i-1}(x_{s+1})\omega_{2s}(t)\big) + \widehat{\omega}'_{2i-1}(\widehat{x}_{i+1})\omega_{2q+2k-3}(t) + \widehat{\omega}_{2i-1}(\widehat{x}_{i+1})\omega_{2q+2k-2}(t). \tag{3.25}$$

Since

$$\widehat{\omega}'_{2i-1}(\widehat{x}_i) = \widehat{\omega}_{2i-1}(\widehat{x}_i) = 0 \quad \widehat{\omega}'_{2i-1}(\widehat{x}_{i+1}) = 1, \quad \widehat{\omega}_{2i-1}(\widehat{x}_{i+1}) = 0$$

in view of (3.8) and (3.9), from (3.25) we obtain (3.20).

Finally, let us consider the case $j = 2i$. In this case, (3.22) takes the form

$$\widehat{\omega}_{2i}(t) = \widehat{\omega}'_{2i}(\widehat{x}_i)\omega_{2q-3}(t) + \widehat{\omega}_{2i}(\widehat{x}_i)\omega_{2q-2}(t) + \sum_{s=q}^{q+k-2} \big(\widehat{\omega}'_{2i}(x_{s+1})\omega_{2s-1}(t) + \widehat{\omega}_{2i}(x_{s+1})\omega_{2s}(t)\big) + \widehat{\omega}'_{2i}(\widehat{x}_{i+1})\omega_{2q+2k-3}(t) + \widehat{\omega}_{2i}(\widehat{x}_{i+1})\omega_{2q+2k-2}(t). \tag{3.26}$$

From (3.8) and (3.9) we find

$$\widehat{\omega}'_{2i}(\widehat{x}_i) = \widehat{\omega}_{2i}(\widehat{x}_i) = \widehat{\omega}'_{2i}(\widehat{x}_{i+1}) = 0, \quad \widehat{\omega}_{2i}(\widehat{x}_{i+1}) = 1,$$

and, consequently, (3.25) can be written in the form (3.21). □

Corollary 3.1. *If the assumptions of Theorem* 2.1 *hold and* $k = 2$, *then the relations* (3.12) *can be written as*

$$\widehat{\omega}_{2i-3}(t) = \omega_{2q-2}(t) + \widehat{\omega}'_{2i-3}(x_{q+1})\omega_{2q-1}(t) + \widehat{\omega}_{2i-3}(x_{q+1})\omega_{2q}(t), \tag{3.27}$$

$$\widehat{\omega}_{2i-2}(t) = \omega_{2q-2}(t) + \widehat{\omega}'_{2i-2}(x_{q+1})\omega_{2q-1}(t) + \widehat{\omega}_{2i-2}(x_{q+1})\omega_{2q}(t), \tag{3.28}$$

$$\widehat{\omega}_{2i-1}(t) = \widehat{\omega}'_{2i-1}(x_{q+1})\omega_{2q-1}(t) + \widehat{\omega}_{2i-1}(x_{q+1})\omega_{2q}(t) + \omega_{2q+1}(t), \tag{3.29}$$

$$\widehat{\omega}_{2i}(t) = \widehat{\omega}'_{2i}(x_{q+1})\omega_{2q-1}(t) + \widehat{\omega}_{2i}(x_{q+1})\omega_{2q}(t) + \omega_{2q+2}(t). \tag{3.30}$$

Proof. Setting $k = 2$ in the relations (3.18), (3.19), (3.20), and (3.21), we obtain the identities (3.27), (3.28), (3.29), and (3.30) respectively. $\square$

Remark 3.1. When implementing the algorithm, it is useful to take into account that the case $k = 1$ corresponds to the mapping χ under which, between the nodes $\widehat{x}_i$ and $\widehat{x}_{i+1}$ there are no nodes of the grid X, i.e., $\chi(i) = q$, $\chi(i+1) = q+1$, so that $\widehat{x}_i = x_q$, $\widehat{x}_{i+1} = x_{q+1}$ (cf. (3.1) and (3.10)); moreover, the formulas in Theorems 2.1 and 3.1 are also preserved in the case $k = 1$ if we agree that for $m > n$ sums of the form $\sum\limits_{i=m}^{n} a_i$ are zero.

Now, we assume that $q = \chi(i)$, $q+k = \chi(i+1)$, $q-k' = \chi(i-1)$, so that between the nodes $\widehat{x}_{i-1}$ and $\widehat{x}_i$ there are the nodes x_j, $j = q-1, q-2, \ldots, q-k'+1$, whereas between the nodes $\widehat{x}_i$ and $\widehat{x}_{i+1}$ there are the nodes x_j, $j = q+1, q+2, \ldots, q+k-1$:

$$\begin{aligned}\widehat{x}_{i-1} = x_{q-k'} < x_{q-k'+1} < \ldots < x_{q-2} < x_{q-1} < \widehat{x}_i \\ = x_q < x_{q+1} << x_{q+2} < \ldots < x_{q+k-1} < x_{q+k} = \widehat{x}_{i+1}.\end{aligned} \tag{3.31}$$

Theorem 3.3. *If Assumption* (A) *holds, then for* $t \in (\alpha, \beta)$ *and any* $i \in \mathbb{Z}$

$$\widehat{\omega}_j(t) = \sum_{s=q'}^{q+k-2} \big(\widehat{\omega}\,'_j(x_{s+1})\omega_{2s-1}(t) + \widehat{\omega}_j(x_{s+1})\omega_{2s}(t)\big), \tag{3.32}$$

where $j \in \{2i-3, 2i-2\}$, $q = \chi(i)$, $q' = \chi(i-1)$, $k = \chi(i+1) - q$.

Proof. The supports of the functions $\widehat{\omega}_j$, $j = 2i-3, 2i-2$, belong to the segment $[\widehat{x}_{i-1}, \widehat{x}_{i+1}]$. For $t \in (\widehat{x}_i, \widehat{x}_{i+1})$ formula (3.32) is valid by Theorem 2.1. We consider the interval $t \in (\widehat{x}_{i-1}, \widehat{x}_i)$. Replacing i with $i-1$, q with q', and k with $k' \stackrel{\text{def}}{=} \chi(i) - \chi(i-1)$ in Theorem 2.1, we get

$$\widehat{\omega}_j(t) = \sum_{s'=q'-1}^{q'+k'-1} \big(\widehat{\omega}\,'_j(x_{s'+1})\omega_{2s'-1}(t) + \widehat{\omega}_j(x_{s'+1})\omega_{2s'}(t)\big). \tag{3.33}$$

According to the notation (3.31), the nodes $x_{q'+k'}$ and x_k coincide with the node $\widehat{x}_i$ and $q'+k' = k$. Therefore, the term in (3.33) with index $s' = q' + k' - 1$ coincides with the term in (3.12) computed for the index $s = q-1$. There are no other common terms in these sums. Therefore, taking into account that the corresponding terms vanish at the endpoints of $[\widehat{x}_i, \widehat{x}_{i+1}]$, we conclude that formula (3.32) can be obtained by combining the sums (3.12) and (3.33). $\square$

Remark 3.2. Replacing $i' = i-1$, we get $q = \chi(i'+1)$, $q' = \chi(i')$, $k = \chi(i'+2) - q$. Setting $s' = s+1$, we can write formula (3.32) in the following equivalent form:

$$\widehat{\omega}_j(t) = \sum_{s'=q'+1}^{q+k-1} \big(\widehat{\omega}\,'_j(x_{s'})\omega_{2s'-3}(t) + \widehat{\omega}_j(x_{s'})\omega_{2s'-2}(t)\big), \tag{3.34}$$

where $j \in \{2i'-1, 2i'\}$ for all $i' \in \mathbb{Z}$.

For each $i \in \mathbb{Z}$ we consider $j \in \{2i-1, 2i\}$ and set $q = \chi(i+1)$, $q' = \chi(i)$, $k = \chi(i+2) - q$. By (3.34), we have

$$\widehat{\omega}_j(t) = \sum_{s=\chi(i)}^{\chi(i+2)} \big(\widehat{\omega}\,'_j(x_s)\omega_{2s-3}(t) + \widehat{\omega}_j(x_s)\omega_{2s-2}(t)\big).$$

Since it is obvious that $\widehat{\omega}'_j(x_{\chi(i)}) = \widehat{\omega}'_j(x_{\chi(i+2)}) = 0$ and $\widehat{\omega}_j(x_{\chi(i)}) = \widehat{\omega}_j(x_{\chi(i+2)}) = 0$, the above relation can be written as

$$\widehat{\omega}_j(t) = \sum_{s=\chi(i)+1}^{\chi(i+2)-1} \big(\widehat{\omega}'_j(x_s)\omega_{2s-3}(t) + \widehat{\omega}_j(x_s)\omega_{2s-2}(t)\big). \tag{3.35}$$

We consider $\mathfrak{p}_{j,l}$, $j,l \in \mathbb{Z}$, defined by

$$\mathfrak{p}_{2i-1,2s-1} = \widehat{\omega}'_{2i-1}(x_{s+1}), \quad \mathfrak{p}_{2i-1,2s} = \widehat{\omega}_{2i-1}(x_{s+1}), \tag{3.36}$$

$$\mathfrak{p}_{2i,2s-1} = \widehat{\omega}'_{2i}(x_{s+1}), \quad \mathfrak{p}_{2i,2s} = \widehat{\omega}_{2i}(x_{s+1}) \tag{3.37}$$

for all $s \in \{\chi(i), \dots, \chi(i+2)-2\}$ and $i \in \mathbb{Z}$, and assume that all the numbers $\mathfrak{p}_{j,l}$ that are not mentioned in this list are equal to zero:

$$\mathfrak{p}_{2i-1,k} = \mathfrak{p}_{2i,k} = 0 \quad \forall i \in \mathbb{Z},\ \forall k \notin \{2\chi(i)-1, 2\chi(i), \dots, 2\chi(i+2)-4\}. \tag{3.38}$$

We introduce the infinite matrix, $\mathfrak{P} \stackrel{\text{def}}{=} (\mathfrak{p}_{j,l})_{j,l\in\mathbb{Z}}$ whose entries are defined by (3.36)–(3.38). Thus, the $(2i-1)$th row of the matrix $\mathfrak{P}$ has the form

$$\dots, 0, 0, \widehat{\omega}'_{2i-1}(x_{\chi(i)+1}), \widehat{\omega}_{2i-1}(x_{\chi(i)+1}), \dots, \widehat{\omega}'_{2i-1}(x_{\chi(i+2)-1}), \widehat{\omega}_{2i-1}(x_{\chi(i+2)-1}), 0, 0, \dots,$$

and the following row (numbered by $2i$) differs from the above one by only the fact that instead of $\widehat{\omega}_{2i-1}$ we should write $\widehat{\omega}_{2i}$. We write the numbers of columns where these nonzero entries are located:

$$2\chi(i)-1,\ 2\chi(i),\ 2\chi(i)+1, 2\chi(i)+2,\ \dots,\ 2\chi(i+2)-5,\ 2\chi(i+2)-4, \tag{3.39}$$

and the total number of such columns is equal to $2(\chi(i+2)-\chi(i))-2$.

If i is replaced with $i+1$, then we have to consider the rows with numbers $j \in \{2i+1, 2i+2\}$. The set of their nonzero entries is translated in such a way that the start point turns out to be in the column numbered by $2\chi(i+1)-1$:

$$2\chi(i+1)-1,\ 2\chi(i+1),\ 2\chi(i+1)+1,\ \dots,\ 2\chi(i+3)-5,\ 2\chi(i+3)-4. \tag{3.40}$$

The number of the common columns in (3.39) and (3.40) is equal to the quantity

$$2\chi(i+1)-1,\ 2\chi(i+1),\ 2\chi(i+1)+1,\ \dots,\ 2\chi(i+2)-5,\ 2\chi(i+2)-4. \tag{3.41}$$

TABLE 1

rows	$2\chi(i+1)+1$	$2\chi(i+1)+2$	…	$2\chi(i+2)-3$	$2\chi(i+2)-2$
$2i-1$	$\widehat{\omega}'_{2i-1}(x_{\chi(i+1)+1})$	$\widehat{\omega}_{2i-1}(x_{\chi(i+1)+1})$	…	$\widehat{\omega}'_{2i-1}(x_{\chi(i+2)-1})$	$\widehat{\omega}_{2i-1}(x_{\chi(i+2)-1})$
$2i$	$\widehat{\omega}'_{2i}(x_{\chi(i+1)+1})$	$\widehat{\omega}_{2i}(x_{\chi(i+1)+1})$	…	$\widehat{\omega}'_{2i}(x_{\chi(i+2)-1})$	$\widehat{\omega}_{2i}(x_{\chi(i+2)-1})$
$2i+1$	$\widehat{\omega}'_{2i+1}(x_{\chi(i+1)+1})$	$\widehat{\omega}_{2i+1}(x_{\chi(i+1)+1})$	…	$\widehat{\omega}'_{2i+1}(x_{\chi(i+2)-1})$	$\widehat{\omega}_{2i+1}(x_{\chi(i+2)-1})$
$2i+2$	$\widehat{\omega}'_{2i+2}(x_{\chi(i+1)+1})$	$\widehat{\omega}_{2i+2}(x_{\chi(i+1)+1})$	…	$\widehat{\omega}'_{2i+2}(x_{\chi(i+2)-1})$	$\widehat{\omega}_{2i+2}(x_{\chi(i+2)-1})$

Remark 3.3. Since the multiplicity of covering by supports of coordinate functions $\widehat{\omega}_j$ is equal to 4, the columns of the matrix $\mathfrak{P}$ contain at most four nonzero entries (in successive four rows), whereas the matrix itself has the step-like structure.

Thus, nonzero entries of the matrix $\mathfrak{P}$ can be found only on the intersection of the columns with the above-mentioned numbers and the row with the numbers $i-1$, $2i$, $2i+1$, $2i+2$ for all $i \in \mathbb{Z}$. They are the values of the functions $\widehat{\omega}_j$ and $\widehat{\omega}'_j$ at nodes of the grid X. For given $i \in \mathbb{Z}$ Table 1 presents nonzero blocks of the matrix $\mathfrak{P}$. The upper row of the table indicates the column numbers, whereas the left column indicates the row numbers. Entries of the matrix $\mathfrak{P}$ are the values of the functions $\widehat{\omega}_j$ and $\widehat{\omega}'_j$ at the nodes of the grid X. Such nonzero blocks are separated by the columns of the matrix $\mathfrak{P}$ where only one entry is different from zero; moreover, this entry is equal to 1 (cf. Table 2).

TABLE 2

rows	$2\chi(i+1)-1$	$2\chi(i+1)$	$2\chi(i+2)-1$	$2\chi(i+2)$
$2i-1$	1	0	0	0
$2i$	0	1	0	0
$2i+1$	0	0	1	0
$2i+2$	0	0	0	1

For the infinite-dimensional column vectors $\omega(t)$ and $\widehat{\omega}(t)$ whose components are the functions $\omega_j(t)$ and $\widehat{\omega}_j(t)$ respectively we have

$$\omega(t) \stackrel{\text{def}}{=} (\ldots, \omega_{-2}(t), \omega_{-1}(t), \omega_0(t), \omega_1(t), \omega_2(t), \ldots)^T,$$

$$\widehat{\omega}(t) \stackrel{\text{def}}{=} (\ldots, \widehat{\omega}_{-2}(t), \widehat{\omega}_{-1}(t), \widehat{\omega}_0(t), \widehat{\omega}_1(t), \widehat{\omega}_2(t), \ldots)^T,$$

which implies that (3.35)–(3.38) can be written in the form[1)]

$$\omega(t) = \mathfrak{P}\widehat{\omega}(t). \tag{3.42}$$

4 Biorthogonal System of Functionals and Their Values on ω_j

On the space $C^1(\alpha, \beta)$, we consider two systems of linear functionals $\{g_i\}_{i\in\mathbb{Z}}$ and $\{\widehat{g}_i\}_{i\in\mathbb{Z}}$ defined by

$$\langle g_{2p-1}, u\rangle \stackrel{\text{def}}{=} u'(x_{p+1}), \quad \langle g_{2p}, u\rangle \stackrel{\text{def}}{=} u(x_{p+1}) \quad \forall p \in \mathbb{Z}, \tag{4.1}$$

$$\langle \widehat{g}_{2r-1}, u\rangle \stackrel{\text{def}}{=} u'(\widehat{x}_{r+1}), \quad \langle \widehat{g}_{2r}, u\rangle \stackrel{\text{def}}{=} u(\widehat{x}_{r+1}) \quad \forall r \in \mathbb{Z}. \tag{4.2}$$

[1)] Multiplication of infinite matrices is performed by the standard formulas (the indices of their entries are taken into account here). In the cases under consideration, the sums have finitely many terms and, consequently, there are no problems with convergence of series.

By (2.9)–(2.12),

$$\langle g_s, \omega_j \rangle = \delta_{s,j}, \quad \langle \widehat{g}_s, \widehat{\omega}_j \rangle = \delta_{s,j}, \quad s, j \in \mathbb{Z}. \tag{4.3}$$

We find the values of the functionals $\widehat{g}_s$ on the coordinate splines ω_j. For this purpose we express the functionals $\widehat{g}_s$ in terms of the functionals g_i. Such a representation exists in view of the structure of the functionals and the fact that the grid $\widehat{X}$ is embedded into the grid X.

Lemma 4.1. *For $u \in C^1(\alpha, \beta)$, $r \in \mathbb{Z}$,*

$$\langle \widehat{g}_{2r-1}, u \rangle = \langle g_{2\chi(r+1)-3}, u \rangle, \tag{4.4}$$

$$\langle \widehat{g}_{2r}, u \rangle = \langle g_{2\chi(r+1)-2}, u \rangle. \tag{4.5}$$

Proof. We denote by $\langle a/b \rangle$ the remainder after division of an integer a by b, and let $\lceil \rho \rceil$ denote the integer part of a real number ρ, i.e.,

$$\lceil \rho \rceil \stackrel{\text{def}}{=} \min\{k \mid k \in \mathbb{Z}, k \geqslant \rho\}.$$

Formula (4.1) can be written as

$$\langle g_s, u \rangle = u^{(\langle s/2 \rangle)}(x_{\lceil s/2 \rceil}) \quad \forall s \in \mathbb{Z}. \tag{4.6}$$

Similarly, formula (4.2) can be written as

$$\langle \widehat{g}_\sigma, u \rangle = u^{(\langle \sigma/2 \rangle)}(\widehat{x}_{\lceil \sigma/2 \rceil}) \quad \forall \sigma \in \mathbb{Z}. \tag{4.7}$$

By the mapping (3.1), from (4.7) we have

$$\langle \widehat{g}_i, u \rangle = u^{(\langle i/2 \rangle)}(x_{\chi(\lceil i/2 \rceil + 1)}) \quad \forall i \in \mathbb{Z}. \tag{4.8}$$

Let $r \in \mathbb{Z}$. If we set $i = 2r - 1$ in (4.8), then from the first relation in (4.1) we find

$$\langle \widehat{g}_{2r-1}, u \rangle = u'(x_{\chi(r+1)}) = \langle g_{2(\chi(r+1)-1)-1}, u \rangle. \tag{4.9}$$

Now, setting $i = 2q - 1$ in (4.8) and using the second relation in (4.1), we have

$$\langle \widehat{g}_{2r}, u \rangle = u(x_{\chi(r+1)}) = \langle g_{2(\chi(r+1)-1)}, u \rangle. \tag{4.10}$$

Formulas (4.9) and (4.10) imply (4.4) and (4.5) respectively. □

Theorem 4.1. *The values of the functional $\widehat{g}_{2r-1}$ and $\widehat{g}_{2r}$ on the coordinate splines ω_j with $j, r \in \mathbb{Z}$ are given by*

$$\langle \widehat{g}_{2r-1}, \omega_j \rangle = \delta_{2\chi(r+1), j+3}, \tag{4.11}$$

$$\langle \widehat{g}_{2r}, \omega_j \rangle = \delta_{2\chi(r+1), j+2}. \tag{4.12}$$

Proof. By (4.4) and (4.3), we have $\langle \widehat{g}_{2r-1}, \omega_j \rangle = \langle g_{2\chi(r+1)-3}, \omega_j \rangle = \delta_{2\chi(r+1), j+3}$. Similarly, by (4.5) and (4.3), we have $\langle \widehat{g}_{2r}, \omega_j \rangle = \langle g_{2\chi(r+1)-2}, \omega_j \rangle = \delta_{2\chi(r+1), j+2}$. Thus, we obtain formulas (4.11) and (4.12). □

From (4.11) and (4.12) the following assertion holds.

Corollary 4.1. *If $i+j$ is odd, then the right-hand sides of* (4.11) *and* (4.12) *vanish*:

$$\langle \widehat{g}_i, \omega_j \rangle = 0, \quad \langle (i+j)/2 \rangle = 1. \tag{4.13}$$

We introduce the matrix $\mathfrak{Q} \stackrel{\text{def}}{=} (\mathfrak{q}_{ij})$ with entries

$$\mathfrak{q}_{ij} = \langle \widehat{g}_i, \omega_j \rangle \tag{4.14}$$

defined by formulas (4.11)–(4.13).

Corollary 4.2. *The matrix $\mathfrak{Q}$ is the left inverse of the matrix $\mathfrak{P}^T$.*

Proof. Passing to the transposed relations (3.42), we obtain the following equality for the row vectors

$$(\omega)^T(t) = (\widehat{\omega})^T(t)\mathfrak{P}^T. \tag{4.15}$$

Multiplying the equality (4.15) by the column vector $g \stackrel{\text{def}}{=} (g_i)_{i\in\mathbb{Z}}$ and taking into account the biorthogonality property (4.1), we obtain the identity matrix I on the left-hand side and (by formula (4.14)) the matrix $\mathfrak{Q}$ on the right-hand side. Thus, from (4.15) it follows that

$$I = \mathfrak{Q}\mathfrak{P}^T, \tag{4.16}$$

which is required to prove. □

5 Wavelet Decomposition

According to Corollary 3.1, the space $\mathbb{S}^1_\varphi(\widehat{X})$ is contained in the space $\mathbb{S}^1_\varphi(X)$ which, in turn, is contained in the space $C^1(\alpha,\beta)$.

We consider the projection operator P from the space $C^1(\alpha,\beta)$ onto the subspace $\mathbb{S}^1_\varphi(\widehat{X})$ given by the formula

$$Pu \stackrel{\text{def}}{=} \sum_j a_j\,\widehat{\omega}_j \quad \forall u \in C^1(\alpha,\beta), a_j = \langle \widehat{g}_j, u\rangle. \tag{5.1}$$

Hence for the space $\mathbb{S}^1_\varphi(\widehat{X})$ we have the wavelet decomposition

$$\mathbb{S}^1_\varphi(X) = \mathbb{S}^1_\varphi(\widehat{X}) \dotplus W, \tag{5.2}$$

where $W \stackrel{\text{def}}{=} (I-P)\mathbb{S}^1_\varphi(X)$ is the wavelet space.

Let $\widetilde{u} \in \mathbb{S}^1_\varphi(X)$. By formula (5.2), we have[2)] two representations of $\widetilde{u}$:

$$\widetilde{u} = \sum_{j\in\mathbb{Z}} c_j\omega_j, \tag{5.3}$$

$$\widetilde{u} = \sum_{i\in\mathbb{Z}} a_i\widehat{\omega}_i \sum_{j\in\mathbb{Z}} b_j\omega_j = \sum_j \Big(\sum_{i\in\mathbb{Z}} a_i\mathfrak{p}_{i,j} + b_j\Big)\omega_j. \tag{5.4}$$

---

2) In the case under consideration, all the sums have finitely many terms and, as was already mentioned, no problems with convergence arise.

From (5.3)–(5.4) and the linear independence of $\{\omega_j\}_{j\in\mathbb{Z}}$ we obtain the *reconstruction formula*

$$c_j = \sum_{i\in\mathbb{Z}} a_i \mathfrak{p}_{i,j} + b_j, \quad j \in \mathbb{Z}. \tag{5.5}$$

Using (5.1) and (5.3), we find

$$a_i = \langle \widehat{g}_i, \widetilde{u}\rangle = \sum_{j\in\mathbb{Z}} c_j \langle \widehat{g}_i, \omega_j\rangle = \sum_{j\in\mathbb{Z}} \mathfrak{q}_{ij} c_j. \tag{5.6}$$

Now, from (5.5) and (5.6) we have

$$b_j = c_j - \sum_{i\in\mathbb{Z}} a_i \mathfrak{p}_{i,j} = c_j - \sum_{i\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} c_s \langle \widehat{g}_i, \omega_s\rangle \mathfrak{p}_{i,j} = c_j - \sum_{s\in\mathbb{Z}} c_s \sum_{i\in\mathbb{Z}} \mathfrak{q}_{i,s}\mathfrak{p}_{i,j}, \quad j\in\mathbb{Z}. \tag{5.7}$$

Introducing the vectors $\mathbf{a} = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)^T$, $\mathbf{b} = (\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots)^T$, and $\mathbf{c} = (\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots)^T$, we write the reconstruction formula (5.5) in the form

$$\mathbf{c} = \mathfrak{P}^T \mathbf{a} + \mathbf{b}. \tag{5.8}$$

Using the relations (5.6) and (5.7), we obtain the *decomposition formula*

$$\begin{aligned} \mathbf{a} &= \mathfrak{Q}\mathbf{c}, \\ \mathbf{b} &= \mathbf{c} - \mathfrak{P}^T\mathfrak{Q}\mathbf{c}. \end{aligned} \tag{5.9}$$

The vectors $\mathbf{c}$, $\mathbf{a}$, $\mathbf{b}$ are called the original, basic, wavelet flows respectively.

Theorem 5.1. *For the basic flow the following relation holds*:

$$a_{2r-1} = c_{2\chi(r+1)-3}, \quad a_{2r} = c_{2\chi(r+1)-2} \quad \forall r \in \mathbb{Z}. \tag{5.10}$$

Proof. From (4.14) and (5.6) we have

$$a_i = \langle \widehat{g}_i, \widetilde{u}\rangle = \sum_{j\in\mathbb{Z}} c_j \langle \widehat{g}_i, \omega_j\rangle. \tag{5.11}$$

For $i = 2r - 1$, using formula (4.11), we get

$$a_{2r-1} = \sum_{j\in\mathbb{Z}} c_j \langle \widehat{g}_{2r-1}, \widetilde{\omega}_j\rangle = \sum_{j\in\mathbb{Z}} c_j \delta_{2\chi(r+1),j+3} = c_{2\chi(r+1)-3}.$$

Setting $i = 2r$ in (5.11) and using (4.12), we find

$$a_{2r} = \sum_{j\in\mathbb{Z}} c_j \delta_{2\chi(r+1),j+2} = c_{2\chi(r+1)-2}.$$

Thus, formula (5.10) is proved. □

Thus, we have obtained the first formula for computing the decomposition. To obtain the second formula, we need to compute entries of the product matrix $\mathfrak{P}^T\mathfrak{Q}$. The entries will be denoted by $[\mathfrak{P}^T\mathfrak{Q}]_{j,s}$, where j is the row number and s is the column number.

Theorem 5.2. *For $j, p \in \mathbb{Z}$ the following relations hold:*

$$\begin{cases} [\mathfrak{P}^T\mathfrak{Q}]_{j,2p-1} = \mathfrak{p}_{2\chi^{-1}(p+1)-3,j} \quad \forall p+1 \in \mathbb{Z}^*, \\ [\mathfrak{P}^T\mathfrak{Q}]_{j,2p-1} = 0 \quad \forall p+1 \in \mathbb{Z}\backslash\mathbb{Z}^*, \end{cases} \tag{5.12}$$

$$\begin{cases} [\mathfrak{P}^T\mathfrak{Q}]_{j,2p} = \mathfrak{p}_{2\chi^{-1}(p+1)-2,j} \quad \forall p+1 \in \mathbb{Z}^*, \\ [\mathfrak{P}^T\mathfrak{Q}]_{j,2p-1} = 0 \quad \forall p+1 \in \mathbb{Z}\backslash\mathbb{Z}^*. \end{cases} \tag{5.13}$$

Proof. Using formula (4.14), we have

$$\begin{aligned} [\mathfrak{P}^T\mathfrak{Q}]_{js} &= \sum_{i\in\mathbb{Z}} \mathfrak{p}_{i,j}\mathfrak{q}_{i,s} = \sum_{r\in\mathbb{Z}} \mathfrak{p}_{2r-1,j}\langle \widehat{g}_{2r-1}, \omega_s\rangle + \sum_{r\in\mathbb{Z}} \mathfrak{p}_{2r,j}\langle \widehat{g}_{2r}, \omega_s\rangle \\ &= \sum_{r\in\mathbb{Z}} \mathfrak{p}_{2r-1,j}\delta_{2\chi(r+1),s+3} + \sum_{r\in\mathbb{Z}} \mathfrak{p}_{2r,j}\delta_{2\chi(r+1),s+2}. \end{aligned} \tag{5.14}$$

We consider the cases of odd and even s. If $s = 2p-1$, then from (5.14) and (4.11)–(4.12) it follows that

$$[\mathfrak{P}^T\mathfrak{Q}]_{j,2p-1} = \sum_{r\in\mathbb{Z}} \mathfrak{p}_{2r-1,j}\delta_{2\chi(r+1),2p+2} + \sum_{r\in\mathbb{Z}} \mathfrak{p}_{2r,j}\delta_{2\chi(r+1),2p+1}. \tag{5.15}$$

The second sum in (5.15) is equal to zero since the indices of the Kronecker symbol cannot coincide there (cf. also formula (4.13)). Therefore, looking for $r = \chi^{-1}(p+1)-1$ from $\chi(r+1) = p+1$ with $p+1 \in \mathbb{Z}^*$ and using (5.15), we find

$$[\mathfrak{P}^T\mathfrak{Q}]_{j,2p-1} = \sum_{r\in\mathbb{Z}} \mathfrak{p}_{2r-1,j}\delta_{2\chi(r+1),2p+2} = \mathfrak{p}_{2(\chi^{-1}(p+1)-1)-1,j} \tag{5.16}$$

for $p+1 \in \mathbb{Z}^*$ and $[\mathfrak{P}^T\mathfrak{Q}]_{j,2p-1} = 0$ for $p+1 \in \mathbb{Z}\backslash\mathbb{Z}^*$. Considering the case $s = 2p$ and using (4.11)–(4.12) in the relation (5.14), we have

$$[\mathfrak{P}^T\mathfrak{Q}]_{j,2p} = \sum_{r\in\mathbb{Z}} \mathfrak{p}_{2r-1,j}\delta_{2\chi(r+1),2p+3} + \sum_{r\in\mathbb{Z}} \mathfrak{p}_{2r,j}\delta_{2\chi(r+1),2p+2}. \tag{5.17}$$

It is easy to see that the first sum in (5.17) vanishes. Thus, for $p+1 \in \mathbb{Z}^*$

$$[\mathfrak{P}^T\mathfrak{Q}]_{j,2p} = \sum_{r\in\mathbb{Z}} \mathfrak{p}_{2r,j}\delta_{2\chi(r+1),2p+2} = \mathfrak{p}_{2(\chi^{-1}(p+1)-1),j}, \tag{5.18}$$

and $[\mathfrak{P}^T\mathfrak{Q}]_{j,2p} = 0$ if $p+1 \in \mathbb{Z}\backslash\mathbb{Z}^*$. Formulas (5.16) and (5.18) are equivalent to formulas (5.12) and (5.13). ☐

Theorem 5.3. *For the wavelet flow*

$$b_j = c_j - \sum_{p+1\in\mathbb{Z}^*} \left(\mathfrak{p}_{2\chi^{-1}(p+1)-3,j}c_{2p-1} + \mathfrak{p}_{2\chi^{-1}(p+1)-2,j}c_{2p}\right) \quad \forall j \in \mathbb{Z}. \tag{5.19}$$

Proof. In the representation $[\mathfrak{P}^T\mathfrak{Q}\mathbf{c}]_j$, we extract summation with respect to even and odd indices

$$[\mathfrak{P}^T\mathfrak{Q}\mathbf{c}]_j = \sum_{s\in\mathbb{Z}}[\mathfrak{P}^T\mathfrak{Q}]_{j,s}c_s = \sum_{p\in\mathbb{Z}}[\mathfrak{P}^T\mathfrak{Q}]_{j,2p-1}c_{2p-1} + \sum_{p\in\mathbb{Z}}[\mathfrak{P}^T\mathfrak{Q}]_{j,2p}c_{2p}.$$

Using formulas (5.12)–(5.13), we find

$$[\mathfrak{P}^T\mathfrak{Q}\mathbf{c}]_j = \sum_{p+1\in\mathbb{Z}^*}\big(\mathfrak{p}_{2\chi^{-1}(p+1)-3,j}c_{2p-1} + \mathfrak{p}_{2\chi^{-1}(p+1)-2,j}c_{2p}\big) \quad \forall j\in\mathbb{Z}. \tag{5.20}$$

Using the second equality in (5.9) and applying (5.20), we obtain (5.19). □

Corollary 5.1. *The following relations hold*:

$$b_{2s-1} = c_{2s-1} - \sum_{p+1\in\mathbb{Z}^*}\big(\widehat{\omega}\,'_{2\chi^{-1}(p+1)-3}(x_{s+1})c_{2p-1} + \widehat{\omega}\,'_{2\chi^{-1}(p+1)-2}(x_{s+1})c_{2p}\big), \tag{5.21}$$

$$b_{2s} = c_{2s} - \sum_{p+1\in\mathbb{Z}^*}\big(\widehat{\omega}_{2\chi^{-1}(p+1)-3}(x_{s+1})c_{2p-1} + \widehat{\omega}_{2\chi^{-1}(p+1)-2}(x_{s+1})c_{2p}\big). \tag{5.22}$$

Proof. For $p+1\in\mathbb{Z}^*$, substituting $i=\chi^{-1}(p+1)-1$ into (3.36)–(3.37), we find the numbers $\mathfrak{p}_{p,q}$ for $p\in\{2i-1,2i\}$, $q\in\{2s-1,2s\}$ and substitute them into formula (5.19) with $j\in\{2s-1,2s\}$. Then we obtain the relations (5.21)–(5.22). □

Theorem 5.4. *The reconstruction formulas have the form*

$$c_{2s-1} = \sum_{i\in\mathbb{Z}} a_{2i-1}\omega\,'_{2i-1}(x_{s+1}) + \sum_{i\in\mathbb{Z}} a_{2i}\omega\,'_{2i}(x_{s+1}) + b_{2s-1}, \tag{5.23}$$

$$c_{2s} = \sum_{i\in\mathbb{Z}} a_{2i-1}\omega_{2i-1}(x_{s+1}) + \sum_{i\in\mathbb{Z}} a_{2i}\omega_{2i}(x_{s+1}) + b_{2s}. \tag{5.24}$$

Proof. Using the relation (5.5), we find

$$c_{2s-1} = \sum_{l\in\mathbb{Z}} a_l\mathfrak{p}_{l,2s-1} + b_{2s-1} = \sum_{i\in\mathbb{Z}} a_{2i-1}\mathfrak{p}_{2i-1,2s-1} + \sum_{i\in\mathbb{Z}} a_{2i}\mathfrak{p}_{2i,2s-1} + b_{2s-1},$$

$$c_{2s} = \sum_{l\in\mathbb{Z}} a_l\mathfrak{p}_{l,2s} + b_{2s} = \sum_{i\in\mathbb{Z}} a_{2i-1}\mathfrak{p}_{2i-1,2s} + \sum_{i\in\mathbb{Z}} a_{2i}\mathfrak{p}_{2i,2s} + b_{2s}.$$

Applying formulas (3.36) and (3.37), we obtain the representations (5.23)–(5.24). □

The case where the grid $\widehat{X}$ is obtained from the gird X by removing one node was considered in [3]. The results obtained in this paper, generalize the corresponding assertions in [3] (cf., for example, Corollary 3.1). Let us illustrate this generalization by discussing formula (5.10) for determining the basic flow $\{a_i\}_{i\in\mathbb{Z}}$ from the initial flow $\{c_j\}_{j\in\mathbb{Z}}$. Without loss of generality we can assume that the grid $\widehat{X}$ is obtained from the grid X by removing the node $\xi \stackrel{\text{def}}{=} x_{\widehat{k}+1}$, where $\widehat{k}$ is an integer. In the notation (3.1), we can write

$$\chi(j) = \begin{cases} j, & j\leqslant \widehat{k}, \\ j+1, & j\geqslant \widehat{k}+1. \end{cases}$$

The following relations were proved in [3]:

$$a_i = \begin{cases} c_i, & i\leqslant 2\widehat{k}-2, \\ a_i = c_{i+2}, & i\geqslant 2\widehat{k}-1. \end{cases} \tag{5.25}$$

We show that (5.25) follows from the relations (5.10) in Theorem 5.1. Indeed, if $r+1 \leqslant \widehat{k}$, then $\chi(r+1) = r+1$. Hence from (5.10) it follows that $a_{2r-1} = c_{2(r+1)-3} = c_{2r-1}$ and $a_{2r} = c_{2r}$, which implies

$$a_j = c_j, \quad j \leqslant 2\widehat{k} - 2. \tag{5.26}$$

If $r+1 \geqslant \widehat{k}+1$, then $\chi(r+1) = r+2$. Consequently, $a_{2r-1} = c_{2(r+2)-3} = c_{2r+1}$ and $a_{2r} = c_{2r+2}$, which implies

$$a_j = c_j, \quad j \geqslant 2\widehat{k} - 1. \tag{5.27}$$

Thus, formulas (5.26) and (5.26) coincide with (5.25).

Similarly, in the case $k = 2$, from the results of this paper we obtain the decomposition and reconstruction formulas established earlier in [3].

References

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