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ON ONE INTEGRABLE DISCRETE SYSTEM

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***Abstract*.** In this paper, we study a system of nonlinear equations on a square graph related to the affine algebra $A_1^{(1)}$. This system is the simplest representative of the class of discrete systems corresponding to affine Lie algebras. We find the Lax representation and construct hierarchies of higher symmetries. In neighborhoods of singular points $\lambda = 0$ and $\lambda = \infty$, we construct formal asymptotic expansions of eigenfunctions of the Lax pair and, based on these expansions, find series of local conservation laws for the system considered.

***Keywords and phrases*:** Lax pair, higher symmetry, conservation law, recursion operator, formal diagonalization.

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CONTENTS

1. Introduction

System of differential equations of exponential type

$$\frac{\partial^2}{\partial x \partial y} u_i = \exp\left\{ \sum_j a_{ij} u_j \right\}, \quad i, j = 1, 2, \dots, N, \tag{1}$$

that generalize the Liouville and sine-Gordon equations have important applications in the conformal field theory (see [17]). In the last years, various discrete versions of the system (1) are widely discussed in the literature in connection with the quantum field theory (see the review [8] and the references therein). Under the condition that the matrix $A = (a_{ij})$ of coefficients of the system (1) coincides with the generalized Cartan matrix of an affine Lie algebra, the system is integrable. In this case, the Lax representation for the system (1) is expressed in the closed form in terms of generators of the corresponding Lie algebra (see [1]). The problem of description of integrable discrete versions of the system (1) remains poorly studied. In this paper, we discuss a discrete system related to the affine

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Lie algebra $A_1^{(1)}$:

$$\begin{cases} z_{n+1,m+1} = \dfrac{w_{n+1,m}^2}{z_{n,m}} - \dfrac{w_{n+1,m}^2}{z_{n,m+1}} + z_{n+1,m}, \\ w_{n+1,m+1} = \dfrac{z_{n,m+1}^2}{w_{n,m}} - \dfrac{z_{n,m+1}^2}{w_{n,m+1}} + w_{n+1,m}, \end{cases} \tag{2}$$

(see [2]), where $z_{n,m} = z(n,m)$ and $w_{n,m} = w(n,m)$ are unknown functions. We construct a Lax pair for the system (2) and describe hierarchies of its higher symmetries and local conservation law. For the construction of Lax pairs for the system (1), its symmetries, and recursion operators, we use the algorithm proposed in [5].

In [2], a higher symmetry of the system (2) with respect to the direction n was found:

$$\begin{cases} z_{n,m,t} = \dfrac{z_{n,m}^2}{w_{n,m}} + w_{n+1,m}, \\ w_{n,m,t} = \dfrac{w_{n,m}^2}{z_{n-1,m}} + z_{n,m}. \end{cases} \tag{3}$$

One can verify that the chain (3) is a compatibility condition for the following system of linear equations:

$$\varphi_{n,m,t} = p\varphi_{n,m}, \tag{4}$$

$$\varphi_{n+1,m} = f\varphi_{n,m}, \tag{5}$$

where

$$p = \begin{pmatrix} \dfrac{w_{n+1,m}}{z_{n,m}} - \dfrac{1}{2}\lambda & \dfrac{w_{n+1,m}}{z_{n,m}} \\ \lambda & \dfrac{z_{n,m}}{w_{n,m}} + \frac{1}{2}\lambda \end{pmatrix}, \quad f = \begin{pmatrix} \dfrac{z_{n+1,m}}{w_{n+1,m}} + \lambda & -\dfrac{w_{n+1,m}}{z_{n,m}} \\ -\lambda & \dfrac{w_{n+1,m}}{z_{n,m}}. \end{pmatrix} \tag{6}$$

In the present paper, for the discrete system (2) we obtain the Lax pair defined by the equations

$$\begin{aligned} \varphi_{n+1,m} &= f\varphi_{n,m}, \\ \varphi_{n,m+1} &= g\varphi_{n,m}, \end{aligned} \tag{7}$$

where f is the same as in (6) and the matrix g has the form

$$g = \begin{pmatrix} 1 + \dfrac{z_{n,m+1}(z_{n,m+1} - z_{n,m})(w_{n,m+1} - w_{n,m})}{z_{n,m}w_{n,m}w_{n,m+1}\lambda} & \dfrac{z_{n,m+1}^2(w_{n,m+1} - w_{n,m})}{z_{n,m}w_{n,m}w_{n,m+1}\lambda} \\ \dfrac{z_{n,m+1} - z_{n,m}}{z_{n,m}} & \dfrac{z_{n,m+1}}{z_{n,m}} \end{pmatrix}. \tag{8}$$

In Sec. 2, we find a higher symmetry of the discrete system (1) with respect to the direction m:

$$\begin{cases} z_{n,m,\tau} = \dfrac{w_{n,m}w_{n,m-1}}{w_{n,m} - w_{n,m-1}}, \\ w_{n,m,\tau} = \dfrac{w_{n,m}^2}{z_{n,m+1} - z_{n,m}}, \end{cases} \tag{9}$$

for which the Lax pair is presented:

$$\begin{aligned} \varphi_{n,m+1} &= g\varphi_{n,m}, \\ \varphi_{n,m,\tau} &= q\varphi_{n,m}, \end{aligned} \tag{10}$$

where g has the form (8) and q is determined by the equality

$$q = \begin{pmatrix} 0 & \dfrac{z_{n,m+1}}{(z_{n,m+1} - z_{n,m})\lambda} \\ \dfrac{w_{n,m} w_{n,m-1}}{z_{n,m}(w_{n,m} - w_{n,m-1})} & -\dfrac{1}{\lambda} + \frac{w_{n,m} w_{n,m-1}}{z_{n,m}(w_{n,m} - w_{n,m-1})} \end{pmatrix}. \tag{11}$$

Formal asymptotic representations of eigenfunctions of the linear equations (5) and (7) in neighborhoods of the singular points $\lambda = \infty$ and $\lambda = 0$ are constructed in Secs. 3 and 4. Based on these representations, we describe infinity series of conservation laws for the system (2).

2. Recursion Operators and Higher Symmetries

To construct a higher symmetry of the system (1), we apply the dressing method developed by V. E. Zakharov and A. B. Shabat (see [15, 16]). We search for a second-order system of differential equations

$$\varphi_{n,m,\tau} = q \varphi_{n,m} \tag{12}$$

compatible with Eq. (7), assuming that q rationally depends on the spectral parameter λ:

$$q = A \frac{1}{\lambda} + B. \tag{13}$$

The compatibility condition of Eqs. (7) and (12) is equivalent to the equation of the form

$$g_\tau + g\left(A \frac{1}{\lambda} + B\right) - D_m \left(A \frac{1}{\lambda} + B\right) g = 0, \tag{14}$$

where g is defined by Eq. (8) and D_m is the shift operator acting by the rule $D_m u_{n,m} = u_{n,m+1}$. Equating coefficients of powers of λ in (14), we arrive at the system of equations for the matrices A and B and also deduce the solvability conditions for these equations in the form of differential equations for the variables $z_{n,m}$ and $w_{n,m}$. Omitting simple but cumbersome calculation, we present the final result:

$$\begin{gathered} A = \begin{pmatrix} c_1 & c_6 \dfrac{z_{n,m+1}}{z_{n,m+1} - z_{n,m}} \\ 0 & c_1 - c_6 \end{pmatrix}, \\ B = \begin{pmatrix} c_3 + \dfrac{c_2}{z_{n,m}} & \dfrac{c_2}{z_{n,m}} \\ c_6 \dfrac{w_{n,m} w_{n,m-1}}{z_{n,m}(w_{n,m} - w_{n,m-1})} & c_3 - c_5 z_{n,m} + c_6 \dfrac{w_{n,m} w_{n,m-1}}{z_{n,m}(w_{n,m} - w_{n,m-1})} \end{pmatrix}, \end{gathered} \tag{15}$$

$$\begin{cases} \dfrac{d}{d\tau} z_{n,m} = c_2 - c_5 z_{n,m}^2 - c_4 z_{n,m} + c_6 \dfrac{w_{n,m} w_{n,m-1}}{w_{n,m} - w_{n,m-1}}, \\ \dfrac{d}{d\tau} w_{n,m+1} = \dfrac{w_{n,m+1}^2}{w_{n,m}^2} \dfrac{d}{d\tau} w_{n,m} + c_4 \dfrac{w_{n,m+1}(w_{n,m+1} - w_{n,m})}{w_{n,m}} \\ \qquad + 2c_5 \dfrac{z_{n,m+1} w_{n,m+1}(w_{n,m+1} - w_{n,m})}{w_{n,m}} - c_6 \dfrac{w_{n,m+1}^2 (z_{n,m} - 2z_{n,m+1} + z_{n,m+2})}{(z_{n,m+2} - z_{n,m+1})(z_{n,m+1} - z_{n,m})}, \end{cases} \tag{16}$$

where c_i are arbitrary constants, $i = \overline{1,6}$. From the compatibility condition for the systems (2) and (16) we obtain $c_2 = 0$ and $c_5 = 0$. Without loss of generality, we can set $c_3 = c_4 = 0$ and $c_6 = 1$

in the formulas (15) and (16) . As a result, we obtain

$$\begin{cases} \dfrac{d}{d\tau} z_{n,m} = \dfrac{w_{n,m} w_{n,m-1}}{w_{n,m} - w_{n,m-1}}, \\ \dfrac{d}{d\tau} w_{n,m} = \dfrac{w_{n,m}^2}{(z_{n,m+1} - z_{n,m})}, \end{cases} \qquad q = \begin{pmatrix} 0 & \dfrac{z_{n,m+1}}{(z_{n,m+1} - z_{n,m})\lambda} \\ \dfrac{w_{n,m} w_{n,m-1}}{z_{n,m}(w_{n,m} - w_{n,m-1})} & -\dfrac{1}{\lambda} + \dfrac{w_{n,m} w_{n,m-1}}{z_{n,m}(w_{n,m} - w_{n,m-1})} \end{pmatrix}.$$

One can prove that the recursion operators for the systems (3) and (9) have the form

$$R_n = \begin{pmatrix} 2\dfrac{z_{n,m}}{w_{n,m}} & D_n - \dfrac{z_{n,m}^2}{w_{n,m}^2} + 2\dfrac{z_{n,m}}{z_{n-1,m}} \\ \dfrac{1}{w_{n,m}^2} + \dfrac{1}{z_{n-1,m}^2} D_n^{-1} & \dfrac{2}{w_{n,m} z_{n-1,m}} \end{pmatrix}$$
$$- 2 \begin{pmatrix} z_{n,m} \\ \dfrac{1}{w_{n,m}} \end{pmatrix} (D_n - 1)^{-1} \left(\frac{w_{n+1,m}}{z_{n,m}^2} - \frac{1}{w_{n,m}}, \quad \frac{z_{n,m}}{w_{n,m}^2} - \frac{1}{z_{n-1,m}} \right), \quad (17)$$

respectively, where D_n is the shift operator acting by the rule $D_n u_{n,m} = u_{n+1,m}$ and

$$R_m = \begin{pmatrix} 2r_{n,m-1}s_{n,m} & s_{n,m}^2(D_m^{-1} + 1) \\ r_{n,m}^2(D_m + 1) & 2r_{n,m}s_{n,m} \end{pmatrix}$$
$$+ 2 \begin{pmatrix} s_{n,m}(D_m - 1)^{-1}(r_{n,m-1} - r_{n,m}) & s_{n,m}(D_m - 1)^{-1}(s_{n,m} - s_{n,m+1}) \\ r_{n,m}(D_m - 1)^{-1}D_m(r_{n,m-1} - r_{n,m}) & r_{n,m}(D_m - 1)^{-1}(s_{n,m} - s_{n,m+1}) \end{pmatrix}, \quad (18)$$

where

$$r_{n,m} = \frac{1}{z_{n,m+1} - z_{n,m}}, \quad s_{n,m} = \frac{1}{v_{n,m} - v_{n,m-1}}, \quad v_{n,m} = \frac{1}{w_{n,m}}.$$

Acting by the recursion operators (17) and (18) on the right-hand sides of Eqs. (3) and (9), respectively, we obtain the following higher symmetries of the system (2):

$$\begin{cases} z_{n,m,t_2} = \dfrac{z_{n,m}^3}{w_{n,m}^2} + \dfrac{z_{n,m}^2}{z_{n-1,m}} + z_{n+1,m} + \dfrac{w_{n+1,m}^2}{z_{n,m}} + 2\dfrac{z_{n,m}w_{n+1,m}}{w_{n,m}}, \\ w_{n,m,t_2} = \dfrac{z_{n,m}^2}{w_{n,m}^3} + \dfrac{w_{n+1,m}}{w_{n,m}^2} + \dfrac{1}{w_{n-1,m}} + \dfrac{w_{n,m}}{z_{n-1,m}^2} + 2\dfrac{z_{n,m}}{w_{n,m}z_{n-1,m}}, \end{cases} \quad (19)$$

$$\begin{cases} z_{n,m,\tau_2} = -\dfrac{1}{(v_{n,m} - v_{n,m-1})^2} \left(\dfrac{1}{z_{n,m+1} - z_{n,m}} + \dfrac{1}{z_{n,m} - z_{n,m-1}} \right), \\ w_{n,m,\tau_2} = -\dfrac{1}{(z_{n,m+1} - z_{n,m})^2} \left(\dfrac{1}{v_{n,m+1} - v_{n,m}} + \dfrac{1}{v_{n,m} - v_{n,m-1}} \right). \end{cases} \quad (20)$$

3. Formal Asymptotic Expansion of Eigenfunctions of the Lax Pair in a Neighborhood of the Singular Point $\lambda = \infty$

Using the linear substitution $\varphi_{n,m} = \tilde{T}\tilde{\varphi}_{n,m}$, where

$$\tilde{T} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

we transform Eqs. (4), (5), and (7), respectively, to the following equations:

$$\tilde{\varphi}_{n,m,t} = P\tilde{\varphi}_{n,m}, \tag{21}$$

$$\tilde{\varphi}_{n+1,m} = F\tilde{\varphi}_{n,m}, \tag{22}$$

$$\tilde{\varphi}_{n,m+1} = G\tilde{\varphi}_{n,m}. \tag{23}$$

This substitution transforms the principal (as $\lambda \to \infty$) part of the potential $P = P_0\lambda + P_1$ of Eq. (21) to the diagonal form $P_0 = \operatorname{diag}\left(\frac{1}{2}, -\frac{1}{2}\right)$. Moreover, we have

$$P_1 = \begin{pmatrix} \dfrac{w_{n+1,m}}{z_{n,m}} + \frac{z_{n,m}}{w_{n,m}} & \dfrac{z_{n,m}}{w_{n,m}} \\ -\dfrac{w_{n+1,m}}{z_{n,m}} & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & -\dfrac{z_{n+1,m}}{w_{n+1,m}} \\ \dfrac{w_{n+1,m}}{z_{n,m}} & \lambda + \dfrac{w_{n+1,m}}{z_{n,m}} + \dfrac{z_{n+1,m}}{w_{n+1,m}} \end{pmatrix}, \tag{24}$$

$$G = \begin{pmatrix} \dfrac{z_{n,m+1}}{z_{n,m}} + \dfrac{z_{n,m+1}^2(w_{n,m+1} - w_{n,m})}{z_{n,m}w_{n,m}w_{n,m+1}\lambda} & \dfrac{z_{n,m+1}(w_{n,m+1} - w_{n,m})}{w_{n,m}w_{n,m+1}} \\ -\dfrac{z_{n,m+1}^2(w_{n,m+1} - w_{n,m})}{z_{n,m}w_{n,m}w_{n,m+1}\lambda} & 1 - \dfrac{z_{n,m+1}(w_{n,m+1} - w_{n,m})}{w_{n,m}w_{n,m+1}\lambda} \end{pmatrix}. \tag{25}$$

Further, we apply the method of asymptotic diagonalization of a linear system (see [1, 11]). By a formal change of variables of the form

$$\tilde{\Phi}_{n,m} = T\tilde{\varphi}_{n,m}, \tag{26}$$

where T is a formal power series

$$T = E + \sum_{i=1}^{\infty} T_i\lambda^{-i} \tag{27}$$

and E is the identity (2×2)-matrix, we transform Eq. (21) to the diagonal form:

$$\tilde{\Phi}_{n,m,t} = \left(P_0\lambda - \sum_{k=0}^{\infty} h_k\lambda^{-k}\right)\tilde{\Phi}_{n,m}, \tag{28}$$

where h_k is a diagonal matrix for all $k \geq 0$ and all matrices T_i, $i \geq 1$, have zero diagonal. From (21), (26), and (28) we have

$$T_t - \left(P_0\lambda - \sum_{k=0}^{\infty} h_k\lambda^{-k}\right)T = -T(P_0\lambda + P_1). \tag{29}$$

Replacing T in (29) using (27) and equating coefficients of the same powers of λ, we obtain recurrent relations for the coefficients h_k and T_k:

$$\begin{gathered} \frac{d}{dt}E - P_0T_1 + h_0E = -EP_1 - T_1P_0, \\ \frac{d}{dt}T_1 - P_0T_2 + h_0T_1 + h_1E = -T_1P_1 - T_2P_0, \\ \frac{d}{dt}T_2 - P_0T_3 + h_0T_2 + h_1T_1 + h_2E = -T_2P_1 - T_3P_0, \quad \ldots. \end{gathered} \tag{30}$$

Solving Eqs. (30) we consecutively find

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \dfrac{z_{n,m}}{w_{n,m}} \\ \dfrac{w_{n+1,m}}{z_{n,m}} & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} 0 & -\dfrac{z_{n,m}^2}{w_{n,m}^2} - \dfrac{z_{n,m}}{z_{n-1,m}} \\ -\dfrac{w_{n+1,m}^2 + z_{n+1,m}z_{n,m}}{z_{n,m}^2} & 0 \end{pmatrix} \lambda^{-2}$$
$$+ \begin{pmatrix} 0 & \dfrac{z_{n,m}}{w_{n-1,m}} + \dfrac{z_{n,m}(z_{n,m}z_{n-1,m} + w_{n,m}^2)^2}{w_{n,m}^3 z_{n-1,m}^2} \\ \dfrac{w_{n+2,m}}{z_{n,m}} + \dfrac{(z_{n,m}z_{n+1,m} + w_{n+1,m}^2)^2}{z_{n,m}^3 w_{n+1,m}} & 0 \end{pmatrix} \lambda^{-3} + \dots, \quad (31)$$

$$h = \sum_{k=0}^{\infty} h_k \lambda^{-k} = \begin{pmatrix} -\dfrac{w_{n+1,m}w_{n,m} + z_{n,m}^2}{z_{n,m}w_{n,m}} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \dfrac{w_{n+1,m}}{w_{n,m}} & 0 \\ 0 & -\dfrac{w_{n+1,m}}{w_{n,m}} \end{pmatrix} \lambda^{-1}$$
$$+ \begin{pmatrix} -\dfrac{w_{n+1,m}z_{n,m}}{w_{n,m}^2} - \dfrac{w_{n+1,m}}{z_{n-1,m}} & 0 \\ 0 & \dfrac{w_{n+1,m}^2}{z_{n,m}w_{n,m}} + \dfrac{z_{n+1,m}}{w_{n,m}} \end{pmatrix} \lambda^{-2}$$
$$+ \begin{pmatrix} \dfrac{w_{n+1,m}}{w_{n-1,m}} + \dfrac{w_{n+1,m}(z_{n,m}z_{n-1,m} + w_{n,m}^2)^2}{w_{n,m}^3 z_{n-1,m}^2} & 0 \\ 0 & \dfrac{(z_{n,m}z_{n+1,m} + w_{n+1,m}^2)^2}{w_{n,m}z_{n,m}^2 w_{n+1,m}} - \dfrac{w_{n+1,m}}{w_{n,m}} \end{pmatrix} \lambda^{-3} + \dots. \quad (32)$$

Since Eqs. (21)–(23) are compatible, the formal change of variables (26) also leads Eq. (22) to the diagonal form (see [9]):

$$\tilde{\Phi}_{n+1,m} = S\tilde{\Phi}_{n,m}, \quad (33)$$

where S is a formal series whose coefficients can be found from the equality

$$S = D_n(T)FT^{-1} \quad (34)$$

and T is the formal series (31). From the formula (34) we find

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \lambda + \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{z_{n+1,m}z_{n,m} + w_{n+1,m}^2}{z_{n,m}w_{n+1,m}} \end{pmatrix} + \begin{pmatrix} \dfrac{z_{n+1,m}}{z_{n,m}} & 0 \\ 0 & -\dfrac{w_{n+2,m}}{w_{n+1,m}} \end{pmatrix} \lambda^{-1}$$
$$+ \begin{pmatrix} -\dfrac{z_{n+1,m}^2}{z_{n,m}w_{n+1,m}} - \dfrac{w_{n+1,m}z_{n+1,m}}{z_{n,m}^2} & 0 \\ 0 & \dfrac{w_{n+2,m}^2}{z_{n+1,m}w_{n+1,m}} + \dfrac{z_{n+2,m}}{w_{n+1,m}} \end{pmatrix} \lambda^{-2} + \dots. \quad (35)$$

Since the system (5), (7) is compatible, the system (22), (23) is also compatible. By the formal change of variable (26) we reduce Eq. (23) to the diagonal form:

$$\tilde{\Phi}_{n,m+1} = K\tilde{\Phi}_{n,m}, \quad (36)$$

where K is a formal series determined by the equality

$$K = D_m(T)GT^{-1} \tag{37}$$

(the formal series T was defined above, see (31)). From Eq. (37) we easily obtain

$$K = \begin{pmatrix} \dfrac{z_{n,m+1}}{z_{n,m}} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \dfrac{z_{n,m+1}^2(w_{n,m+1}-w_{n,m})}{z_{n,m}w_{n,m}w_{n,m+1}} & 0 \\ 0 & -\dfrac{z_{n,m+1}(w_{n,m+1}-w_{n,m})}{w_{n,m}w_{n,m+1}} \end{pmatrix}\lambda^{-1}$$
$$+ \begin{pmatrix} -\dfrac{z_{n,m+1}^3(w_{n,m+1}-w_{n,m})}{z_{n,m}w_{n,m}w_{n,m+1}^2} & 0 \\ 0 & \dfrac{(w_{n,m+1}-w_{n,m})(z_{n,m+1}^2(w_{n,m+1}-w_{n,m})+w_{n,m}w_{n,m+1}w_{n+1,m})}{w_{n,m}w_{n,m+1}^2} \end{pmatrix}\lambda^{-2}$$
$$+ \begin{pmatrix} \dfrac{z_{n,m+1}^3(w_{n,m+1}-w_{n,m})}{z_{n,m}w_{n,m}^3w_{n,m+1}}\left(\dfrac{z_{n,m+1}(w_{n,m}^2-w_{n,m+1}^2)}{w_{n,m+1}^2}+\dfrac{w_{n,m}^2+z_{n,m}z_{n-1,m}}{z_{n-1,m}}\right) & 0 \\ 0 & k_{22} \end{pmatrix}\lambda^{-3}+\ldots, \tag{38}$$

where

$$k = \frac{w_{n,m}-w_{n,m+1}}{w_{n,m}w_{n,m+1}}\left(\frac{z_{n,m+1}^3(w_{n,m+1}-w_{n,m})^2}{w_{n,m}^2w_{n,m+1}^2}\right.$$
$$\left.+\frac{2z_{n,m+1}w_{n+1,m}(w_{n,m+1}-w_{n,m})}{w_{n,m}w_{n,m+1}}+\frac{w_{n+1,m}^2+z_{n+1,m}z_{n,m}}{z_{n,m}}\right). \tag{39}$$

The compatibility condition of Eqs.(28) and (33) can be represented in the form

$$(D_n-1)h = \frac{d}{dt}\log S. \tag{40}$$

This shows that the functions h and $\log S$ are generating functions of conservation laws for the system (3). We expand the function $\log S$ in the series with respect to negative powers of λ and then write Eq. (40) in the expanded form:

$$\begin{aligned}
\log S &= \log\left(S_1\lambda + S_0 + S_{-1}\lambda^{-1} + S_{-2}\lambda^{-2}+\ldots\right) \\
&= \log S_1\lambda\left(1+\frac{S_0}{S_1}\lambda^{-1}+\frac{S_{-1}}{S_1}\lambda^{-2}+\frac{S_{-2}}{S_1}\lambda^{-3}+\ldots\right) \\
&= \log S_1 + \log\lambda + \frac{S_0}{S_1}\lambda^{-1} + \left(\frac{S_{-1}}{S_1}-\frac{1}{2}\left(\frac{S_0}{S_1}\right)^2\right)\lambda^{-2} \\
&\quad + \left(\frac{S_{-2}}{S_1}-\frac{S_{-1}S_0}{S_1^2}+\frac{1}{3}\left(\frac{S_0}{S_1}\right)^3\right)\lambda^{-3}+\ldots.
\end{aligned} \tag{41}$$

Taking into account (41), we rewrite (40) in the form

$$\begin{aligned}
&(D_n-1)\left(h_0+h_1\lambda^{-1}+h_2\lambda^{-2}+h_3\lambda^{-3}+\ldots\right) \\
&= \frac{d}{dt}\left(\log S_1+\frac{S_0}{S_1}\lambda^{-1}+\left(\frac{S_{-1}}{S_1}-\frac{1}{2}\left(\frac{S_0}{S_1}\right)^2\right)\lambda^{-2}+\left(\frac{S_{-2}}{S_1}-\frac{S_{-1}S_0}{S_1^2}+\frac{1}{3}\left(\frac{S_0}{S_1}\right)^3\right)\lambda^{-3}+\ldots\right).
\end{aligned} \tag{42}$$

Comparing coefficients of the same powers of λ in Eq. (42), we obtain a new infinity series of conservation laws for the system (3). We present the first three of them:

$$(D_n - 1)\left(-\frac{w_{n+1,m}}{w_{n,m}}\right) = \frac{d}{dt}\left(\frac{z_{n+1,m}}{w_{n+1,m}} + \frac{w_{n+1,m}}{z_{n,m}}\right),$$

$$(D_n - 1)\left(\frac{w_{n+1,m}^2}{w_{n,m}z_{n,m}} + \frac{z_{n+1,m}}{w_{n,m}}\right) = \frac{d}{dt}\left(-\frac{w_{n+2,m}}{w_{n+1,m}} - \frac{1}{2}\left(\frac{z_{n+1,m}}{w_{n+1,m}} + \frac{w_{n+1,m}}{z_{n,m}}\right)^2\right),$$

$$(D_n - 1)\left(-\frac{\left(w_{n+1,m}^2 + z_{n+1,m}z_{n,m}\right)^2}{z_{n,m}^2 w_{n,m}w_{n+1,m}} - \frac{w_{n+2,m}}{w_{n,m}}\right)$$
$$= \frac{d}{dt}\left(\frac{w_{n+2,m}^2 + z_{n+1,m}z_{n+2,m}}{z_{n+1,m}w_{n+1,m}} + \frac{w_{n+2,m}(w_{n+1,m}^2 + z_{n+1,m}z_{n,m})}{w_{n+1,m}^2 z_{n,m}} + \frac{1}{3}\left(\frac{z_{n+1,m}}{w_{n+1,m}} + \frac{w_{n+1,m}}{z_{n,m}}\right)^3\right).$$

From the compatibility of Eqs. (33) and (36) we obtain the equality

$$(D_n - 1)\ln K = (D_m - 1)\ln S, \tag{43}$$

which determines an infinity series of conservation laws for the system (2). Expanding both sides of Eq. (43) in series, we obtain

$$(D_n - 1)\left(\ln K_0 + \frac{K_1}{K_0}\lambda^{-1} + \left(\frac{K_2}{K_0} - \frac{1}{2}\left(\frac{K_1}{K_0}\right)^2\right)\lambda^{-2} + \left(\frac{K_3}{K_0} - \frac{K_2K_1}{K_0^2} + \frac{1}{3}\left(\frac{K_1}{K_0}\right)^3\right)\lambda^{-3} + \dots\right)$$
$$= (D_m - 1)\left(\ln S_1 + \frac{S_0}{S_1}\lambda^{-1} + \left(\frac{S_{-1}}{S_1} - \frac{1}{2}\left(\frac{S_0}{S_1}\right)^2\right)\lambda^{-2} + \left(\frac{S_{-2}}{S_1} - \frac{S_{-1}S_0}{S_1^2} + \frac{1}{3}\left(\frac{S_0}{S_1}\right)^3\right)\lambda^{-3} + \dots\right).$$

The first two conservation laws for (2) have the form

$$(D_n - 1)\left(\frac{1}{2}\frac{(w_{n,m+1} - w_{n,m})(z_{n,m+1}^2(w_{n,m+1} - w_{n,m}) + 2w_{n,m}w_{n+1,m}w_{n,m+1})}{w_{n,m}^2 w_{n,m+1}^2}\right)$$
$$= (D_m - 1)\left(-\frac{w_{n+2,m}}{w_{n+1,m}} - \frac{1}{2}\left(\frac{z_{n+1,m}}{w_{n+1,m}} + \frac{w_{n+1,m}}{z_{n,m}}\right)^2\right),$$

$$(D_n - 1)\left(-\frac{(w_{n,m+1} - w_{n,m})^2 z_{n,m+1}w_{n+1,m}}{w_{n,m}^2 w_{n,m+1}^2}\right.$$
$$\left.-\frac{(w_{n,m+1} - w_{n,m})(w_{n+1,m}^2 + z_{n+1,m}z_{n,m})}{w_{n,m+1}w_{n,m}z_{n,m}} - \frac{1}{3}\frac{z_{n,m+1}^3(w_{n,m+1} - w_{n,m})^3}{w_{n,m}^3 w_{n,m+1}^3}\right)$$
$$= (D_m - 1)\left(\frac{w_{n+2,m}^2 + z_{n+1,m}z_{n+2,m}}{z_{n+1,m}w_{n+1,m}} + \frac{w_{n+2,m}(w_{n+1,m}^2 + z_{n+1,m}z_{n,m})}{w_{n+1,m}^2 z_{n,m}}\right.$$
$$\left. + \frac{1}{3}\left(\frac{z_{n+1,m}}{w_{n+1,m}} + \frac{w_{n+1,m}}{z_{n,m}}\right)^3\right).$$

4. Formal Asymptotic Expansion of Eigenfunctions of the Lax Pair in a Neighborhood of the Singular Point $\lambda = 0$

Now we examine the solution to Eq. (7) in a neighborhood of the singular point $\lambda = 0$. Using the substitution of the spectral parameter $\xi = 1/\lambda$, we move the singular point to infinity. Apply the method of asymptotic diagonalization of discrete operator developed in [3, 6, 7]. We decompose the potential g of Eq. (7) into the product of three factors $g = \alpha Z \gamma$, where α and γ are lower- and upper-triangular matrix-valued functions, respectively, bounded and nondegenerate at $\xi = \infty$. The singularity of the function g is concentrated in the diagonal factor Z:

$$\alpha = \begin{pmatrix} \dfrac{z_{n,m+1}(z_{n,m+1}-z_{n,m})(w_{n,m+1}-w_{n,m})}{z_{n,m}w_{n,m}w_{n,m+1}} + \xi^{-1} & 0 \\ \dfrac{z_{n,m+1}-z_{n,m}}{z_{n,m}}\xi^{-1} & \dfrac{z_{n,m+1}w_{n,m}w_{n,m+1}\xi}{z_{n,m+1}(z_{n,m+1}-z_{n,m})(w_{n,m+1}-w_{n,m})\xi + z_{n,m}w_{n,m}w_{n,m+1}} \end{pmatrix},$$

$$Z = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & \dfrac{z_{n,m+1}^2(w_{n,m+1}-w_{n,m})\xi}{z_{n,m+1}(z_{n,m+1}-z_{n,m})(w_{n,m+1}-w_{n,m})\xi + z_{n,m}w_{n,m}w_{n,m+1}} \\ 0 & 1 \end{pmatrix}.$$

We reduce the linear equation (7) to a certain special form, which is convenient for the construction of asymptotic expansions. Namely, by the change $\varphi_{n,m} = \gamma^{-1}\psi_{n,m}$ we transform Eq. (7) to the form

$$\psi_{n,m+1} = PZ\psi_{n,m}, \tag{44}$$

where $P = D_m(\gamma)\alpha$. Note that the function $P(z_{n,m}, w_{n,m}, \xi)$ is analytical and nondegenerate in a neighborhood of the point $\xi = \infty$:

$$P = P_0 + P_1\xi^{-1} + P_2\xi^{-2} + P_3\xi^{-3} + \dots,$$

and the regularity condition holds: obviously, the principal minor of the matrix $P(z_{n,m}, w_{n,m}, \infty)$ coinciding with the element P_{11} of this matrix is nonzero. Then by [7, Proposition 1] there exists a formal series

$$T = \sum_{i\geq 0} T_i\xi^{-1},$$

such that the operator $L_0 := T^{-1}LT$ has the form $L_0 = D_m^{-1}hZ$, where $L = D_m^{-1}P(z_{n,m}, w_{n,m}, \xi)Z$ and h is a formal series with diagonal coefficients:

$$h = h_0 + h_1\xi^{-1} + h_2\xi^{-2} + h_3\xi^{-3} + \dots.$$

Due to [7, Proposition 1] we conclude that the required coefficients of the formal series T and h are defined by the equality

$$D_m(T)h = P\overline{T}, \tag{45}$$

where the function

$$\overline{T} := ZTZ^{-1} = \overline{T}_0 + \overline{T}_1\xi^{-1} + \overline{T}_2\xi^{-2} + \dots$$

is also analytic at infinity. We rewrite (45) in the expanded form:

$$D_m\Big(T_0 + T_1\xi^{-1} + T_2\xi^{-2} + T_3\xi^{-3} + \dots\Big)\Big(h_0 + h_1\xi^{-1} + h_2\xi^{-2} + h_3\xi^{-3} + \dots\Big) = \\ = \Big(P_0 + P_1\xi^{-1} + P_2\xi^{-2} + P_3\xi^{-3} + \dots\Big)\Big(\overline{T}_0 + \overline{T}_1\xi^{-1} + \overline{T}_2\xi^{-2} + \dots\Big). \tag{46}$$

Comparing coefficients of the same powers of ξ in Eq. (46), we obtain the following infinite chain of equations for the unknowns T_k and h_k:

$$\begin{aligned}
&D_m(T_0)h_0 = P_0\overline{T}_0,\\
&D_m(T_0)h_1 + D_m(T_1)h_0 = P_0\overline{T}_1 + P_1\overline{T}_0,\\
&D_m(T_0)h_2 + D_m(T_1)h_1 + D_m(T_2)h_0 = P_0\overline{T}_2 + P_1\overline{T}_1 + P_2\overline{T}_0, \quad \ldots.
\end{aligned} \tag{47}$$

We set $\operatorname{diag} T_0 = (1,1)$ and for all $i > 0$ we take diagonal elements of the matrix T_i equal zero. Consecutive solution of the equations of the system (47) yields

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \dfrac{w_{n,m}w_{n,m-1}}{z_{n,m}(w_{n,m}-w_{n,m-1})} & 0 \end{pmatrix}\xi^{-1} + \begin{pmatrix} 0 & t_{12} \\ t_{21} & 0 \end{pmatrix}\xi^{-2} + \ldots, \tag{48}$$

where

$$\begin{aligned}
t_{12} &= -\frac{z_{n,m}w_{n,m}^2 w_{n,m+1}^2 z_{n,m+2}}{z_{n,m+1}(z_{n,m+1}-z_{n,m})^2(w_{n,m+1}-w_{n,m})^2(z_{n,m+2}-z_{n,m+1})},\\
t_{21} &= -\frac{w_{n,m}^2 w_{n,m-1}^2(z_{n,m+1}-z_{n,m-1})}{z_{n,m}(z_{n,m}-z_{n,m-1})(z_{n,m+1}-z_{n,m})(w_{n,m}-w_{n,m-1})^2},
\end{aligned}$$

$$\begin{aligned}
h = &\begin{pmatrix} \dfrac{z_{n,m+1}(z_{n,m+1}-z_{n,m})(w_{n,m+1}-w_{n,m})}{z_{n,m}w_{n,m}w_{n,m+1}} & 0 \\ 0 & \dfrac{w_{n,m}w_{n,m+1}}{(z_{n,m+1}-z_{n,m})(w_{n,m+1}-w_{n,m})} \end{pmatrix}\\
&+ \begin{pmatrix} 1 + \dfrac{z_{n,m+2}(z_{n,m+1}-z_{n,m})}{z_{n,m}(z_{n,m+2}-z_{n,m+1})} & 0 \\ 0 & -\dfrac{w_{n,m}^2 w_{n,m+1}^2(z_{n,m+2}-z_{n,m})}{(z_{n,m+1}-z_{n,m})^2(w_{n,m+1}-w_{n,m})^2(z_{n,m+2}-z_{n,m+1})} \end{pmatrix}\xi^{-1}\\
&+ \begin{pmatrix} -\dfrac{z_{n,m+1}w_{n,m+2}w_{n,m+1}(z_{n,m+1}-z_{n,m})}{z_{n,m}(z_{n,m+2}-z_{n,m+1})^2(w_{n,m+2}-w_{n,m+1})} & 0 \\ 0 & h_{22} \end{pmatrix}\xi^{-2} + \ldots,
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
h_{22} = &\frac{w_{n,m}^2 w_{n,m+1}^3}{(z_{n,m+1}-z_{n,m})(w_{n,m+1}-w_{n,m})^2(z_{n,m+2}-z_{n,m+1})^2}\\
&\times\left(\frac{w_{n,m}(z_{n,m+2}-z_{n,m})^2}{(z_{n,m+1}-z_{n,m})^2(w_{n,m+1}-w_{n,m})} + \frac{w_{n,m+2}}{(w_{n,m+2}-w_{n,m+1})}\right).
\end{aligned}$$

Now we transform Eq. (5) by using the same change $\varphi_{n,m} = \gamma^{-1}\psi_{n,m}$:

$$\psi_{n+1,m} = F\psi_{n,m}, \tag{50}$$

where $F = \gamma_{n+1,m} f \gamma^{-1}$. Equation (50) can be written in the form

$$\psi_{n,m} = M\psi_{n,m}, \tag{51}$$

where $M = D_n^{-1}F$. Conjugating by the formal series $M_0 = T^{-1}MT$, we reduce the operator M to the diagonal form $M_0 = D_n^{-1}S$, where

$$S = \begin{pmatrix} \dfrac{z_{n+1,m}}{w_{n+1,m}} & 0 \\ 0 & \dfrac{w_{n+1,m}}{z_{n,m}} \end{pmatrix} + \begin{pmatrix} -\dfrac{z_{n+1,m}z_{n,m+1}z_{n,m}}{w_{n+1,m}^2(z_{n,m+1}-z_{n,m})} & 0 \\ 0 & \dfrac{z_{n,m+1}}{(z_{n,m+1}-z_{n,m})} \end{pmatrix}\xi^{-1}$$
$$+ \begin{pmatrix} \dfrac{z_{n+1,m}z_{n,m}^2(z_{n,m+1}^2(w_{n,m+1}-w_{n,m})+w_{n+1,m}w_{n,m}w_{n,m+1})}{w_{n+1,m}^3(z_{n,m+1}-z_{n,m})^2(w_{n,m+1}-w_{n,m})} & 0 \\ 0 & -\dfrac{z_{n,m}w_{n,m}w_{n,m+1}}{(z_{n,m+1}-z_{n,m})^2(w_{n,m+1}-w_{n,m})} \end{pmatrix}\xi^{-2} + \dots,$$

For the system (3), we write several first conservation laws from the infinite sequence obtained by the diagonalization:

$$(D_m - 1)\ln\left(\frac{z_{n+1,m}}{w_{n+1,m}}\right) = (D_n - 1)\ln\left(\frac{z_{n,m+1}(z_{n,m+1}-z_{n,m})(w_{n,m+1}-w_{n,m})}{z_{n,m}w_{n,m}w_{n,m+1}}\right),$$
$$(D_m - 1)\ln\left(\frac{w_{n+1,m}}{z_{n,m}}\right) = (D_n - 1)\ln\left(\frac{w_{n,m}w_{n,m+1}}{(z_{n,m+1}-z_{n,m})(w_{n,m+1}-w_{n,m})}\right),$$
$$(D_m - 1)\left(\frac{z_{n,m+1}z_{n,m}}{w_{n+1,m}(z_{n,m+1}-z_{n,m})}\right)$$
$$= (D_n - 1)\left(\frac{w_{n,m}w_{n,m+1}(z_{n,m+2}-z_{n,m})}{(z_{n,m+1}-z_{n,m})(w_{n,m+1}-w_{n,m})(z_{n,m+2}-z_{n,m+1})}\right),$$
$$(D_m - 1)\left(\frac{1}{2}\frac{z_{n,m}^2(z_{n,m+1}^2(w_{n,m+1}-w_{n,m})+2w_{n+1,m}w_{n,m}w_{n,m+1})}{w_{n+1,m}^2(z_{n,m+1}-z_{n,m})^2(w_{n,m+1}-w_{n,m})}\right)$$
$$= (D_n - 1)\left(-\frac{w_{n,m}w_{n,m+1}^2}{(z_{n,m+2}-z_{n,m+1})^2(w_{n,m+1}-w_{n,m})}\left(\frac{w_{n,m+2}}{(w_{n,m+2}-w_{n,m+1})} + \frac{1}{2}\frac{(z_{n,m+2}-z_{n,m})^2 w_{n,m}}{(z_{n,m+1}-z_{n,m})^2(w_{n,m+1}-w_{n,m})}\right)\right).$$

Now we transform (10) by the change $\varphi_{n,m} = \gamma^{-1}\psi_{n,m}$:

$$\psi_{n,m,\tau} = Q\psi_{n,m}, \tag{52}$$

where $Q = \gamma(q\gamma^{-1} - (\gamma^{-1})_\tau)$.

The compatibility of the systems (44) and (52) implies the equality

$$D_\tau(P)Z = D_m(Q)PZ - PZQ,$$

equivalent to the equality $[D_\tau - Q, L] = 0$, where $L = D_m^{-1}PZ$ and D_τ is the operator of total differentiation with respect to τ. Using the conjugation transform $x \to T^{-1}xT$, we rewrite the last equality in the form

$$[D_\tau - Q_0, L_0] = 0, \tag{53}$$

where $Q_0 = -T^{-1}D_\tau(T) + T^{-1}QT$ is a formal series with diagonal coefficients and the operator $L_0 = D_m^{-1}hZ = T^{-1}LT$ is defined above. The series Q_0 has the form

$$Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\xi + \begin{pmatrix} \dfrac{z_{n,m+1}w_{n,m}w_{n,m-1}}{z_{n,m}(z_{n,m+1}-z_{n,m})(w_{n,m}-w_{n,m-1})} & 0 \\ 0 & -\dfrac{w_{n,m}w_{n,m-1}}{(z_{n,m+1}-z_{n,m})(w_{n,m}-w_{n,m-1})} \end{pmatrix}$$

$$+ \begin{pmatrix} -\dfrac{w_{n,m}^2 w_{n,m-1}w_{n,m+1}}{(z_{n,m+1}-z_{n,m})^2(w_{n,m}-w_{n,m-1})(w_{n,m+1}-w_{n,m})} & 0 \\ 0 & \dfrac{w_{n,m}^2 w_{n,m-1}w_{n,m+1}}{(z_{n,m+1}-z_{n,m})^2(w_{n,m}-w_{n,m-1})(w_{n,m+1}-w_{n,m})} \end{pmatrix}\xi^{-1}$$

$$+ \begin{pmatrix} \dfrac{w_{n,m}^3 w_{n,m-1}w_{n,m+1}^2(z_{n,m+2}-z_{n,m})}{(z_{n,m+1}-z_{n,m})^3(w_{n,m}-w_{n,m-1})(w_{n,m+1}-w_{n,m})^2(z_{n,m+2}-z_{n,m+1})} & 0 \\ 0 & q_{22} \end{pmatrix}\xi^{-2} + \ldots,$$

where

$$q_{22} = -\frac{w_{n,m}^3 w_{n,m-1}w_{n,m+1}^2(z_{n,m+2}-z_{n,m})}{(z_{n,m+1}-z_{n,m})^3(w_{n,m}-w_{n,m-1})(w_{n,m+1}-w_{n,m})^2(z_{n,m+2}-z_{n,m+1})}.$$

Equation (53) implies the equality

$$D_\tau \ln h = (D_m - 1)Q_0. \tag{54}$$

from which we can find an infinite series of conservation laws of the chain (9). We present several of them:

$$D_\tau \ln\left(\frac{z_{n,m+1}(z_{n,m+1}-z_{n,m})(w_{n,m+1}-w_{n,m})}{z_{n,m}w_{n,m}w_{n,m+1}}\right) = (D_m - 1)\left(\frac{z_{n,m+1}w_{n,m}w_{n,m-1}}{z_{n,m}(z_{n,m+1}-z_{n,m})(w_{n,m}-w_{n,m-1})}\right),$$

$$D_\tau \ln\left(\frac{w_{n,m}w_{n,m+1}}{(z_{n,m+1}-z_{n,m})(w_{n,m+1}-w_{n,m})}\right) = (D_m - 1)\left(-\frac{w_{n,m}w_{n,m-1}}{(z_{n,m+1}-z_{n,m})(w_{n,m}-w_{n,m-1})}\right),$$

$$D_\tau\left(\frac{w_{n,m}w_{n,m+1}(z_{n,m+2}-z_{n,m})}{(z_{n,m+1}-z_{n,m})(w_{n,m+1}-w_{n,m})(z_{n,m+2}-z_{n,m+1})}\right) = (D_m - 1)\left(-\frac{w_{n,m}^2 w_{n,m-1}w_{n,m+1}}{(z_{n,m+1}-z_{n,m})^2(w_{n,m}-w_{n,m-1})(w_{n,m+1}-w_{n,m})}\right).$$

5. Conclusion

The system (2) examined in the present paper is the simplest particular case of discrete systems of exponential type:

$$e^{p^i_{n+1,m+1}} - e^{p^i_{n+1,m}} = e^{-\sum_{j=1}^{j=i-1} a_{ij}p^j_{n,m+1} - \sum_{j=i+1}^{j=N} a_{ij}p^j_{n+1,m} - \frac{1}{2}a_{ii}(p^i_{n,m+1}+p^i_{n,m})}\left(e^{p^i_{n,m+1}} - e^{p^i_{n,m}}\right), \tag{55}$$

where $i = 1, \ldots, N$.

We conjecture that the system (55) is integrable if the matrix $A = \{a_{ij}\}$ of the coefficients of the system (55) coincides with the generalized Cartan matrix of a certain affine Lie algebra (see also [2]). In this paper we verify our hypothesis for the case of the affine algebra $A_1^{(1)}$. Setting in (55)

$$N = 2, \quad a_{11} = 2, \quad a_{12} = -2, \quad a_{21} = -2, \quad a_{22} = 2,$$

i.e., choosing as A the corresponding Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \tag{56}$$

we obtain a system of the form

$$\begin{cases} e^{p^1_{n+1,m+1}} - e^{p^1_{n+1,m}} = e^{2p^2_{n+1,m} - p^1_{n,m}} - e^{2p^2_{n+1,m} - p^1_{n,m+1}}, \\ e^{p^2_{n+1,m+1}} - e^{p^2_{n+1,m}} = e^{2p^1_{n,m+1} - p^2_{n,m}} - e^{2p^1_{n,m+1} - p^2_{n,m+1}}, \end{cases} \tag{57}$$

related to (2) by the substitution $e^{p^1_{n,m}} = z_{n,m}$, $e^{p^2_{n,m}} = w_{n,m}$. We prove that in the case (56) the system (55) is integrable. Therefore, it has two hierarchies of higher symmetries, infinite series of local conservation laws, and admits a Lax representation.

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REFERENCES

1. V. G. Drinfeld and V. V. Sokolov, "Lie algebras and equations of the Korteweg–de Vries type," *J. Sov. Math.*, **30**, No. 2, 1975–2036 (1985).
2. R. N. Garifullin, I. T. Habibullin, and M. V. Yangubaeva, "Affine and finite Lie algebras and integrable Toda field equations on discrete space-time," *SIGMA*, **8**, 062 (2012).
3. I. T. Habibullin, "Discrete Zakharov–Shabat systems and integrable equations," *Zap. Nauchn. Sem. LOMI*, **146**, 137–146 (1985).
4. I. T. Habibullin and A. R. Khakimova, "Invariant manifolds and Lax pairs for integrable nonlinear chains," *Teor. Mat. Fiz.* (2016), in print.
5. I. T. Habibullin, A. R. Khakimova, and M. N. Poptsova, "On a method for constructing the Lax pairs for nonlinear integrable equations," *J. Phys. A: Math. Theor.*, **49**, 035202 (2016).
6. I. T. Habibullin and M. N. Poptsova, "Asymptotic diagonalization of the discrete Lax pair around singularities and conservation laws for dynamical systems," *J. Phys. A: Math. Theor.*, **48**, 115203 (2015).
7. I. T. Habibullin and M. V. Yangubaeva, "Formal diagonalization of a discrete Lax operator and conservation laws and symmetries of dynamical systems," *Teor. Mat. Fiz.*, **177**, No. 3, 441–467 (2013).
8. A. Kuniba, T. Nakanishi, and J. Suzuki, "T-systems and Y-systems in integrable systems," *J. Phys. A: Math. Theor.*, **44**, 103001 (2011); `arXiv:1010.1344`.
9. A. V. Mikhailov, "Formal diagonalisation of Lax–Darboux schemes," *Model. Anal Inform. Syst.*, **22**, No. 6, 795–817 (2015).
10. Yu. B. Suris, "Integrable discretizations for lattice system: local equations of motion and their Hamiltonian properties," *Rev. Math. Phys.*, **11**, No. 6, 727–822 (1999).
11. W. Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, Pure Appl. Math., **14**, Interscience, New York–London–Sydney (1965).
12. R. S. Ward, "Discrete Toda field equations," *Phys. Lett. A*, **199**, 45–48 (1995).
13. R. Willox and M. Hattori, *Discretisations of constrained KP hierarchies*, Preprint `arXiv:1406.5828` (2014).

14. R. I. Yamilov, "Symmetries as integrability criteria for differential difference equations," *J. Phys. A: Math. Gen.*, **39**, R541–R623 (2006).
15. V. E. Zakharov and A. B. Shabat, "A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem, I," *Funct. Anal. Appl.*, **8**, No. 3, 226–235 (1974).
16. V. E. Zakharov and A. B. Shabat, "Integration of nonlinear equations of mathematical physics by the method of inverse scattering, II," *Funct. Anal. Appl.*, **13**, No. 3, 166–174 (1979).
17. A. B. Zamolodchikov and Al. B. Zamolodchikov, *Conformal Field Theory and Critical Phenomena in Two-Dimensional Systems* [in Russian], MCCME, Moscow (2009).

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