DOI 10.1007/s10958-019-04423-6
Journal of Mathematical Sciences, Vol. 241, No. 3, September, 2019

OSCILLATORY AND NONOSCILLATORY CONDITIONS FOR A SECOND-ORDER HALF-LINEAR DIFFERENTIAL EQUATION

L. K. Kussainova and B. S. Koshkarova
UDC 517.926, 517.98

Abstract. In this paper, we obtain oscillatory and nonoscillatory conditions for a certain second-order half-linear differential equation.

***Keywords and phrases*:** half-linear differential equation, oscillatory property, nonoscillatory property, variational method.

***AMS Subject Classification*:** 34C10

1. Introduction. In this paper, we discuss oscillatory and nonoscillatory conditions for the second-order half-linear differential equation

$$\Big(\rho(t)\,|y'(t)|^{p-2}\,y'(t)\Big)' + q(t)\,|y(t)|^{p-2}\,y(t) = 0, \quad t \geq a, \quad 1 < p < \infty,\ p \neq 2, \tag{1}$$

in the case where the sign of $q(\cdot)$ alternates in any neighborhood of $+\infty$. We assume that the function $q(\cdot)$ can be represented in the form $q = u - v$, where u and v are positive and continuous on $\bar{I} = [a, \infty)$ functions, and the function $\rho(t) > 0$ is continuous on I and is such that $\rho\,|y'|^{p-2}$ is differentiable almost everywhere in I for all $y \in C^2(I)$.

Problems on the oscillation of solutions of Eq. (1) in the case where $q > 0$ are well studied (see [3, 5]). Introduce the following notions.

In the present paper, by a *solution* of Eq. (1) we mean a twice differentiable in I solution $y(t)$ for which the function $\rho(t)\,|y'(t)|^{p-2}\,y(t)$ is differentiable in I.

A solution $y(t)$ of Eq. (1) is said to be *oscillatory* on I if there exists a sequence of points $x_k \xrightarrow[k\to\infty]{} \infty$ at which $y(x_k) = 0$. Equation (1) is said to be *oscillatory* if all its solutions are oscillatory.

There exist various methods of study of the oscillatory property of equations of the form (1). The main methods are the so-called "Riccati techniques" and the variational principle (see [1–3, 5]).

The variational principle is based on the following theorem.

Theorem 1 (see [3]). *Equation* (1) *is nonoscillatory if and only if there exists* $T \in \mathbb{R}$ *such that for all* $T \leq c \leq d < \infty$ *the condition*

$$F(y, c, d) = \int\limits_c^d \big(\rho(t)|y'(t)|^p - q(t)|y(t)|^p\big)\,dt > 0$$

holds for any nontrivial function $y \in \dot{W}_p^1(T, \infty)$.

We denote by $W_p^1(\Omega)$ the space of all absolutely continuous on the segment $\Omega = [\Omega^-, \Omega^+]$ functions for which

$$\|f; W_p^1(\Omega)\| = \left(\int\limits_\Omega \big(|y'|^p + |y|^p\big)\,dt\right)^{\frac{1}{p}} < \infty,$$

and set $\dot{W}_p^1(\Omega) = \{y \in W_p^1(\Omega) : y(\Omega^-) = y(\Omega^+) = 0\}$.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory, Vol. 139, Differential Equations. Mathematical Physics, 2017.

2. Auxiliary assertions. On the segment $[0,1]$, the following well-known estimate is valid:

$$\max_{t\in[0,1]}|z(t)| \le A_0\int_0^1 \left(|z'|+|z|\right)dt. \qquad (2)$$

Performing a change of variables, we obtain from (2) that for all $y\in W_p^1(x,x+h)$, the following inequality holds:

$$\max_{t\in[x,x+h]}|y(t)| \le A_0\int_x^{x+h} \left(|y'|+h^{-1}|y|\right)dt, \qquad (3)$$

where A_0 is the best constant in (2).

Lemma. *Assume that the following condition holds*:

$$\left(\int_x^{x+h}\rho^{-\frac{p'}{p}}dt\right)^{\frac{p}{p'}}\int_x^{x+h} v\,dt \ge 1.$$

Then for all $y\in W_p^1(x,x+h)$ the following estimate is valid:

$$h^{-1}\int_x^{x+h}|y(t)|dt \le 2^p\left(\int_x^{x+h}\left(\rho(t)|y'|^p+v(t)|y|^p\right)dt\right)^{\frac{1}{p}}\left(\int_x^{x+h}\rho^{-\frac{p'}{p}}dt\right)^{\frac{1}{p'}}. \qquad (4)$$

Proof. We can assume that

$$h^{-1}\int_x^{x+h}|y|dt = 1,$$

and then we must prove the inequality

$$1 \le 2^p\int_x^{x+h}\left(\rho(t)|y'|^p+v(t)|y|^p\right)dt\left(\int_x^{x+h}\rho^{-\frac{p'}{p}}\right)^{\frac{p}{p'}}. \qquad (5)$$

Estimate (5) is nontrivial if

$$\left(\int_x^{x+h}\rho^{-\frac{p'}{p}}\right)^{\frac{p}{p'}}\int_x^{x+h}\rho(t)|y'|^p dt < 2^{-p}. \qquad (6)$$

Since y is continuous on $[x,x+h]$, we have

$$1 = h^{-1}\int_x^{x+h}|y(t)|^p dt = |y(t_0)|^p, \quad t_0\in[x,x+h].$$

From (6) we conclude that

$$|y(t)-y(t_0)| = \left|\int_{t_0}^{t} y'(\xi)d\xi\right| \le \int_x^{x+h}|y'(\xi)|d\xi \le \left(\int_x^{x+h}\rho|y'(\xi)|^p d\xi\right)^{\frac{1}{p}}\left(\int_x^{x+h}\rho^{-\frac{p'}{p}}d\xi\right)^{\frac{1}{p'}} < 2^{-1}$$

for any point $t\in[x,x+h]$. Therefore, for all $t\in[x,x+h]$ we have

$$\big|y(t)\big| = \Big|y(t_0)-(y(t_0)-y(t))\Big| \ge \Big||y(t_0)|-|y(t_0)-y(t)|\Big| \ge \frac{1}{2},$$

which implies

$$\left(\int_x^{x+h} \rho^{-\frac{p'}{p}} d\xi\right)^{\frac{p}{p'}} \int_x^{x+h} v(t)|y(t)|^p dt \geq \frac{1}{2^p} \left(\int_x^{x+h} \rho^{-\frac{p'}{p}} d\xi\right)^{\frac{p}{p'}} \int_x^{x+h} v(t)dt \geq \frac{1}{2^p}.$$

The proof is complete. □

A weight pair (ρ, v) on $\bar{I}$ is said to be *admissible* if

$$h^*(x) = h^*(x|\rho, v) = \sup\left\{h > 0 : \left(\int_x^{x+h} \rho^{1-p'} dt\right)^{\frac{p}{p'}} \int_x^{x+h} v dt \leq 1\right\} < \infty$$

for all $x \geq a$.

Let $\Delta^*(x) = [x, x + h^*(x)]$ $(\Delta^*(x) < \infty)$. Then the following equality holds:

$$\left(\int_{\Delta^*(x)} \rho^{1-p'}\right)^{\frac{p'}{p}} \int_{\Delta^*(x)} v = 1; \tag{7}$$

this follows from the absolute continuity of the Lebesgue integral.

3. Main results.

Theorem 2. *Let (ρ, v) be an admissible pair on $\bar{I}$. If there exists $T > a$ such that*

$$\int_{\Delta^*(x)} u(t)dt < \frac{1}{2^p(1+2^p)A_0^p} \int_{\Delta^*(x)} v(t)dt, \tag{8}$$

for all $x \geq T$, then Eq. (1) *is nonoscillatory.*

Proof. By Theorem 1, Eq. (1) is nonoscillatory if there exists $T > a$ such that for all $c < d$ $(c \geq T)$ and any function $y \in \dot{W}_p^1(c, d)$ we have

$$\int_c^d \left(\rho(t)|y'(t)^p + v(t)|y|^p\right) dt > \int_c^d u(t)|y|^p dt.$$

Since $y(d) = 0$, we can assume that $y(t) = 0$ for $t \geq d$.

Consider a system of segments $\{\Delta_k\}_{k=1}^N$, $N \leq \infty$, that cover the segment $[c, d]$, where $\Delta_k = [x_k, x_k + h_k]$, $x_{k+1} = x_k + h_k$, $x_1 = c$, $h_k = h^*(x_k)$. From (3)–(5), (7), and (8) we conclude that

$$\int_c^d u(t)|y|^p dt \leq A_0^p \sum_{k=1}^N \int_{\Delta_k} u(t)dt \left(\int_{\Delta_k} \left(|y'| + h^{-1}|y|\right) dt\right)^p$$

$$\leq A_0^p \sum_{k=1}^N \int_{\Delta_k} u(t)dt \left[\left(\int_{\Delta_k} \rho^{-\frac{p'}{p}}\right)^{\frac{1}{p'}} \left(\int_{\Delta_k} \rho(t)|y|^p dt\right)^{\frac{1}{p}}\right.$$

$$\left. + 2\left(\int_{\Delta_k} \rho^{-\frac{p'}{p}}\right)^{\frac{1}{p'}} \left(\int_{\Delta_k} \left(\rho(t)|y'|^p + v(t)|y|^p\right) dt\right)^{\frac{1}{p}}\right]^p$$

$$\le (1+2^p)(2A_0)^p \sup_{x\ge T} \int\limits_{\Delta^*(x)} u(t)dt \left(\int\limits_{\Delta_k^*} v(t)dt\right)^{-1} \int\limits_c^d \left(\rho(t)|y'|^p + v(t)|y|^p\right) dt$$

$$< \int\limits_c^d \left(\rho(t)|y'|^p + v(t)|y|^p\right) dt.$$

The proof is complete. □

We set

$$\Omega^*(x) = \Big[x + \tau h^*(x),\ x + (1-\tau)h^*(x)\Big], \quad \tau = \frac{1}{4}.$$

Theorem 3. *Let (ρ, v) be an admissible pair on $\bar{I}$. Equation (1) is oscillatory if there exists a sequence of points $x_k \xrightarrow[k\to\infty]{} \infty$ satisfying the following conditions:*

$$\begin{aligned}
&(1) \quad \left(\int\limits_{\Delta_k} \rho^{-\frac{p'}{p}} dt\right)^{\frac{1}{p'}} \left(\int\limits_{\Delta_k} \rho(t)dt\right)^{\frac{1}{p}} \le C_\rho h^*(x_k), \\
&(2) \quad \int\limits_{\Omega^*(x_k)} u(t)dt \ge (1 + (2\pi C_\rho)^p) \int\limits_{\Delta^*(x_k)} v(t)dt.
\end{aligned} \tag{9}$$

Proof. By Theorem 1, Eq. (1) is oscillatory if for any $T > 0$ there exists a function $y \in \dot{W}_p^1(c,d)$, $T < c < d < \infty$, such that

$$\int\limits_c^d |y|^p u(t)dt \ge \int\limits_c^d \left(\rho(t)|y'|^p + v(t)|y|^p\right) dt.$$

We take

$$\theta(t) = \frac{\pi}{2} \int\limits_0^t \sin(\pi\xi)d\xi.$$

Let $\Delta_k = [x_k, x_k + h_k]$, $h_k = h^*(x_k)$, and

$$y_k(t) = \begin{cases} \theta\left(\dfrac{t-\Delta_k^-}{\tau h_k}\right), & \Delta_k^- \le t \le \Omega_k^-, \\ 1, & t \in \Omega_k, \\ \theta\left(\dfrac{\Delta_k^+ - t}{\tau h_k}\right), & \Omega_k^+ \le t \le \Delta_k^+. \end{cases}$$

Then

$$y_k'(t) = \begin{cases} \dfrac{\pi}{2\tau h_k} \sin \dfrac{t-\Delta_k^-}{\tau h_k}, & \Delta_k^- \le t \le \Omega_k^-, \\ 0, & t \in \Omega_k, \\ -\dfrac{\pi}{2\tau h_k} \sin \dfrac{\Delta_k^+ - t}{\tau h_k}, & \Omega_k^+ \le t \le \Delta_k^+. \end{cases}$$

Therefore, $|y'(t)| \leq 2\pi/h_k$. We see that $y_k \in \dot{W}_p^1(\Delta_k)$, and by the condition (2) of the theorem we have

$$\frac{\int\limits_{\Delta_k} u(t)|y_k|^p dt}{\int\limits_{\Delta_k} \left(\rho(t)|y_k'(t)|^p + v(t)|y_k(t)|^p\right) dt} \geq \left(\int\limits_{\Delta_k} (\rho(t))^{-\frac{p'}{p}} dt\right)^{\frac{p}{p'}} \int\limits_{\Omega_k} u(t)dt \cdot \frac{1}{\frac{\pi}{2\tau}C_\rho + 1} = 1.$$

Since $x_k \to \infty$, for any $T > 0$ we take $c = x_k$ and $d = x_k + h^*(x_k)$, where $x_k > T$. □

Remark 1. The condition of Theorem 2 is equivalent to the following:

$$\lim_{x\to\infty} \sup \frac{\int\limits_{\Delta^*(x)} u(t)dt}{\int\limits_{\Delta^*(x)} v(t)dt} < \frac{1}{2^p(1+2^p)A_0^p}.$$

Remark 2. Let ρ be a positive and continuous on the whole real axis function. If ρ satisfies the condition (A_p), then in the condition (9) of Theorem 3 we can take $C_\rho = \|\rho\|_{A_p}$ (see [4]). For example, $\rho(x) = x^\alpha$, $-1 < \alpha < p - 1$, satisfies the condition (9) on $\bar{I} = [a, \infty)$, $a > 0$.

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L. K. Kussainova
L. N. Gumilyov Eurasian National University, Astana, Kazakhstan
E-mail: `leili2006@mail.ru`
B. S. Koshkarova
L. N. Gumilyov Eurasian National University, Astana, Kazakhstan
E-mail: `b-koshkarova@yandex.kz`