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# EXISTENCE OF ENTROPIC SOLUTIONS OF AN ELLIPTIC PROBLEM IN ANISOTROPIC SOBOLEV–ORLICZ SPACES

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***Abstract.*** We consider the Dirichlet problem in an arbitrary unbounded domain with inhomogeneous boundary conditions for a certain class of anisotropic elliptic equations whose right-hand sides belong to the class $L_1$ and prove the existence of entropic solutions in anisotropic Sobolev–Orlicz spaces.

***Keywords and phrases:*** anisotropic elliptic equation, entropic solution, existence of solution, Sobolev–Orlicz space, $N$-function, pseudo-monotonic operator.

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## CONTENTS



## 1. Introduction

Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n = \{\boldsymbol{x} = (x_1, x_2, \dots, x_n)\}$, $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$. Consider the Dirichlet problem for the equation

$$\sum_{i=1}^{n} (a_i(\boldsymbol{x}, u, \nabla u))_{x_i} = a_0(\boldsymbol{x}, u), \quad \boldsymbol{x} \in \Omega, \tag{1.1}$$

with the following inhomogeneous boundary condition:

$$u(\boldsymbol{x})\Big|_{\partial\Omega} = \psi(\boldsymbol{x})\Big|_{\partial\Omega}. \tag{1.2}$$

Since the 1980s, second-order nonlinear elliptic equations of the form

$$\sum_{i=1}^{n} (a_i(\boldsymbol{x}, u, \nabla u))_{x_i} - a_0(\boldsymbol{x}, u, \nabla u) = f(\boldsymbol{x}), \quad f \in L_1, \tag{1.3}$$

with measures in the right-hand side have been intensively examined. Weak solutions of Eqs. (1.3) with power nonlinearities in the space $\mathbb{R}_n$ with $f \in L_{1,\mathrm{loc}}(\mathbb{R}_n)$, were studied by H. Brezis (see [10]), L. Bokkardo, T. Galuet, and J. Vazquez (see [8]), M. Bendahmane and K. Karlsen (see [3]), et al. The existence of weak solutions of the Dirichlet problem in a bounded domain $\Omega$ with a function $f \in L_1(\Omega)$ was found by L. Bokkardo and T. Galuet (see [7]) and L. Bokkardo, T. Galuet, and P. Marcellini (see [9]).

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F. Benilan, L. Bokkardo, T. Galluet, R. Garieri, M. Pierre, and J. Vazquez (see [4]) and L. Bokkardo (see [6]) in their works proposed the concept of an entropic solution of the Dirichlet problem and proved its existence and uniqueness for elliptic equations with power nonlinearities and an $L_1$-right-hand side. These authors pointed out that instead of entropic solutions first proposed by S. N. Kruzhkov (see [18]), renormalized solutions can be considered for first-order equations. Such solutions belong to the functional class containing entropic solutions, but, in contrast to them, renormalized solutions satisfy another family of integral relations. In some cases, the concepts of entropic and renormalized solutions are equivalent. The problems on the existence and uniqueness of renormalized solutions of elliptic problems in Orlicz spaces were studied in [2, 13].

Summability properties and estimates of entropic solutions for the Dirichlet problem in bounded regions for the nonlinear elliptic equation (1.3) satisfying the condition of degenerating coercivity were found by A. A. Kovalevsky (see [15]).

In [5], A. Benkirane and J. Bennouna studied the existence of entropic solutions for the Dirichlet problem in Orlicz spaces for elliptic equations with second-order nonpolynomial nonlinearities and $f \in L_1(\Omega)$ ($\Omega$ is a bounded domain).

Note that in the works familiar to the author, the results are obtained for entropic and renormalized solutions of elliptic problems in bounded domains (except for [4]) with homogeneous boundary conditions. In [4], the authors proved the existence of entropic solutions of the Dirichlet problem (1.1), (1.2) in anisotropic Sobolev–Orlicz spaces without the assumption of the boundedness of the domain $\Omega$.

## 2. $N$-Functions and Orlicz Spaces

In this section, we present necessary information from the theory of $N$-functions and Orlicz spaces (see [20]). A nonnegative, continuous, downward-convex function $B(z)$, $z \in \mathbb{R}$, is called an *$N$-function* if it is even and satisfies the relations

$$\lim_{z\to 0} \frac{B(z)}{z} = 0, \quad \lim_{z\to\infty} \frac{B(z)}{z} = \infty.$$

Note that $B(\varepsilon z) \le \varepsilon B(z)$, $z \in \mathbb{R}$, for $0 < \varepsilon \le 1$.

An $N$-function

$$\overline{B}(z) = \sup_{y\ge 0}(y|z| - B(y)), \quad z \in \mathbb{R},$$

it called *complementary* to $N$-function $B(z)$. The following *Young inequality* holds:

$$|zy| \le B(y) + \overline{B}(z), \quad z, y \in \mathbb{R}. \tag{2.1}$$

In addition, we have the equality

$$zB'(z) = \overline{B}(B'(z)) + B(z), \quad z \in \mathbb{R}, \tag{2.2}$$

where $B'(z)$ is the right-hand side derivative of the $N$-function $B(z)$.

For $N$-functions $B(z)$ and $M(z)$ we write $B(z) \prec M(z)$ if there exist numbers $l > 0$ and $z_0 > 0$ such that

$$B(z) \le M(lz), \quad |z| \ge z_0.$$

We say that an $N$-function $B(z)$ *grows considerably faster* than an $N$-function $M(z)$ and write $M(z) \prec\prec B(z)$ if

$$\lim_{z\to\infty} \frac{M(z)}{B(lz)} = 0$$

for any number $l > 0$.

We say that an $N$-function $B(z)$ *satisfies the* $\Delta_2$-*condition* for large values of $z$ if there exist numbers $c > 0$ and $z_0 \geq 0$ such that $B(2z) \leq cB(z)$ for any $|z| \geq z_0$. The $\Delta_2$-condition is equivalent to the inequality

$$B(lz) \leq c(l)B(z) \tag{2.3}$$

for $|z| \geq z_0$, where $l$ is any number greater than one and $c(l) > 0$.

An $N$-function $B(z)$ satisfies the $\Delta_2$-condition if and only if there exist numbers $c > 1$ and $z_0 \geq 0$ such that for $|z| \geq z_0$ the following inequality holds:

$$zB'(z) \leq cB(z) \tag{2.4}$$

(see [20, Chap. I, Sec. 4, Theorem 4.1]). In the sequel we assume that all $N$-functions considered satisfy the $\Delta_2$-condition for all values of $z \in \mathbb{R}$ (i.e., $z_0 = 0$).

For an $N$-function $B(z)$, due to the convexity and the inequality (2.3), there exists $c > 0$ such that the following inequality holds:

$$B(y + z) \leq cB(z) + cB(y), \quad z, y \in \mathbb{R}. \tag{2.5}$$

Assume that $Q$ is an arbitrary domain of $\mathbb{R}^n$. We consider the Orlicz space $L_B(Q)$ with the Luxembourg norm

$$\|v\|_{B,Q} = \inf \left\{ k > 0 \;\middle|\; \int_Q B\left(\frac{v(\boldsymbol{x})}{k}\right) d\boldsymbol{x} \leq 1 \right\}.$$

The following inequalities hold (see [20, Chap. II, Sec. 9, inequalities (9.21) and (9.12)]):

$$\int_Q B\left(\frac{v(\boldsymbol{x})}{\|v\|_{B,Q}}\right) d\boldsymbol{x} \leq 1, \tag{2.6}$$

$$\|v\|_{B,Q} \leq \int_Q B(v)d\boldsymbol{x} + 1. \tag{2.7}$$

Moreover, if an $N$-function $B(z)$ satisfies the $\Delta_2$-condition, then the inequality

$$\int_Q B(v)d\boldsymbol{x} \leq c(\|v\|_{B,Q}) \tag{2.8}$$

is fulfilled for $v \in L_B(Q)$. For functions $u \in L_B(Q)$ and $v \in L_{\overline{B}}(Q)$, the Hölder inequality holds (see [20, Chap. II, Sec. 9, inequalities (9.24) and (9.27)]:

$$\left| \int_Q u(\boldsymbol{x})v(\boldsymbol{x})d\boldsymbol{x} \right| \leq 2\|u\|_{B,Q}\|v\|_{\overline{B},Q}. \tag{2.9}$$

We denote the norm in the spaces $L_p(Q)$, $p \in [1, \infty]$, by $\|\cdot\|_{p,Q}$. For brevity, we will omit the subscript $Q = \Omega$ in the notation $\|\cdot\|_{p,Q}$ and $\|\cdot\|_{B,Q}$. For any $N$-function $B(z)$, if $\operatorname{mes} Q < \infty$, then $L_B(Q) \subset L_1(Q)$ and the following inequality holds:

$$\|v\|_{1,Q} \leq A_0(\operatorname{mes} Q)\|v\|_{B,Q}, \quad v \in L_B(Q). \tag{2.10}$$

For $N$-functions $B_1(z), \ldots, B_n(z)$, let us define the anisotropic Sobolev–Orlicz space $\mathring{H}^1_{\mathrm{B}}(Q)$ as the completion of $C_0^\infty(Q)$ with respect to the norm

$$\|v\|_{\mathring{H}^1_{\mathrm{B}}(Q)} = \sum_{i=1}^n \|v_{x_i}\|_{B_i,Q}.$$

We recall the following embedding theorem for the anisotropic space $\overset{\circ}{H}{}^1_{\mathrm{B}}(Q)$. Let

$$h(\theta) = \left( \prod_{i=1}^{n} \frac{B_i^{-1}(\theta)}{\theta} \right)^{1/n}.$$

Assume that the integral

$$\int\limits_0^1 \frac{h(\theta)}{\theta} d\theta$$

converges. Then we can define the $N$-function $B_*^{-1}(z)$ by the formula

$$B_*^{-1}(z) = \int\limits_0^{|z|} \frac{h(\theta)}{\theta} d\theta.$$

**Lemma 2.1** ((see [14])). *Let* $v \in \overset{\circ}{H}{}^1_{\mathrm{B}}(Q)$.

(1) *If*

$$\int\limits_1^{\infty} \frac{h(\theta)}{\theta} d\theta = \infty, \tag{2.11}$$

*then* $\overset{\circ}{H}{}^1_{\mathrm{B}}(Q) \subset L_{B_*}(Q)$ *and*

$$\|v\|_{B_*,Q} \leq A_1 \|v\|_{\overset{\circ}{H}{}^1_{\mathrm{B}}(Q)}; \tag{2.12}$$

(2) *if*

$$\int\limits_1^{\infty} \frac{h(\theta)}{\theta} d\theta < \infty, \tag{2.13}$$

*then* $\overset{\circ}{H}{}^1_{\mathrm{B}}(Q) \subset L_{\infty}(Q)$ *and*

$$\|v\|_{\infty,Q} \leq A_2 \|v\|_{\overset{\circ}{H}{}^1_{\mathrm{B}}(Q)}. \tag{2.14}$$

*Here*

$$A_1 = \frac{n-1}{n}, \quad A_2 = \int\limits_0^{\infty} \frac{h(\theta)}{\theta} d\theta.$$

## 3. Assumptions and Statement of Results

Let $N$-functions $B_1(z), \ldots, B_n(z)$ and their complementary functions $\overline{B}_1(z), \ldots, \overline{B}_n(z)$ fulfil the $\Delta_2$-condition. We denote by $\mathrm{L}_{\mathrm{B}}(\Omega)$ the space $L_{B_1}(\Omega) \times \ldots \times L_{B_n}(\Omega)$ with the norm

$$\|\boldsymbol{v}\|_{\mathrm{B}} = \|v_1\|_{B_1} + \ldots + \|v_n\|_{B_n}, \quad \boldsymbol{v} = (v_1, \ldots, v_n) \in \mathrm{L}_{\mathrm{B}}(\Omega). \tag{3.1}$$

Similarly we define the space $\mathrm{L}_{\overline{\mathrm{B}}}(\Omega)$. We assume that $\psi(\boldsymbol{x}) \in L_{\infty}(\Omega)$, $\nabla \psi \in \mathrm{L}_{\mathrm{B}}(\Omega)$.

Let $\boldsymbol{s} \cdot \boldsymbol{t}$ be the scalar product of the vectors $\boldsymbol{s} = (s_1, \ldots, s_n)$ and $\boldsymbol{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and

$$\boldsymbol{a}(\boldsymbol{x}, s_0, \boldsymbol{s}) = \Big( a_1(\boldsymbol{x}, s_0, \boldsymbol{s}), \ \ldots, \ a_n(\boldsymbol{x}, s_0, \boldsymbol{s}) \Big). \tag{3.2}$$

Introduce some conditions on the functions that are involved in Eq. (1.1). Assume that the functions $a_0(\boldsymbol{x}, s_0)$ and $a_i(\boldsymbol{x}, s_0, \boldsymbol{s})$, $\alpha = 1, \ldots, n$, are measurable with respect to $\boldsymbol{x} \in \Omega$ for $s_0 \in \mathbb{R}$ and $\boldsymbol{s} \in \mathbb{R}^n$ and continuous with respect to $s_0 \in \mathbb{R}$ and $\boldsymbol{s} \in \mathbb{R}^n$ for almost all $\boldsymbol{x} \in \Omega$. We also assume that the function $a_0(\boldsymbol{x}, s_0)$ is nondecreasing with respect to $s_0 \in \mathbb{R}$.

Assume that there exist nonnegative measurable functions $\phi(\boldsymbol{x}), \Phi(\boldsymbol{x}) \in L_1(\Omega)$, a continuous positive function $\widehat{a}(k)$, and a positive constant $\overline{a}$ such that the following inequalities hold:

$$\overline{\mathrm{B}}(\boldsymbol{a}(\boldsymbol{x}, s_0, \boldsymbol{s})) \le \widehat{a}(k)\{\Phi(\boldsymbol{x}) + \mathrm{B}(\boldsymbol{s})\}, \quad \overline{\mathrm{B}}(\boldsymbol{a}) = \sum_{i=1}^{n} \overline{B}_i(a_i), \quad \mathrm{B}(\boldsymbol{s}) = \sum_{i=1}^{n} B_i(s_i); \tag{3.3}$$

$$\Big(\boldsymbol{a}(\boldsymbol{x}, s_0, \boldsymbol{s}) - a(\boldsymbol{x}, s_0, \boldsymbol{t})\Big) \cdot (\boldsymbol{s} - \boldsymbol{t}) > 0 \tag{3.4}$$

for almost all $\boldsymbol{x} \in \Omega$ and any $s_0 \in [-k, k]$, $\boldsymbol{s}, \boldsymbol{t} \in \mathbb{R}^n$, $\boldsymbol{s} \neq \boldsymbol{t}$. Assume that the following inequality holds for almost all $\boldsymbol{x} \in \Omega$ and all $s_0 \in \mathbb{R}$, $\boldsymbol{s} \in \mathbb{R}^n$:

$$\boldsymbol{a}(\boldsymbol{x}, s_0, \boldsymbol{s}) \cdot (\boldsymbol{s} - \nabla\psi) \ge \overline{a}\mathrm{B}(\boldsymbol{s}) - \phi(\boldsymbol{x}). \tag{3.5}$$

The functions $a_i(\boldsymbol{x}, s_0, \boldsymbol{s})$, $i = 1, \dots, n$, satisfy the Hölder condition with respect to the variable $s_0$: there exist a continuous function $\widehat{A}(R, \rho)$, $R, \rho > 0$, which increases with respect to each of its arguments, and a number $\alpha \in (0, 1)$ such that for any $\boldsymbol{x} \in \Omega(R) = \{\boldsymbol{x} \in \Omega : |\boldsymbol{x}| < R\}$, $s_0, t_0 \in \mathbb{R}$, $|s_0| < \rho$, $|t_0| < \rho$, $\boldsymbol{s} \in \mathbb{R}^n$, the following inequalities hold:

$$\overline{B}_i\left(\frac{|a_i(\boldsymbol{x}, s_0, \boldsymbol{s}) - a_i(\boldsymbol{x}, t_0, \boldsymbol{s})|}{|s_0 - t_0|^{\alpha}}\right) \le \widehat{A}(R, \rho)\mathrm{B}(\boldsymbol{s}), \quad i = 1, \dots, n. \tag{3.6}$$

Here $N$-functions $B_1(z), \dots, B_n(z)$ and their complementary $N$-functions $\overline{B}_1(z), \dots, \overline{B}_n(z)$ satisfy the $\Delta_2$-condition. The nonclassical condition was used in [1] for nonlinear elliptic variational one-sided problems in Orlicz spaces.

We set $a_0(\boldsymbol{x}, s_0) = a_0(\boldsymbol{x}, \psi) + b(\boldsymbol{x}, s_0)$. Assume there exists $\delta_0 > 0$ such that

$$a_0(\boldsymbol{x}, \psi),\ a_0(\boldsymbol{x}, \psi \pm \delta_0)\ \in L_1(\Omega). \tag{3.7}$$

The function $b(\boldsymbol{x}, s_0)$ is a Carathéordory function, which is nondecreasing with respect to $s_0 \in \mathbb{R}$, $b(\boldsymbol{x}, \psi) = 0$ for almost all $\boldsymbol{x} \in \Omega$, and hence for almost all $\boldsymbol{x} \in \Omega$ and $s_0 \in \mathbb{R}$ the following inequality holds:

$$b(\boldsymbol{x}, s_0)(s_0 - \psi) \ge 0. \tag{3.8}$$

Assume that

$$\sup_{|s_0| \le k} |b(\boldsymbol{x}, s_0)| = G_k(\boldsymbol{x}) \in L_{1,\mathrm{loc}}(\Omega). \tag{3.9}$$

The condition (3.7) implies

$$b(\boldsymbol{x}, \psi \pm \delta_0) \in L_1(\Omega). \tag{3.10}$$

We introduce the function

$$T_k(r) = \begin{cases} k & \text{for } r > k, \\ r & \text{for } |r| \le k, \\ -k & \text{for } r < -k \end{cases}$$

and the notation

$$[v] = \int_{\Omega} v \, d\boldsymbol{x}.$$

**Definition 3.1.** A measurable function $u : \Omega \to \mathbb{R}$ is called an *entropic solution* of the problem (1.1), (1.2) if it satisfies the following conditions:

$$1)\quad A_0(\boldsymbol{x}) = a_0(\boldsymbol{x}, u) \in L_1(\Omega),$$

$$2)\quad T_k(u - \psi) \in \mathring{H}^1_{\mathrm{B}}(\Omega) \quad \text{for all } k > 0,$$

and for all $k>0$ and $\xi(\boldsymbol{x})\in C_0^1(\Omega)$, the following inequality holds:

$$\Big[a_0(\boldsymbol{x},u)T_k(u-\psi-\xi)+\boldsymbol{a}(\boldsymbol{x},u,\nabla u)\cdot\nabla T_k(u-\psi-\xi)\Big]\le 0. \tag{3.11}$$

**Theorem 3.1.** *Assume that the conditions* (2.11), (3.3)–(3.7), (3.9) *are fulfilled. Then there exists an entropic solution of the problem* (1.1), (1.2).

## 4. Preliminary Information

In the sequel, all constants are assumed to be positive.

Consider Carathéordory functions

$$a_{i\psi}(\boldsymbol{x},s_0,\boldsymbol{s})=a_i\Big(\boldsymbol{x},s_0+\psi(\boldsymbol{x}),\boldsymbol{s}+\nabla\psi(\boldsymbol{x})\Big),\quad i=1,\dots,n,$$
$$a_{0\psi}(\boldsymbol{x},s_0)=a_0\big(\boldsymbol{x},s_0+\psi(\boldsymbol{x})\big),\quad \boldsymbol{x}\in\Omega,\quad s_0\in\mathbb{R},\quad \boldsymbol{s}\in\mathbb{R}^n.$$

The function $a_{0\psi}(\boldsymbol{x},s_0)$ is nondecreasing with respect to $s_0$. Applying (3.3), (2.5), and (3.4) for the vector-valued function

$$\boldsymbol{a}_\psi(\boldsymbol{x},s_0,\boldsymbol{s})=\Big(a_{1\psi}(\boldsymbol{x},s_0,\boldsymbol{s}),\ \dots,\ a_{n\psi}(\boldsymbol{x},s_0,\boldsymbol{s})\Big)$$

for almost all $\boldsymbol{x}\in\Omega$ and any $s_0\in[-k,k]$, $\boldsymbol{s},\boldsymbol{t}\in\mathbb{R}^n$, $\boldsymbol{s}\ne\boldsymbol{t}$, we obtain the inequalities

$$\begin{aligned}\overline{\mathrm{B}}\big(\boldsymbol{a}_\psi(\boldsymbol{x},s_0,\boldsymbol{s})\big)&=\overline{\mathrm{B}}\Big(\boldsymbol{a}(\boldsymbol{x},s_0+\psi(\boldsymbol{x}),\boldsymbol{s}+\nabla\psi(\boldsymbol{x}))\Big)\\&\le\widehat{a}\big(k+\|\psi\|_\infty\big)\Big\{\Phi(\boldsymbol{x})+\mathrm{B}\big(\boldsymbol{s}+\nabla\psi(\boldsymbol{x})\big)\Big\}\le\widehat{a}\big(k+\|\psi\|_\infty\big)\Big\{\Phi(\boldsymbol{x})+c\mathrm{B}\big(\nabla\psi(\boldsymbol{x})\big)+c\mathrm{B}(\boldsymbol{s})\Big\}\\&\le\widehat{a}_\psi(k)\Big\{\Phi_\psi(\boldsymbol{x})+\mathrm{B}(\boldsymbol{s})\Big\}\end{aligned}\tag{3.3$\psi$}$$

and

$$\Big(\boldsymbol{a}_\psi(\boldsymbol{x},s_0,\boldsymbol{s})-a_\psi(\boldsymbol{x},s_0,\boldsymbol{t})\Big)\cdot(\boldsymbol{s}-\boldsymbol{t})>0.\tag{3.4$\psi$}$$

Using (3.5) and (2.5), for almost all $\boldsymbol{x}\in\Omega$ and all $s_0\in\mathbb{R}$, $\boldsymbol{s}\in\mathbb{R}^n$, we obtain the inequalities

$$\begin{aligned}\boldsymbol{a}_\psi(\boldsymbol{x},s_0,\boldsymbol{s})\cdot\boldsymbol{s}&=\boldsymbol{a}\Big(\boldsymbol{x},\ s_0+\psi(\boldsymbol{x}),\ \boldsymbol{s}+\nabla\psi(\boldsymbol{x})\Big)\cdot\boldsymbol{s}\\&\ge\overline{a}\mathrm{B}(\boldsymbol{s}+\nabla\psi)-\phi(\boldsymbol{x})\ge\frac{\overline{a}}{c\mathrm{B}(\boldsymbol{s})}-\overline{a}\mathrm{B}\big(\nabla\psi(\boldsymbol{x})\big)-\phi(\boldsymbol{x})=\overline{a}_\psi\mathrm{B}(\boldsymbol{s})-\phi_\psi(\boldsymbol{x}).\end{aligned}\tag{3.5$\psi$}$$

Obviously, the functions $\phi_\psi(\boldsymbol{x})$, $\Phi_\psi(\boldsymbol{x})\in L_1(\Omega)$ are nonnegative, the function $\widehat{a}_\psi(k)$ is positive and continuous, and $\overline{a}_\psi$ is a positive number.

It follows from (3.6) that there exist a continuous function $\widehat{A}_\psi(R,\rho)$ and a number $\alpha\in(0,1)$ such that for any $\boldsymbol{x}\in\Omega(R)$, $s_0,t_0\in\mathbb{R}$, $|s_0|<\rho$, $|t_0|<\rho$, and $\boldsymbol{s}\in\mathbb{R}^n$, the following inequalities hold:

$$\begin{aligned}\overline{B}_i\left(\frac{|a_{i\psi}(\boldsymbol{x},s_0,\boldsymbol{s})-a_{i\psi}(\boldsymbol{x},t_0,\boldsymbol{s})|}{|s_0-t_0|^\alpha}\right)&\le\widehat{A}\big(R,\rho+\|\psi\|_\infty\big)\mathrm{B}\big(\boldsymbol{s}+\nabla\psi\big)\\&\le\widehat{A}_\psi(R,\rho)\big(\mathrm{B}(\boldsymbol{s})+\mathrm{B}(\nabla\psi)\big),\quad i=1,\dots,n.\end{aligned}\tag{3.6$\psi$}$$

Assume that $a_{0\psi}(\boldsymbol{x},s_0)=a_{0\psi}(\boldsymbol{x},0)+b_\psi(\boldsymbol{x},s_0)$, $a_{0\psi}(\boldsymbol{x},0)=a_0(\boldsymbol{x},\psi)$, $b_\psi(\boldsymbol{x},s_0)=b(\boldsymbol{x},s_0+\psi)$. According to (3.7), we obtain

$$A^0_{0\psi}(\boldsymbol{x})=a_{0\psi}(\boldsymbol{x},0)\in L_1(\Omega).\tag{3.7$\psi$}$$

The function $b_\psi(\boldsymbol{x},s_0)$ is a Carathéordory function, which is nondecreasing with respect to $s_0\in\mathbb{R}$ and $b_\psi(\boldsymbol{x},0)=0$ for almost all $\boldsymbol{x}\in\Omega$; therefore, for almost all $\boldsymbol{x}\in\Omega$ and $s_0\in\mathbb{R}$ we have

$$b_\psi(\boldsymbol{x},s_0)s_0\ge 0.\tag{3.8$\psi$}$$

From (3.9) we obtain

$$\sup_{|s_0|\le k}\left|b_\psi(\boldsymbol{x},s_0)\right| \le \sup_{|s_0+\psi|\le k+\|\psi\|_\infty}\left|b(\boldsymbol{x},s_0+\psi)\right| = G_{k+\|\psi\|_\infty}(\boldsymbol{x}) \in L_{1,\mathrm{loc}}(\Omega). \tag{3.9$\psi$}$$

Finally, it follows from (3.10) that there exists $\delta_0 > 0$ such that

$$b_\psi(\boldsymbol{x},\pm\delta_0) \in L_1(\Omega). \tag{3.10$\psi$}$$

Let $u$ be an entropic solution of the problem (1.1), (1.2). Assuming $w = u - \psi$, we can reformulate Definition 3.1 as follows.

**Definition 3.1$\psi$.** An *entropic solution* of the problem (1.1), (1.2) is a measurable function $u : \Omega \to \mathbb{R}$ satisfying the condition $u(\boldsymbol{x}) = w(\boldsymbol{x}) + \psi(\boldsymbol{x})$ and the following conditions:

$$(1\psi)\quad A_{0\psi}(\boldsymbol{x}) = a_{0\psi}(\boldsymbol{x},w) \in L_1(\Omega),$$
$$(2\psi)\quad T_k(w) \in \mathring{H}^1_{\mathrm{B}}(\Omega) \text{ for all } k>0$$

and for all $k > 0$ and $\xi(\boldsymbol{x}) \in C_0^1(\Omega)$, the following inequality holds:

$$\Big[a_{0\psi}(\boldsymbol{x},w)T_k(w-\xi) + \boldsymbol{a}_\psi(\boldsymbol{x},w,\nabla w)\cdot\nabla T_k(w-\xi)\Big] \le 0. \tag{3.11$\psi$}$$

Let $\chi(P)$ be the propositional function which is equal to 1 if $P$ is a true proposition and to 0 if $P$ is false.

It follows from the item $(2\psi)$ of the definition of an entropic solution that

$$\chi(|w|<k)\nabla w \in \mathrm{L}_{\mathrm{B}}(\Omega) \tag{4.1}$$

for any $k > 0$. Hence, applying (3.3$\psi$), we find that

$$\chi(|w|<k)\boldsymbol{a}_\psi(\boldsymbol{x},w,\nabla w) \in \mathrm{L}_{\overline{\mathrm{B}}}(\Omega) \tag{4.2}$$

for any $k > 0$.

**Lemma 4.1.** *If $u = w + \psi$ is an entropic solution of the problem* (1.1), (1.2), *then for all $k \ge 1$, the following inequality holds*:

$$\left\|\mathrm{B}(\nabla T_k w)\right\|_1 = \int\limits_{\{\Omega:|w|<k\}} \mathrm{B}(\nabla w)d\boldsymbol{x} \le C_1 k. \tag{4.3}$$

*Proof.* According to the inequality (3.11$\psi$) for $\xi = 0$ and the condition $(1\psi)$, we obtain

$$\int\limits_{\{\Omega:|w|<k\}} \boldsymbol{a}_\psi(\boldsymbol{x},w,\nabla w)\cdot\nabla w d\boldsymbol{x} = \int\limits_{\Omega} \boldsymbol{a}_\psi(\boldsymbol{x},w,\nabla w)\cdot\nabla T_k(w) d\boldsymbol{x} \le -\int\limits_{\Omega} a_{0\psi}(\boldsymbol{x},w)T_k(w)d\boldsymbol{x} \le k\|A_{0\psi}\|_1.$$

Applying the inequality (3.5$\psi$), we obtain the inequality

$$\overline{a}_\psi \int\limits_{\{\Omega:|w|<k\}} \mathrm{B}(\nabla w)d\boldsymbol{x} \le k\|A_{0\psi}\|_1 + \|\phi_\psi\|_1,$$

which implies (4.3). ☐

**Lemma 4.2.** *Let the condition* (2.11) *hold. Assume that for a measurable function* $v : \Omega \to \mathbb{R}$ *satisfying the condition* $T_k v \in \mathring{H}^1_{\mathrm{B}}(\Omega)$ *for all* $k \geq 1$, *the following inequality holds*:

$$\big\|\mathrm{B}(\nabla T_k v)\big\|_1 = \int\limits_{\{\Omega:|v|<k\}} \mathrm{B}(\nabla v) d\boldsymbol{x} \leq C_2 k. \tag{4.4}$$

*Then*

$$\operatorname{mes}\Big(\big\{\Omega : |v| \geq k\big\}\Big) \to 0, \quad k \to \infty. \tag{4.5}$$

*Proof.* For an $N$-function $\overline{B}$ satisfying the $\Delta_2$-condition, the following relation holds:

$$\lim_{\|\omega\|_B \to \infty} \frac{\|B(\omega)\|_1}{\|\omega\|_B} = \infty \tag{4.6}$$

(see [12, Lemma 3.14]). According to the inequalities (2.12) and (4.4), taking into account (4.6) we obtain

$$\|T_k v\|_{B_*} \leq A_1 \|\nabla T_k v\|_{\mathrm{B}} \leq A_1 \varepsilon(k) \|\mathrm{B}(\nabla T_k v)\|_1 \leq C_3 k \varepsilon(k), \quad k \geq 1, \tag{4.7}$$

$\varepsilon(k) \to 0$ as $k \to \infty$.

The inequality (4.7) is obtained under the condition

$$\|\nabla T_k v\|_{\mathrm{B}} \to \infty, \quad k \to \infty;$$

otherwise,

$$\|\nabla T_k v\|_{\mathrm{B}} \leq C_4 = C_4 k \varepsilon(k), \quad k > 0,$$

so that the inequality (4.7) is also valid.

From (4.7) we obtain

$$B_*\left(\frac{k}{\|T_k v\|_{B_*}}\right) \geq B_*\left(\frac{1}{C_3 \varepsilon(k)}\right) \to \infty, \quad k \to \infty. \tag{4.8}$$

Further, applying (2.6), we have

$$1 \geq \int\limits_{\Omega} B_*\left(\frac{T_k v}{\|T_k v\|_{B_*}}\right) d\boldsymbol{x} \geq B_*\left(\frac{k}{\|T_k v\|_{B_*}}\right) \operatorname{mes}\Big(\big\{\Omega : |v| \geq k\big\}\Big).$$

Using (4.8), from the last inequality we conclude (4.5). □

**Remark 4.1.** If $u = w + \psi$ is an entropic solution of the problem (1.1), (1.2) and the condition (2.11) is fulfilled, then Lemmas 4.1 and 4.2 imply

$$\operatorname{mes}\Big(\big\{\Omega : |w| \geq k\big\}\Big) \to 0, \quad k \to \infty. \tag{4.9}$$

**Lemma 4.3.** *Let the condition* (2.11) *be fulfilled. Assume that for a measurable function* $v : \Omega \to \mathbb{R}$ *satisfying the condition* $T_k v \in \mathring{H}^1_{\mathrm{B}}(\Omega)$ *for all* $k \geq 1$, *the inequality* (4.4) *holds. Then*

$$\operatorname{mes}\Big(\big\{\Omega : \mathrm{B}(\nabla v) \geq \rho\big\}\Big) \to 0, \quad \rho \to \infty. \tag{4.10}$$

*Proof.* We set

$$\Phi(k, \rho) = \operatorname{mes}\Big\{\Omega : \ |v| \geq k, \ \mathrm{B}(\nabla v) \geq \rho\Big\}, \quad k, \rho \geq 0.$$

We have proved above (see (4.5)) that

$$\Phi(k, 0) \to 0, \quad k \to \infty.$$

Since the function $\rho \to \Phi(k,\rho)$ is nonincreasing, we have the following inequalities for $k, \rho > 0$:

$$\Phi(0,\rho) \le \frac{1}{\rho}\int_0^\rho \Phi(0,\varrho)d\varrho \le \Phi(k,0) + \frac{1}{\rho}\int_0^\rho \Big(\Phi(0,\varrho) - \Phi(k,\varrho)\Big)d\varrho. \tag{4.11}$$

Note that

$$\Phi(0,\varrho) - \Phi(k,\varrho) = \operatorname{mes}\Big\{\Omega: \ |v| < k, \ \mathrm{B}(\nabla v) \ge \varrho\Big\}.$$

Therefore, from (4.4) we obtain

$$\int_0^\infty \Big(\Phi(0,\varrho) - \Phi(k,\varrho)\Big)d\varrho = \int_{\{\Omega:|v|<k\}} \mathrm{B}(\nabla v)d\boldsymbol{x} \le C_2 k.$$

Now (4.11) implies

$$\Phi(0,\rho) \le \Phi(k,0) + \frac{C_2 k}{\rho}.$$

Choosing $k$ such that $\Phi(k,0) < \varepsilon$, we can achieve the fulfillment of the inequality $\Phi(0,\rho) < 2\varepsilon$ by selecting $\rho$. Therefore Lemma (4.10) is proved. $\square$

**Lemma 4.4.** *Let an $N$-function $\overline{B}(z)$ satisfy the $\Delta_2$-condition and $v^m(\boldsymbol{x})$, $m = 1, \dots, \infty$, and $v(\boldsymbol{x})$ be functions of $L_B(\Omega)$ such that*

$$\|v^m\|_B \le C, \quad m = 1, 2, \dots,$$
$$v^m \to v \quad \text{almost everywhere in } \Omega, \quad m \to \infty.$$

*Then $v^m \rightharpoonup v$ weakly in $L_B(\Omega)$ as $m \to \infty$.*

The proof of Lemma 4.4 for $B(z) = |z|^a$, $a > 1$, an be found in [19, Chap. I, Sec. 1.4, Lemma 1.3]; for the $N$-function $B(z)$ the lemma can be proved similarly.

**Lemma 4.5.** *If $u = w + \psi$ is an entropic solution of the problem* (1.1), (1.2), *then the inequality* (3.11$\psi$) *is valid for any function $\xi \in \mathring{H}^1_{\mathrm{B}}(\Omega) \cap L_\infty(\Omega)$.*

*Proof.* According to the definition of the space $\mathring{H}^1_{\mathrm{B}}(\Omega)$, there exists a sequence $\xi^m \in C_0^\infty(\Omega)$ such that

$$\nabla\xi^m \to \nabla\xi \quad \text{in } \mathrm{L}_{\mathrm{B}}(\Omega) \text{ for } m \to \infty.$$

Hence, according to (2.12) and (2.10), we conclude the convergence $\xi^m \to \xi$ and $\nabla\xi^m \to \nabla\xi$ in $L_{1,\mathrm{loc}}(\Omega)$ as $m \to \infty$, which means that one can select a subsequence (denote it by the same symbol) that $\xi^m \to \xi$ and $\nabla\xi^m \to \nabla\xi$ almost everywhere in $\Omega$. Then for any $k > 0$ we have the convergences

$$T_k(w - \xi^m) \to T_k(w - \xi), \quad \nabla T_k(w - \xi^m) \to \nabla T_k(w - \xi) \quad \text{almost everywhere in } \Omega \text{ as } m \to \infty. \tag{4.12}$$

Let

$$\widehat{k} = k + \sup_{m=1,2,\dots}\Big(\|\xi^m\|_\infty, \ \|\xi\|_\infty\Big);$$

then

$$\big|\nabla T_k(w - \xi^m)\big| \le \big|\nabla T_{\widehat{k}}(w)\big| + \big|\nabla\xi^m\big|, \quad \boldsymbol{x} \in \Omega, \quad m = 1, 2, \dots.$$

Since the converging sequence $\nabla\xi^m$ is bounded in $\mathrm{L}_{\mathrm{B}}(\Omega)$, we conclude that, according to (4.1), the norms $\|\nabla T_k(w - \xi^m)\|_{\mathrm{B}}$ are bounded. Using (4.12) and Lemma 4.4, for any $k > 0$ we obtain

$$\nabla T_k(w - \xi^m) \rightharpoonup \nabla T_k(w - \xi) \quad \text{in } \mathrm{L}_{\mathrm{B}}(\Omega) \text{ as } m \to \infty. \tag{4.13}$$

Now we pass to the limit as $m \to \infty$ in the inequality

$$\Big[a_{0\psi}(\boldsymbol{x}, w)T_k(w - \xi^m) + \boldsymbol{a}_\psi(\boldsymbol{x}, w, \nabla w) \cdot \nabla T_k(w - \xi^m)\Big] \le 0.$$

Since $a_{0\psi}(\boldsymbol{x}, w) \in L_1(\Omega)$, using (4.12), according to the Lebesgue theorem, we can pass to the limit as $m \to \infty$ in the first term. Due to the inclusion

$$\boldsymbol{a}_\psi(\boldsymbol{x}, w, \nabla w)\chi(|w| < \widehat{k}) \in \mathrm{L}_{\overline{\mathrm{B}}}(\Omega)$$

(see (4.2)), using (4.13), we conclude that the second term of the last inequality also has a limit as $m \to \infty$. □

**Remark 4.2.** In the sequel, in order to avoid awkwardness in arguments, instead of statement like "we can extract a subsequence from the sequence $u^m$ (denote it by the same symbol), which converges almost everywhere in $\Omega$ as $m \to \infty$" we simply write "sequence $u^m$ selectively converges almost everywhere in $\Omega$ as $m \to \infty$." Moreover, we will use the term "selective weak convergence."

Let us denote by $\mathcal{F}$ the following class of functions $T \in: C^2(\mathbb{R}) \cap L_\infty(\mathbb{R})$:

$$T(0) = 0; \quad T'(r) \geq 0,\ r \in \mathbb{R}; \quad T'(r) = 0,\ |r| \geq k;$$
$$T(-r) = -T(r),\ r \in \mathbb{R}; \quad T''(r) \leq 0,\ r \geq 0.$$

**Lemma 4.6.** *An entropic solution $u = w + \psi$ of the problem* (1.1), (1.2) *satisfies the inequality*

$$\Big[a_{0\psi} T(w - \xi) + \boldsymbol{a}_\psi \cdot \nabla T(w - \xi)\Big] \leq 0 \tag{3.11$\psi T$}$$

*for any $\xi \in C_0^1(\Omega)$ and all $T \in \mathcal{F}$.*

*Proof.* Obviously, (3.11$\psi T$) is valid for $T(r) = \sum a_j T_{k_j}(r)$, $a_j \geq 0$. In the general case, we can approximate functions $T \in \mathcal{F}$ in the norm of $C^1(\mathbb{R})$ by linear combinations (see [4, Lemma 3.2]). □

**Lemma 4.7** (see [17, Lemma 4])**.** *Let $Q$ be a bounded domain and, in addition, if the condition* (2.13) *is fulfilled, then let $M(z)$ be an arbitrary $N$-function, whereas if the condition* (2.11) *is fulfilled, then let $M(z) \prec\prec B_*(z)$. Then the embedding operator $\mathring{H}^1_{\mathrm{B}}(Q) \subset L_M(Q)$ is completely continuous.*

**Lemma 4.8** (see [7, lemma 2])**.** *Let $(X, \mathcal{T}, \mathrm{mes})$ be a measurable space such that*$\mathrm{mes}(X) < \infty$*. Assume that $\gamma : X \to [0, +\infty]$ is a measurable function such that* $\mathrm{mes}\big(\{\boldsymbol{x} \in X : \gamma(\boldsymbol{x}) = 0\}\big) = 0$*. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that the inequality*

$$\int\limits_Q \gamma(\boldsymbol{x}) d\boldsymbol{x} \leq \delta$$

*implies* $\mathrm{mes}(Q) \leq \varepsilon$.

## 5. Existence of Generalized Solutions

Consider the Dirichlet problem for the following second-order anisotropic quasilinear elliptic equation:

$$\sum_{i=1}^n \big(a_i(\boldsymbol{x}, u, \nabla u)\big)_{x_i} - a_0(\boldsymbol{x}, u, \nabla u) = 0, \quad \boldsymbol{x} \in \Omega; \tag{5.1}$$

$$u\Big|_{\partial\Omega} = 0. \tag{5.2}$$

Assume that the functions $a_i(\boldsymbol{x}, s_0, \boldsymbol{s})$, $i = 0, \ldots, n$, are measurable with respect to $\boldsymbol{x} \in \Omega$ for $\mathbf{s} = (s_0, \boldsymbol{s}) = (s_0, s_1, \ldots, s_n) \in \mathbb{R}^{n+1}$ and continuous with respect to $\mathbf{s} \in \mathbb{R}^{n+1}$ for almost all $\boldsymbol{x} \in \Omega$. Let $\mathbf{s} \cdot \mathbf{t}$ be the scalar product of the vectors $\mathbf{s} = (s_0, \boldsymbol{s})$ and $\mathbf{t} = (t_0, \boldsymbol{t}) \in \mathbb{R}^{n+1}$ and

$$\mathbf{a}(\boldsymbol{x}, \mathbf{s}) = \Big(a_0(\boldsymbol{x}, \mathbf{s}),\ a_1(\boldsymbol{x}, \mathbf{s}),\ \ldots,\ a_n(\boldsymbol{x}, \mathbf{s})\Big).$$

Assume that there exist nonnegative measurable functions $\phi(\boldsymbol{x})$, $\Phi(\boldsymbol{x}) \in L_1(\Omega)$ such that for almost all $\boldsymbol{x} \in \Omega$ and any $\mathbf{s} = (s_0, \boldsymbol{s}) \in \mathbb{R}^{n+1}$, the inequalitiy (3.4) holds, and

$$\overline{\mathbf{B}}(\mathbf{a}(\boldsymbol{x}, \mathbf{s})) \le \widehat{a}\mathbf{B}(\mathbf{s}) + \Phi(\boldsymbol{x}), \quad \overline{\mathbf{B}}(\mathbf{a}) = \sum_{i=0}^{n} \overline{B}_i(a_i), \quad \mathbf{B}(\mathbf{s}) = \sum_{i=0}^{n} B_i(s_i) = B_0(s_0) + \mathrm{B}(\boldsymbol{s}); \tag{5.3}$$

$$\mathbf{a}(\boldsymbol{x}, \mathbf{s}) \cdot \mathbf{s} \ge \overline{a}\mathbf{B}(\mathbf{s}) - \phi(\boldsymbol{x}), \tag{5.4}$$

where the $N$-functions $B_0(z), B_1(z), \ldots, B_n(z)$ and their complementary $N$-functions $\overline{B}_0(z)$, $\overline{B}_1(z), \ldots,$ $\overline{B}_n(z)$ satisfy the $\Delta_2$-condition.

We denote by $\mathbf{L}_{\overline{\mathbf{B}}}(\Omega)$ the space $L_{\overline{B}_0}(\Omega) \times L_{\overline{B}_1}(\Omega)$ with the norm

$$\|\mathbf{v}\|_{\overline{\mathbf{B}}} = \|v_0\|_{\overline{B}_0} + \|v_1\|_{\overline{B}_1} + \ldots + \|v_n\|_{\overline{B}_n}, \quad \mathbf{v} = (v_0, v_1, \ldots, v_n) \in \mathbf{L}_{\overline{\mathbf{B}}}(\Omega).$$

We define the Sobolev–Orlicz space $\mathring{W}^1_{\mathbf{B}}(\Omega)$ as the completion of the space $C_0^\infty(\Omega)$ with respect to the norm

$$\|v\|_{\mathring{W}^1_{\mathbf{B}}(\Omega)} = \|v\|_{B_0} + \|v\|_{\mathring{H}^1_{\mathrm{B}}(\Omega)}.$$

If the condition (2.11) holds, we assume that

$$B_0(z) \prec\prec B_*(z), \tag{5.5}$$

whereas (2.13) holds, let $B_0(z)$ be an arbitrary $N$-function.

We assume that

$$B_i(z) \prec B_0(z), \quad i = 1, 2, \ldots, n. \tag{5.6}$$

From the condition (5.3), using (2.7), for $u \in \mathring{W}^1_{\mathbf{B}}(\Omega)$ we obtain the estimate

$$\begin{aligned} \big\|\mathbf{a}(\boldsymbol{x}, u, \nabla u)\big\|_{\overline{\mathbf{B}}} &= \sum_{i=0}^{n} \big\|a_i(\boldsymbol{x}, u, \nabla u)\big\|_{\overline{B}_i} \\ &\le \sum_{i=0}^{n} \int_{\Omega} \overline{B}_i\big(a_i(\boldsymbol{x}, u, \nabla u)\big) d\boldsymbol{x} + n + 1 \le \widehat{a}\big\|\mathrm{B}(\nabla u)\big\|_1 + \widehat{a}\big\|B_0(u)\big\|_1 + \|\Phi\|_1 + n + 1. \end{aligned} \tag{5.7}$$

Introduce the notation $v_{x_0} = v$. Further, by an element $\mathbf{a}(\boldsymbol{x}, u, \nabla u) \in \mathbf{L}_{\overline{\mathbf{B}}}(\Omega)$, for $v(\boldsymbol{x}) \in \mathring{W}^1_{\mathbf{B}}(\Omega)$ we define the functional $\mathbf{A}(u)$ by the formula

$$\langle \mathbf{A}(u), v \rangle = \Big[\mathbf{a}(\boldsymbol{x}, u, \nabla u) \cdot (v, \nabla v)\Big] = \sum_{i=0}^{n} \big[a_i v_{x_i}\big]. \tag{5.8}$$

Using the Hölder inequality (2.9), for functions $u(\boldsymbol{x}), v(\boldsymbol{x}) \in \mathring{W}^1_{\mathbf{B}}(\Omega)$ we obtain the inequality

$$\Big|\langle \mathbf{A}(u), v \rangle\Big| \le 2\sum_{i=0}^{n} \|a_i\|_{\overline{B}_i} \|v_{x_i}\|_{B_i} \le 2\big\|\mathbf{a}(\boldsymbol{x}, u, \nabla u)\big\|_{\overline{\mathbf{B}}} \|v\|_{\mathring{W}^1_{\mathbf{B}}(\Omega)}. \tag{5.9}$$

It follows from (5.9) and (5.7) that the functional of $\mathbf{A}(u)$ defined by (5.8) in the space $\mathring{W}^1_{\mathbf{B}}(\Omega)$ is bounded.

**Definition 5.1.** A *generalized solution* of the problem (5.1), (5.2) is a function $u(\boldsymbol{x}) \in \mathring{W}^1_{\mathbf{B}}(\Omega)$ satisfying the integral identity

$$\langle \mathbf{A}(u), v \rangle = 0 \tag{5.10}$$

for any function $v(\boldsymbol{x}) \in \mathring{W}^1_{\mathbf{B}}(\Omega)$.

**Theorem 5.1.** *If the conditions* (3.4), (5.3)–(5.6) *are fulfilled, then there exists a generalized solution of the problem* (5.1), (5.2).

The existence of a solution of the problem (5.1), (5.2) with a monotonic operator $\mathbf{A}$ is proved in [16]. For anisotropic equation with power nonlinearities, the existence of a solution of the Dirichlet problem was proved by F. Browder (see [11]); it was based on an abstract theorem for pseudo-monotonic operators.

**Definition 5.2.** An operator $A : V \to V'$ is called *pseudo-monotonic* if

(i) $A$ is a bounded operator;

(ii) the conditions "$u^j \rightharpoonup u$ weakly in $V$" and

$$\lim_{j\to\infty} \sup \langle A(u^j), u^j - u\rangle \le 0$$

imply that for any $v \in V$, the following inequality holds:

$$\lim_{j\to\infty} \inf \langle A(u^j), u^j - v\rangle \ge \langle A(u), u - v\rangle. \tag{5.11}$$

**Lemma 5.1** (see [19, Chap. II, Sec. 2, Theorem 2.7])**.** *Let $V$ be a reflexive, separable Banach space. Assume that an operator $A : V \to V'$ is pseudo-monotonic and coercive, i.e.,*

$$\frac{\langle A(u), u\rangle}{\|u\|} \to \infty, \quad \|u\| \to \infty. \tag{5.12}$$

*Then the mapping $A : V \to V'$ is surjective, i.e., for any $F \in V'$ there exists $u \in V$ such that $A(u) = F$.*

Before checking the conditions of Lemma 5.1, we present some additional estimates and remarks.

**Remark 5.1.** It follows from (2.7) and (2.8) that, if a $N$-function $B(z)$ satisfies the $\Delta_2$-condition, then the boundedness of the set of functions in the space $L_B(\Omega)$ is equivalent to the boundedness on the average. Therefore, the boundedness of the set $\Theta \subset \mathring{W}^1_{\mathbf{B}}(\Omega)$ with respect to the norm is equivalent to the boundedness of the set $\{\|\mathbf{B}(u)\|_1,\ u \in \Theta\}$.

**Remark 5.2.** The space $\mathring{W}^1_{\mathbf{B}}(\Omega)$ is a reflexive, separable Banach space.

**Proposition 5.1.** *Assume that the conditions* (3.4), (5.3)–(5.6) *are fulfilled. Then the operator*

$$\mathbf{A} : \mathring{W}^1_{\mathbf{B}}(\Omega) \to \left(\mathring{W}^1_{\mathbf{B}}(\Omega)\right)',$$

*defined by* (5.8)*, is pseudo-monotonic.*

*Proof.* The boundedness of the operator $\mathbf{A}$ follows from the estimates (5.9) and (5.7). Consider a sequence $\{u^j\}_{j=1}^{\infty}$ in the space $\mathring{W}^1_{\mathbf{B}}(\Omega)$ such that

$$u^j \rightharpoonup u \quad \text{weakly in } \mathring{W}^1_{\mathbf{B}}(\Omega), \quad j \to \infty; \tag{5.13}$$

$$\lim_{j\to\infty} \sup \langle \mathbf{A}(u^j), u^j - u\rangle \le 0. \tag{5.14}$$

We show that

$$\mathbf{A}(u^j) \rightharpoonup \mathbf{A}(u) \quad \text{weakly in } \left(\mathring{W}^1_{\mathbf{B}}(\Omega)\right)', \quad j \to \infty; \tag{5.15}$$

$$\langle \mathbf{A}(u^j), u^j - u\rangle \to 0, \quad j \to \infty. \tag{5.16}$$

Obviously, (5.15) and (5.16) imply (5.11).

First, the convergence (5.13) and the inequality (2.8) imply the estimtes

$$\|u^j\|_{\mathring{W}^1_{\mathbf{B}}(\Omega)} \le C_1, \quad j = 1, 2, \dots; \tag{5.17}$$

$$\|B_0(u^j)\|_1 + \|\mathrm{B}(\nabla u^j)\|_1 \le C_2, \quad j = 1, 2, \dots. \tag{5.18}$$

In addition, combining (5.7) and (5.18), we obtain the estimate

$$\big\|\mathbf{a}(\boldsymbol{x},u,\nabla u)\big\|_{\overline{\mathbf{B}}}=\sum_{i=0}^{n}\big\|a_i(\boldsymbol{x},u^j,\nabla u^j)\big\|_{\overline{B}_i}\le C_3,\quad j=1,2,\dots. \tag{5.19}$$

Fix arbitrary $R>0$. In the case of (2.11), by Lemma 4.7, the space $\mathring{W}^1_{\mathbf{B}}(\Omega(R+1))$ is compactly embedded in $L_P(\Omega(R+1))$ for any $N$-function $M(z)$ satisfying the condition $M(z)\prec\prec B_*(z)$. In the case of (2.13), for any $N$-function $M(z)$, by Lemma 4.7, the space $\mathring{W}^1_{\mathbf{B}}(\Omega(R+1))$ is compactly embedded in $L_M(\Omega(R+1))$. Under the conditions (5.5) and (5.6), in both cases (2.11) and (2.13), the space $\mathring{W}^1_{\mathbf{B}}(\Omega(R+1))$ is compactly embedded in the space $L_{B_i}(\Omega(R+1))$, $i=0,\dots,n$.

From the condition (5.6), using (2.3), we establish the existence of $z_0>0$ such that

$$B_i(z)\le C_4B_0(z),\quad |z|\ge z_0,\quad i=1,2,\dots,n. \tag{5.20}$$

Let $\eta_R(r)=\min(1,\max(0,R+1-r))$. Using (2.5), (5.20), and (5.18), we deduce the inequalities

$$\int\limits_{\Omega(R+1)}\Big(\mathrm{B}\Big(\nabla\big(u^j\eta_R(|\boldsymbol{x}|)\big)\Big)+B_0\Big(u^j\eta_R\big(|\boldsymbol{x}|\big)\Big)\Big)\,d\boldsymbol{x}=\int\limits_{\Omega(R+1)}\Big(\mathrm{B}\Big(\nabla u^j\eta_R+u^j\nabla\eta_R\Big)+B_0\big(u^j\eta_R\big)\Big)\,d\boldsymbol{x}$$
$$\le\int\limits_{\Omega(R+1)}\Big(C_5\Big\{\mathrm{B}(\nabla u^j)+\mathrm{B}(u^j)\Big\}+B_0(u^j)\Big)\,d\boldsymbol{x}\le C_6\int\limits_{\Omega(R+1)}\big(\mathrm{B}(\nabla u^j)+\mathrm{B}(z_0)+B_0(u^j)\big)\,d\boldsymbol{x}$$
$$\le C_6\Big(\big\|B_0(u^j)\big\|_{1,\Omega(R+1)}+\big\|\mathrm{B}(\nabla u^j)\big\|_{1,\Omega(R+1)}\Big)+C_7\operatorname{mes}\Omega(R+1)\le C_8(R),\quad j=1,2,\dots.$$

Therefore (see Remark 5.1), the sequence $\{u^j\eta_R\}_{j=1}^{\infty}$ is bounded in the space $\mathring{W}^1_{\mathbf{B}}(\Omega(R+1))$. Due to the compactness of the embeddings

$$\mathring{W}^1_{\mathbf{B}}(\Omega(R+1))\subset L_{B_i}(\Omega(R+1)),\quad i=0,\dots,n,$$

the following strict convergences hold:

$$u^j\eta_R\to u\eta_R\quad\text{in } L_{B_i}(\Omega(R+1)),\ i=0,1,\dots,n,\quad j\to\infty.$$

Therefore, we conclude the strict convergences

$$u^j\to u\quad\text{in } L_{B_i}(\Omega(R)),\ i=0,1,\dots,n,\quad j\to\infty, \tag{5.21}$$

and the selective convergence $u^j\to u$ almost everywhere in $\Omega(R)$. The convergence

$$u^j\to u\quad\text{almost everywhere in }\Omega,\quad j\to\infty, \tag{5.22}$$

can be proved by the diagonal process.

We set

$$A^j(\boldsymbol{x})=\sum_{i=0}^{n}\Big(a_i\big(\boldsymbol{x},u^j,\nabla u^j\big)-a_i\big(\boldsymbol{x},u,\nabla u\big)\Big)(u^j-u)_{x_i},\quad j=1,\dots;$$

then

$$\langle\mathbf{A}(u^j)-\mathbf{A}(u),u^j-u\rangle=\int\limits_{\Omega}A^j(\boldsymbol{x})d\boldsymbol{x},\quad j=1,\dots.$$

According to (5.13), (5.14), we have

$$\limsup_{j\to\infty}\int\limits_{\Omega}A^j(\boldsymbol{x})d\boldsymbol{x}\le 0. \tag{5.23}$$

We write $A^j(\boldsymbol{x})$ as follows:

$$A^j(\boldsymbol{x}) = \sum_{i=1}^{n} \Big( a_i\big(\boldsymbol{x}, u^j, \nabla u^j\big) - a_i\big(\boldsymbol{x}, u^j, \nabla u\big)\Big)(u^j - u)_{x_i}$$

$$+ \sum_{i=1}^{n} \Big( a_i\big(\boldsymbol{x}, u^j, \nabla u\big) - a_i\big(\boldsymbol{x}, u, \nabla u\big)\Big)(u^j - u)_{x_i}$$

$$+ \Big( a_0\big(\boldsymbol{x}, u^j, \nabla u^j\big) - a_0\big(\boldsymbol{x}, u, \nabla u\big)\Big)(u^j - u)$$

$$= q^j(\boldsymbol{x}) + r^j(\boldsymbol{x}) + s^j(\boldsymbol{x}), \quad j = 1, \dots. \tag{5.24}$$

We show that

$$r^j(\boldsymbol{x}) \to 0 \quad \text{almost everywhere in } \Omega, \quad j \to \infty, \tag{5.25}$$

$$s^j(\boldsymbol{x}) \to 0 \quad \text{almost everywhere in } \Omega, \quad j \to \infty. \tag{5.26}$$

Consider the Nemytsky operators $A_i(u) = a_i(\boldsymbol{x}, u, \nabla v)$, $i = 1, 2, \dots, n$, for fixed $v \in \mathring{H}^1_{\mathrm{B}}(\Omega)$ for $\boldsymbol{x} \in \Omega(R)$, $R > 0$. Applying the estimate (5.3), we deduce the inequality

$$\overline{B}_i(a_i(\boldsymbol{x}, u, \nabla v)) \leq \widehat{a}\mathrm{B}(\nabla v) + B_0(u) + \Phi(\boldsymbol{x}),$$

with the function $\widehat{a}\mathrm{B}(\nabla v) + \Phi(\boldsymbol{x}) \in L_1(\Omega)$. According to [20, Chap. III, Sec. 17, Theorem 17.5], the operators $A_i$ act from $L_{B_0}(\Omega(R))$ into $L_{\overline{B}_i}(\Omega(R))$. Moreover, from [20, Chap. III, Sec. 17, Theorem 17.3] we conclude the continuity of the operators $A_i$, $i = 1, 2, \dots, n$, in $L_{B_0}(\Omega(R))$ for any $R > 0$.

Applying the inequality (2.9), we obtain

$$\int\limits_{\Omega(R)} \big|r^j(\boldsymbol{x})\big| d\boldsymbol{x} \leq 2 \sum_{i=1}^{n} \Big\| a_i\big(\boldsymbol{x}, u^j, \nabla u\big) - a_i\big(\boldsymbol{x}, u, \nabla u\big) \Big\|_{\overline{B}_i, \Omega(R)} \big\| (u^j - u)_{x_i} \big\|_{B_i, \Omega(R)}.$$

Due to the convergence of $u^j \to u$ in $L_{B_0}(\Omega(R))$ as $j \to \infty$ (see (5.21)) and the continuity of the operators $A_i : L_{B_0}(\Omega(R)) \to L_{\overline{B}_i}(\Omega(R))$, $i = 1, 2, \dots, n$, the first factor tends to zero and the second factor is uniformly bounded (see (5.17)). Thus, we see that for any $R > 0$

$$r^j(\boldsymbol{x}) \to 0, \quad j \to \infty,$$

in $L_1(\Omega(R))$. Hence, using the diagonal process, we conclude the convergence (5.25).

Using the inequality (2.9), we deduce

$$\int\limits_{\Omega(R)} \big|s^j(\boldsymbol{x})\big| d\boldsymbol{x} \leq 2 \Big\| a_0\big(\boldsymbol{x}, u^j, \nabla u^j\big) - a_0\big(\boldsymbol{x}, u, \nabla u\big) \Big\|_{\overline{B}_0, \Omega(R)} \| u^j - u \|_{B_0, \Omega(R)}.$$

The first factor is uniformly bounded (see (5.19))), while the second factor tends to zero (see (5.21)); therefore, for any $R > 0$

$$s^j(\boldsymbol{x}) \to 0, \quad j \to \infty,$$

in $L_1(\Omega(R))$. Hence, using the diagonal process, we conclude the convergence (5.26).

Next, we write $A^j(\boldsymbol{x})$ in the form

$$A^j(\boldsymbol{x}) = \sum_{i=1}^{n} a_i\big(\boldsymbol{x}, u^j, \nabla u^j\big) u^j_{x_i} + a_0\big(\boldsymbol{x}, u^j, \nabla u^j\big) u^j - g^j(\boldsymbol{x}), \quad j = 1, \dots, \tag{5.27}$$

where

$$g^j(\boldsymbol{x}) = \sum_{i=1}^{n} a_i\big(\boldsymbol{x}, u, \nabla u\big)(u^j - u)_{x_i} + a_0\big(\boldsymbol{x}, u, \nabla u\big)(u^j - u)$$
$$+ \sum_{i=1}^{n} a_i\big(\boldsymbol{x}, u^j, \nabla u^j\big)u_{x_i} + a_0\big(\boldsymbol{x}, u^j, \nabla u^j\big)u \in L_1(\Omega), \quad j = 1, \dots .$$

Using the inequality (2.1) for $\varepsilon \in (0,1)$, we obtain

$$\big|g^j(\boldsymbol{x})\big| \le \varepsilon\Big(B_0(u^j) + \mathrm{B}(\nabla u^j) + \overline{\mathbf{B}}(\mathbf{a}(\boldsymbol{x}, u^j, \nabla u^j))\Big) + C_9(\varepsilon)\Big(B_0(u) + \mathrm{B}(\nabla u) + \overline{\mathbf{B}}(\mathbf{a}(\boldsymbol{x}, u, \nabla u))\Big).$$

Applying (5.3), we deduce the inequality

$$\big|g^j(\boldsymbol{x})\big| \le \varepsilon C_{10}\Big(\mathrm{B}(\nabla u^j) + B_0(u^j)\Big) + C_{11}(\varepsilon)\Big(\mathrm{B}(\nabla u) + B_0(u) + \Phi(\boldsymbol{x})\Big). \tag{5.28}$$

Using (5.4), from (5.27) we deduce the inequality

$$A^j(\boldsymbol{x}) \ge \overline{a}\Big(\mathrm{B}(\nabla u^j) + B_0(u^j)\Big) - \phi(\boldsymbol{x}) - \big|g^j(\boldsymbol{x})\big|. \tag{5.29}$$

Combining (5.28), (5.29) and choosing $\varepsilon < \overline{a}/C_{10}$, we obtain the estimates

$$A^j(\boldsymbol{x}) \ge C_{12}\Big(\mathrm{B}(\nabla u^j) + B_0(u^j)\Big) - \Phi_u(\boldsymbol{x}), \quad j = 1, \dots, \tag{5.30}$$

with the nonnegative function

$$\Phi_u(\boldsymbol{x}) = \phi(\boldsymbol{x}) + C_{11}\Big(\mathrm{B}(\nabla u) + B_0(u) + \Phi(\boldsymbol{x})\Big) \in L_1(\Omega),$$

which is finite almost everywhere in $\Omega$.

Let $A^j(\boldsymbol{x}) = A^{j+}(\boldsymbol{x}) - A^{j-}(\boldsymbol{x})$, where $A^{j+}(\boldsymbol{x})$ and $A^{j-}(\boldsymbol{x})$ are the positive and negative parts of $A^j(\boldsymbol{x})$, respectively. From (5.30) we have the estimates

$$A^{j+}(\boldsymbol{x}) \ge C_{12}\Big(\mathrm{B}(\nabla u^j) + B_0(u^j)\Big) - \Phi_u(\boldsymbol{x}), \quad j = 1, \dots . \tag{5.31}$$

If $\chi^j(\boldsymbol{x})$ is the characteristic function of the set $\{\boldsymbol{x} : A^{j-}(\boldsymbol{x}) > 0\}$, then

$$-A^{j-} = \chi^j q^j + \chi^j r^j + \chi^j s^j,$$

and, according to (5.25) and (5.26),

$$\chi^j r^j(\boldsymbol{x}) \to 0, \quad \chi^j s^j(\boldsymbol{x}) \to 0$$

almost everywhere in $\Omega$ as $j \to \infty$. Due to (5.4), $\chi^j q^j(\boldsymbol{x}) \ge 0$ almost everywhere in $\Omega$; then $A^{j-}(\boldsymbol{x}) \to 0$ almost everywhere in $\Omega$ as $j \to \infty$.

Therefore, from (5.30) we obtain the estimate

$$A^j(\boldsymbol{x}) \ge -\Phi_u(\boldsymbol{x}), \quad j = 0, 1, \dots .$$

Hence we have $A^{j-}(\boldsymbol{x}) \le \Phi_u(\boldsymbol{x})$, $j = 1, \dots$. Then according to the Lebesgue theorem,

$$A^{j-}(\boldsymbol{x}) \to 0 \quad \text{in } L_1(\Omega), \quad j \to \infty. \tag{5.32}$$

Therefore, according to (5.23),

$$0 \le \lim_{j\to\infty} \sup \int_\Omega A^{j+}(\boldsymbol{x}) d\boldsymbol{x} = \lim_{j\to\infty} \sup \int_\Omega A^j(\boldsymbol{x}) d\boldsymbol{x} + \lim_{j\to\infty} \sup \int_\Omega A^{j-}(\boldsymbol{x}) d\boldsymbol{x} \le 0.$$

Consequently,

$$A^{j+}(\boldsymbol{x}) \to 0 \quad \text{in } L_1(\Omega), \quad j \to \infty. \tag{5.33}$$

Thus, from (5.32) and (5.33) we conclude the convergence

$$A^j(\boldsymbol{x}) \to 0 \quad \text{in } L_1(\Omega), \quad j \to \infty, \tag{5.34}$$

and also the selective convergences

$$A^{j+}(\boldsymbol{x}) \to 0, \quad A^j(\boldsymbol{x}) \to 0 \quad \text{almost everywhere in } \Omega, \quad j \to \infty. \tag{5.35}$$

Now we prove the convergence

$$u^j_{x_i}(\boldsymbol{x}) \to u_{x_i}(\boldsymbol{x}) \quad \text{almost everywhere in } \Omega, \quad \alpha = 1,2,\dots,n, \quad j \to \infty. \tag{5.36}$$

We denote by $\Omega' \subset \Omega$ the subset of points of full measure for which the convergences (5.22) and (5.35) hold and the inequalities (3.4) (5.3), and (5.4) are valid.

On the contrary, assume that at some point $\boldsymbol{x}^* \in \Omega'$ the convergence is violated. We introduce the notation

$$s^j_0 = u^j(\boldsymbol{x}^*), \quad s_0 = u(\boldsymbol{x}^*),$$
$$\boldsymbol{s}^j = \Big(s^j_1,\ s^j_2,\ \dots,\ s^j_n\Big) = \Big(u^j_{x_1}(\boldsymbol{x}^*),\ u^j_{x_2}(\boldsymbol{x}^*),\ \dots,\ u^j_{x_n}(\boldsymbol{x}^*)\Big),$$
$$\boldsymbol{s} = \Big(s_1,\ s_2,\ \dots,\ s_n\Big) = \Big(u_{x_1}(\boldsymbol{x}^*),\ u_{x_2}(\boldsymbol{x}^*),\ \dots,\ u_{x_n}(\boldsymbol{x}^*)\Big).$$

Assume that the sequence $\{\mathrm{B}(\boldsymbol{s}^j)\}_{j=1}^{\infty}$ is unbounded. Then the estimate (5.31) implies the unboundedness of the sequence $A^{j+}(\boldsymbol{x}^*)$, $j = 1,2,\dots$, which contradicts (5.35). Therefore, the sequence $\{s^j\}_{j=1}^{\infty}$ is bounded.

Let $\boldsymbol{s}^* = \big(s^*_1,\ s^*_2,\ \dots,\ s^*_n\big)$ be one of the partial limits of $\boldsymbol{s}^j = \big(s^j_1,\ s^j_2,\ \dots,\ s^j_n\big)$ as $j \to \infty$. Then, taking into account (5.22), we obtain

$$s^j_0 \to s_0, \quad s^j_i \to s^*_i, \quad i = 1,2,\dots,n, \quad j \to \infty.$$

Therefore, using (5.25), (5.26), and (5.35) from (5.24) and the continuity of $a_i(\boldsymbol{x}^*, s_0, \boldsymbol{s})$ with respect to $\mathbf{s} = (s_0, \boldsymbol{s})$ we obtain

$$A^j(\boldsymbol{x}^*) \to \sum_{i=1}^{n} \Big(a_i(\boldsymbol{x}^*, s_0, \boldsymbol{s}^*) - a_i(\boldsymbol{x}^*, s_0, \boldsymbol{s})\Big)(s^*_i - s_i) = 0;$$

therefore, according to (3.4), we have $\boldsymbol{s} = \boldsymbol{s}^*$. This contradicts the fact that there is no convergence at the point $\boldsymbol{x}^*$.

Thus, from (5.22) and (5.36) and the continuity of $a_i(\boldsymbol{x}, s_0, \boldsymbol{s})$ with respect to $\mathbf{s} = (s_0, \boldsymbol{s})$ we conclude that as $j \to \infty$

$$a_i\big(\boldsymbol{x}, u^j, \nabla u^j\big) \to a_i\big(\boldsymbol{x}, u, \nabla u\big) \quad \text{almost everywhere in } \Omega, \quad i = 0,1,\dots,n.$$

In addition, the boundedness of $a_i\big(\boldsymbol{x}, u^i, \nabla u^i\big)$ in $L_{\overline{B}_i}(\Omega)$, $i = 0,1,\dots,n$, follows from (5.19). Using Lemma 4.4, we find the weak convergences

$$a_i\big(\boldsymbol{x}, u^j, \nabla u^j\big) \rightharpoonup a_i\big(\boldsymbol{x}, u, \nabla u\big) \quad \text{in } L_{\overline{B}_i}(\Omega), \quad i = 0,1,2,\dots,n. \tag{5.37}$$

The weak convergence (5.15) follows from (5.37).

To complete the proof, we note that (5.16) is implied from (5.13) and (5.34):

$$\langle \mathbf{A}(u^j), u^j - u \rangle = \langle \mathbf{A}(u^j) - \mathbf{A}(u), u^j - u \rangle + \langle \mathbf{A}(u), u^j - u \rangle \to 0, \quad j \to \infty.$$

Proposition 5.1 is proved. □

*Proof of Theorem* 5.1. The coercirity of the operator $\mathbf{A}$ is proved in [17]. From Proposition 5.1, according to Lemma 5.1, it follows that there exists a function $u \in \mathring{W}^1_{\mathbf{B}}(\Omega)$ such that $\mathbf{A}(u) = \mathbf{0}$. Thus, for any $v \in \mathring{W}^1_{\mathbf{B}}(\Omega)$ the identity (5.10) is valid. □

## 6. Existence of Entropic Solutions

*Proof of Theorem* 3.1.

*Step* 1. Choose a sequence of functions $A^m_{0\psi}(\boldsymbol{x}) \in C_0^\infty(\Omega)$ such that

$$A^m_{0\psi} \to A^0_{0\psi} \quad \text{in } L_1(\Omega), \quad m \to \infty, \tag{6.1}$$

and

$$\|A^m_{0\psi}\|_1 \leq \|A^0_{0\psi}\|_1, \quad m = 1, 2, \ldots. \tag{6.2}$$

Consider the equation

$$\sum_{i=1}^n \big(a_i^m(\boldsymbol{x}, w, \nabla w)\big)_{x_i} = a_0^m(\boldsymbol{x}, w), \quad \boldsymbol{x} \in \Omega, \tag{6.3}$$

with the functions $a_i^m(\boldsymbol{x}, s_0, \boldsymbol{s}) = a_{i\psi}(\boldsymbol{x}, T_m s_0, \boldsymbol{s})$ and $a_0^m(\boldsymbol{x}, s_0) = A^m_{0\psi}(\boldsymbol{x}) + b^m(\boldsymbol{x}, s_0) + B_0'(s_0)/m$. Here $b^m(\boldsymbol{x}, s_0) = T_m b_\psi(\boldsymbol{x}, s_0)\kappa_m(\boldsymbol{x})$ and $\kappa_m(\boldsymbol{x})$ is the characteristic function of the set $\Omega(m) = \{\boldsymbol{x} \in \Omega : |\boldsymbol{x}| < m\}$. We assume that continuously differentiable $N$-functions $B_0$ and $\overline{B}_0$ satisfy the $\Delta_2$-condition and the requirements (5.5), (5.6) are fulfilled.

Obviously,

$$\big|b^m(\boldsymbol{x}, s_0)\big| \leq \big|b_\psi(\boldsymbol{x}, s_0)\big|, \quad s_0 \in \mathbb{R}, \quad \boldsymbol{x} \in \Omega. \tag{6.4}$$

In addition, applying (3.8$\psi$), we obtain the inequalities

$$b^m(\boldsymbol{x}, s_0)s_0 \geq 0, \quad s_0 B_0'(s_0) \geq B_0(s_0) \geq 0, \quad s_0 \in \mathbb{R}, \quad \boldsymbol{x} \in \Omega. \tag{6.5}$$

A generalized solution of the problem (6.3), (5.2) is a function $w^m \in \mathring{W}^1_{\mathbf{B}}(\Omega)$ satisfying the integral identity

$$\Big[A^m_{0\psi}(\boldsymbol{x}) + T_m b_\psi(\boldsymbol{x}, w^m)\kappa_m(\boldsymbol{x}) + \frac{B_0'(w^m)}{m} + \boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big) \cdot \nabla v\Big] = 0 \tag{6.6}$$

for any function $v \in \mathring{W}^1_{\mathbf{B}}(\Omega)$.

For the functions $\boldsymbol{a}^m(\boldsymbol{x}, s_0, \boldsymbol{s}) = \big(a_1^m(\boldsymbol{x}, s_0, \boldsymbol{s}), \ \ldots, \ a_n^m(\boldsymbol{x}, s_0, \boldsymbol{s})\big)$ and $a_0^m(\boldsymbol{x}, s_0)$, we verify the conditions (3.4) (5.3), and (5.4). Obviously,

$$\overline{B}_0(b^m(\boldsymbol{x}, s_0)) = \overline{B}_0\Big(T_m b_\psi(\boldsymbol{x}, s_0)\kappa_m(\boldsymbol{x})\Big) \leq \overline{B}_0(m)\kappa_m(\boldsymbol{x}) \in L_1(\Omega).$$

Therefore, using (2.5), (2.2), and (2.4), we obtain

$$\overline{B}_0(a_0^m(\boldsymbol{x}, s_0)) \leq c\overline{B}_0(A^m_{0\psi}(\boldsymbol{x})) + c\overline{B}_0(b^m(\boldsymbol{x}, s_0)) + c\overline{B}_0\left(\frac{B_0'(s_0)}{m}\right)$$
$$\leq \widehat{a}_m B_0(s_0) + \Phi_m(\boldsymbol{x}), \quad \Phi_m(\boldsymbol{x}) \in L_1(\Omega). \tag{6.7}$$

From (3.3$\psi$) and (6.7) we obtain the inequality (5.3).

Then, applying (2.1) and (6.5), we obtain

$$a_0^m(\boldsymbol{x}, s_0)s_0 = \left(A^m_{0\psi}(\boldsymbol{x}) + b^m(\boldsymbol{x}, s_0) + \frac{B_0'(s_0)}{m}\right)s_0 \geq \frac{B_0(s_0)}{m} - \varepsilon B_0(s_0) - C(\varepsilon)\overline{B}_0(A^m_{0\psi}).$$

Hence, choosing $\varepsilon < 1/m$, we obtain the inequality

$$a_0^m(\boldsymbol{x}, s_0)s_0 \geq \overline{a}_m B_0(s_0) - \phi_m(\boldsymbol{x}), \quad \phi_m(\boldsymbol{x}) \in L_1(\Omega). \tag{6.8}$$

Combining (3.5$\psi$) and (6.8), we deduce the inequality (5.4).

In addition, taking into account (3.4$\psi$), we see that (3.4) is valid. According to Theorem 5.1, there exists a generalized solution $w^m \in \mathring{W}^1_{\mathbf{B}}(\Omega)$ of the problem (6.3), (5.2).

*Step* 2. Consider the function $T_{k,h}(r) = T_k(r - T_h(r))$. Obviously,

$$T_{k,h}(r) = \begin{cases} 0 & \text{for } |r| < h, \\ r - h\,\mathrm{sign}\, r & \text{for } h \le |r| < k + h, \\ k\ \mathrm{sign}\, r & \text{for } |r| \ge k + h. \end{cases}$$

Setting $v = T_{k,h}w^m$ in (6.6) and taking into account (6.5), we obtain

$$\int\limits_{\{\Omega: h \le |w^m| < k+h\}} \boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big) \cdot \nabla w^m d\boldsymbol{x} + k \int\limits_{\{\Omega: |w^m| \ge k+h\}} \left( \big|b^m(\boldsymbol{x}, w^m)\big| + \frac{\big|B_0'(w^m)\big|}{m} \right) d\boldsymbol{x}$$
$$+ \int\limits_{\{\Omega: h \le |w^m| < k+h\}} \left( b^m\big(\boldsymbol{x}, w^m\big) + \frac{B_0'(w^m)}{m} \right) \Big( w^m - h\,\mathrm{sign}\, w^m \Big) d\boldsymbol{x} \le k \int\limits_{\{\Omega: |w^m| \ge h\}} |A^m_{0\psi}| d\boldsymbol{x}. \quad (6.9)$$

Due to (6.5), the following inequality holds for $h \le |w^m|$:

$$\left( b^m(\boldsymbol{x}, w^m) + \frac{B_0'(w^m)}{m} \right) \Big( w^m - h\,\mathrm{sign}\, w^m \Big) \ge 0.$$

Taking this into account, from (6.9) we deduce

$$\int\limits_{\{\Omega: h \le |w^m| < k+h\}} \boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big) \cdot \nabla w^m d\boldsymbol{x}$$
$$+ k \int\limits_{\{\Omega: |w^m| \ge k+h\}} \left( \big|b^m(\boldsymbol{x}, w^m)\big| + \frac{\big|B_0'(w^m)\big|}{m} \right) d\boldsymbol{x} \le k \int\limits_{\{\Omega: |w^m| \ge h\}} |A^m_{0\psi}| d\boldsymbol{x}. \quad (6.10)$$

Applying ($3.5\psi$) and taking into account (6.2), we reduce the inequality (6.10) to the form

$$\overline{a}_\psi \int\limits_{\{\Omega: h \le |w^m| < k+h\}} \mathrm{B}(\nabla w^m) d\boldsymbol{x} + k \int\limits_{\{\Omega: |w^m| \ge k+h\}} \left( \big|b^m(\boldsymbol{x}, w^m)\big| + \frac{\big|B_0'(w^m)\big|}{m} \right) d\boldsymbol{x}$$
$$\le k \int\limits_{\{\Omega: |w^m| \ge h\}} |A^m_{0\psi}| d\boldsymbol{x} + \int\limits_{\{\Omega: h \le |w^m| < k+h\}} \phi_\psi d\boldsymbol{x} \le k \|A^0_{0\psi}\|_1 + \|\phi_\psi\|_1. \quad (6.11)$$

Now, taking $T_k w^m$ as a test function in (6.6), we obtain

$$\int\limits_\Omega \left\{ \boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big) \cdot \nabla T_k w^m + \left( A^m_{0\psi}(\boldsymbol{x}) + b^m(\boldsymbol{x}, w^m) + \frac{B_0'(w^m)}{m} \right) T_k w^m \right\} = 0.$$

Applying (6.2) and (6.5), we deduce

$$\int\limits_{\{\Omega: |w^m| < k\}} \boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big) \cdot \nabla w^m d\boldsymbol{x} + k \int\limits_{\{\Omega: |w^m| \ge k\}} \left( \big|b^m(\boldsymbol{x}, w^m)\big| + \frac{\big|B_0'(w^m)\big|}{m} \right) d\boldsymbol{x}$$
$$\le k \|A^m_{0\psi}\|_1 \le k \|A^0_{0\psi}\|_1.$$

Hence, using the inequality ($3.5\psi$), we obtain

$$\overline{a}_\psi \int\limits_{\{\Omega: |w^m| < k\}} \mathrm{B}(\nabla w^m) d\boldsymbol{x} + k \int\limits_{\{\Omega: |w^m| \ge k\}} \left( \big|b^m(\boldsymbol{x}, w^m)\big| + \frac{\big|B_0'(w^m)\big|}{m} \right) d\boldsymbol{x} \le k \|A^0_{0\psi}\|_1 + \|\phi_\psi\|_1. \quad (6.12)$$

According to (6.4) and (3.9$\psi$) we obtain

$$\sup_{|w^m|\le k}\left(\left|b^m(\boldsymbol{x},w^m)\right|+\frac{\left|B_0'(w^m)\right|}{m}\right)\le\sup_{|w^m|\le k}\Big(\left|b_\psi(\boldsymbol{x},w^m)\right|+\left|B_0'(w^m)\right|\Big)$$
$$\le G_{k+\|\psi\|_\infty}(\boldsymbol{x})+|B_0'(k)|\in L_{1,\mathrm{loc}}(\Omega). \quad (6.13)$$

Combining (6.12) and (6.13), we conclude that for any compact $Q\subset\Omega$ the following inequalities are valid:

$$\left\|b^m(\boldsymbol{x},w^m)\right\|_{1,Q}+\frac{\left\|B_0'(w^m)\right\|_{1,Q}}{m}\le C_1,\quad m=1,2,\dots. \tag{6.14}$$

We prove that

$$\left\|b^m(\boldsymbol{x},w^m)\right\|_1\le C_2,\quad m=1,2,\dots. \tag{6.15}$$

Choosing $k=\delta_0$ ($\delta_0$ from (3.10$\psi$)) in (6.12), we obtain

$$\int\limits_{\{\Omega:|w^m|\ge\delta_0\}}\left|b^m(\boldsymbol{x},w^m)\right|d\boldsymbol{x}\le C_3,\quad m=1,2,\dots. \tag{6.16}$$

From (6.4) and (3.10$\psi$) we obtain

$$\int\limits_{\{\Omega:|w^m|<\delta_0\}}\left|b^m(\boldsymbol{x},w^m)\right|d\boldsymbol{x}\le\int\limits_{\{\Omega:|w^m|<\delta_0\}}\left|b_\psi(\boldsymbol{x},w^m)\right|d\boldsymbol{x}$$
$$\le\int\limits_{\{\Omega:0\le w^m<\delta_0\}}b_\psi(\boldsymbol{x},\delta_0)d\boldsymbol{x}+\int\limits_{\{\Omega:-\delta_0<w^m<0\}}\left|b_\psi(\boldsymbol{x},-\delta_0)\right|d\boldsymbol{x}\le C_4. \quad (6.17)$$

Combining (6.16) and (6.17), we obtain (6.15).

*Step* 3. From (6.12) for any $k>0$ we obtain the estimate

$$\int\limits_\Omega \mathrm{B}(\nabla T_k w^m)d\boldsymbol{x}=\int\limits_{\{\Omega:|w^m|<k\}}\mathrm{B}(\nabla w^m)d\boldsymbol{x}\le kC_5+C_6,\quad m=1,2,\dots. \tag{6.18}$$

Hence, according to Lemma 4.2, we obtain

$$\mathrm{mes}\big(\{\Omega:|w^m|\ge k\}\big)\to 0\quad\text{is uniformly in } m,\quad k\to\infty. \tag{6.19}$$

We prove the following convergence:

$$w^m\to w\quad\text{almost everywhere in }\Omega,\quad m\to\infty. \tag{6.20}$$

From the estimate (2.5), using (6.18), we deduce

$$\int\limits_\Omega \mathrm{B}\Big(\nabla(\eta_R(|\boldsymbol{x}|)T_k w^m)\Big)d\boldsymbol{x}\le c\int\limits_{\{\Omega:|w^m|<k\}}\mathrm{B}(\nabla w^m)d\boldsymbol{x}+c\int\limits_\Omega \mathrm{B}\Big(T_k w^m\nabla\eta_R(|\boldsymbol{x}|)\Big)d\boldsymbol{x}\le C_7(k,R).$$

Hence for any fixed $k,R>0$, we obtain the boundedness of the set $\{\eta_R T_k w^m\}$ in $\mathring{H}^1_{\mathrm{B}}(\Omega(R+1))$. For an $N$-function $M\prec\prec B_*$, according to Lemma 4.7, we obtain the compactness of the embedding

$$\mathring{H}^1_{\mathrm{B}}(\Omega(R+1))\subset L_M(\Omega(R+1)).$$

Thus, for any fixed $k,R>0$, the selective convergence $\eta_R T_k w^m\to v$ in $L_M(\Omega(R+1))$ as $m\to\infty$ is proved. This implies the convergence $T_k w^m\to v$, as well as the selective convergence $L_M(\Omega(R))$ almost everywhere in $\Omega(R)$ as $m\to\infty$ for $k=1,2,\dots$. By the diagonal process, we can prove that there is a measurable function $w:\Omega\to\mathbb{R}$ such that $v=T_k w$ and $w^m\to w$ almost everywhere in $\Omega(R)$ for all $R>0$. This implies the convergence (6.20).

The convergence $w^m \to w$ almost everywhere in $\Omega(R)$ for all $R > 0$ implies the local convergence in measure and, therefore, the local Cauchy property of $w^m$ in measure:

$$\operatorname{mes}\Big\{\Omega(R) : |w^m - w^l| \geq \nu\Big\} \to 0 \quad \text{for } m, l \to \infty \text{ for any } \nu > 0. \tag{6.21}$$

*Step* 4. From (6.18) and (3.3$\psi$), for any $k > 0$ we have the estimate

$$\Big\|\overline{\mathrm{B}}\big(\boldsymbol{a}_\psi(\boldsymbol{x}, T_m w^m, \nabla w^m)\big)\chi(|w^m| < k)\Big\|_1 \leq C_8(k), \quad m = 1, 2, \dots. \tag{6.22}$$

From (6.18) according to Lemma 4.3 we obtain

$$\operatorname{mes}\Big\{\Omega : \mathrm{B}(\nabla w^m) \geq \rho\Big\} \to 0 \quad \text{uniformly in } m, \quad \rho \to \infty. \tag{6.23}$$

First, we prove the convergence

$$\nabla w^m \to \nabla w \quad \text{locally in measure}, \quad m \to \infty. \tag{6.24}$$

Consider the set

$$E_{\nu,\theta,\rho}(R) = \Big\{\Omega(R) : |w^l - w^m| < \nu,\ \mathrm{B}(\nabla w^l) \leq \rho,\ \mathrm{B}(\nabla w^m) \leq \rho,$$
$$|w^l| \leq \rho,\ |w^m| \leq \rho,\ |\nabla(w^l - w^m)| \geq \theta\Big\}.$$

Due to the embedding

$$\Big\{\Omega(R) : |\nabla(w^l - w^m)| \geq \theta\Big\} \subset \Big\{\Omega : \mathrm{B}(\nabla w^l) > \rho\Big\} \cup \Big\{\Omega : \mathrm{B}(\nabla w^m) > \rho\Big\}$$
$$\cup \Big\{\Omega(R) : |w^l - w^m| \geq \nu\Big\} \cup \Big\{\Omega : |w^l| > \rho\Big\} \cup \Big\{\Omega : |w^m| > \rho\Big\} \cup E_{\nu,\theta,\rho}(R)$$

and (6.19) and (6.23), by an appropriate choice of $\rho$ we obtain the inequality

$$\operatorname{mes}\Big\{\Omega(R) : |\nabla(w^l - w^m)| \geq \theta\Big\}$$
$$< 4\varepsilon + \operatorname{mes} E_{\nu,\theta,\rho}(R) + \operatorname{mes}\Big\{\Omega(R) : |w^l - w^m| \geq \nu\Big\}, \quad m, l = 1, 2, \dots. \tag{6.25}$$

According to the condition (3.4$\psi$) and the well-known fact that a continuous function on a compact set achieves the lowest value, there exists a function $\gamma(\boldsymbol{x}) > 0$ almost everywhere in $\Omega$ such that for $\mathrm{B}(\boldsymbol{s}) \leq \rho$, $\mathrm{B}(\boldsymbol{t}) \leq \rho$, $|s_0| \leq \rho$, and $|\boldsymbol{s} - \boldsymbol{t}| \geq \theta$, the inequality

$$\Big(\boldsymbol{a}_\psi(\boldsymbol{x}, s_0, \boldsymbol{s}) - \boldsymbol{a}_\psi(\boldsymbol{x}, s_0, \boldsymbol{t})\Big) \cdot (\boldsymbol{s} - \boldsymbol{t}) \geq \gamma(\boldsymbol{x}) \tag{6.26}$$

holds. We introduce the notation

$$A_0^m(\boldsymbol{x}) = A_{0\psi}^m(\boldsymbol{x}) + b^m(\boldsymbol{x}, w^m) + \frac{B_0'(w^m)}{m}.$$

The uniform boundedness of $A_0^m(\boldsymbol{x})$ in $L_{1,\mathrm{loc}}(\Omega)$ with respect to $m$ follows from (6.2) and (6.14). Writing (6.6) twice for $w^m$ and $w^l$ and subtracting the second relation from the first, we obtain

$$\Big[\Big(\boldsymbol{a}_\psi(\boldsymbol{x}, T_m w^m, \nabla w^m) - \boldsymbol{a}_\psi(\boldsymbol{x}, T_l w^l, \nabla w^l)\Big) \cdot \nabla v + (A_0^m - A_0^l)v\Big] = 0.$$

Substituting the test function

$$v = \eta_R(|\boldsymbol{x}|)\eta_\rho(|w^l|)\eta_\rho(|w^m|)T_\nu(w^m - w^l),$$

we obtain

$$\Big[\Big(\boldsymbol{a}_\psi(\boldsymbol{x}, T_m w^m, \nabla w^m) - \boldsymbol{a}_\psi(\boldsymbol{x}, T_l w^l, \nabla w^l)\Big) \cdot \nabla(\eta_R(|\boldsymbol{x}|)\eta_\rho(|w^l|)\eta_\rho(|w^m|)T_\nu(w^m - w^l))\Big]$$
$$= -\Big[(A_0^m - A_0^l)\eta_R(|\boldsymbol{x}|)\eta_\rho(|w^l|)\eta_\rho(|w^m|)T_\nu(w^m - w^l)\Big] \leq C_9(R)\nu. \tag{6.27}$$

Next, using (6.26), we deduce

$$\int\limits_{E_{\nu,\theta,\rho}(R)} \gamma(\boldsymbol{x})d\boldsymbol{x} \le \int\limits_{E_{\nu,\theta,\rho}(R)} \Big(\boldsymbol{a}_\psi(\boldsymbol{x}, T_m w^m, \nabla w^m) - \boldsymbol{a}_\psi(\boldsymbol{x}, T_m w^m, \nabla w^l)\Big)\cdot\nabla\big(w^m - w^l\big)d\boldsymbol{x}$$

$$\le \int\limits_{|w^m-w^l|<\nu} \eta_R(|\boldsymbol{x}|)\eta_\rho\big(|w^l|\big)\eta_\rho\big(|w^m|\big)\Big(\boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big) - \boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^l\big)\Big)\cdot\nabla\big(w^m - w^l\big)d\boldsymbol{x}$$

$$= \int\limits_{|w^m-w^l|<\nu} \eta_R\big(|\boldsymbol{x}|\big)\eta_\rho\big(|w^l|\big)\eta_\rho\big(|w^m|\big)\Big(\boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big) - \boldsymbol{a}_\psi\big(\boldsymbol{x}, T_l w^l, \nabla w^l\big)\big)\cdot\nabla\big(w^m - w^l\big)d\boldsymbol{x}$$

$$+ \int\limits_{|w^m-w^l|<\nu} \eta_R\big(|\boldsymbol{x}|\big)\eta_\rho\big(|w^l|\big)\eta_\rho\big(|w^m|\big)\Big(\boldsymbol{a}_\psi\big(\boldsymbol{x}, T_l w^l, \nabla w^l\big) - \boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^l\big)\big)\cdot\nabla\big(w^m - w^l\big)d\boldsymbol{x}$$

$$= I_1 + I_2. \quad (6.28)$$

To estimate $I_1$, we use (6.27) and (2.1):

$$I_1 \le \sum_{i=1}^n \int\limits_{\substack{|w^m|<\rho+1,\\ |w^l|<\rho+1,\\ |\boldsymbol{x}|<R+1}} \Big(\Big|a_{i\psi}\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big)\Big| + \Big|a_{i\psi}\big(\boldsymbol{x}, T_l w^l, \nabla w^l\big)\Big|\Big)\Big|T_\nu\big(w^m - w^l\big)\Big|d\boldsymbol{x}$$

$$+ \sum_{i=1}^n \int\limits_{\substack{\rho<|w^l|<\rho+1,\\ |w^m|<\rho+1}} \Big(\Big|a_{i\psi}\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big)\Big| + \Big|a_{i\psi}\big(\boldsymbol{x}, T_l w^l, \nabla w^l\big)\Big|\Big)\Big|w^l_{x_i}\Big|\cdot\Big|T_\nu\big(w^m - w^l\big)\Big|d\boldsymbol{x}$$

$$+ \sum_{i=1}^n \int\limits_{\substack{\rho<|w^m|<\rho+1,\\ |w^l|<\rho+1}} \Big(\Big|a_{i\psi}\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big)\Big| + \Big|a_{i\psi}\big(\boldsymbol{x}, T_l w^l, \nabla w^l\big)\Big|\Big)\Big|w^m_{x_i}\Big|\cdot\Big|T_\nu\big(w^m - w^l\big)\Big|d\boldsymbol{x}$$

$$+ C_9(R)\nu \le \nu\Big(3\Big\|\overline{\mathrm{B}}\big(\boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big)\big)\chi\big(|w^m| < \rho+1\big)\Big\|_1$$

$$+ 3\|\overline{\mathrm{B}}\Big(\boldsymbol{a}_\psi\big(\boldsymbol{x}, T_l w^l, \nabla w^l\big)\Big)\chi\Big(|w^l| < \rho+1\Big)\Big\|_1 + 2\Big\|\mathrm{B}(\nabla w^m)\chi\big(|w^m| < \rho+1\big)\Big\|_1$$

$$+ 2\Big\|\mathrm{B}(\nabla w^l)\chi\big(|w^l| < \rho+1\big)\Big\|_1 + C_{10}(R)\big).$$

Using (6.12), (6.22), we deduce

$$I_1 \le C_{11}(R,\rho)\nu. \quad (6.29)$$

For $m, l \ge \rho+1$, from

$$|w^l| < \rho+1, \quad |w^m| < \rho+1, \quad |w^m - w^l| < \nu$$

we obtain $|T_m w^m - T_l w^l| < \nu$. Applying the Hölder condition ($3.6\psi$) and the inequalities (2.1), (2.5) for $m, l \ge \rho+1$, we obtain

$$|I_2| \le \int\limits_{\substack{|w^m-w^l|<\nu,\\ |w^l|<\rho+1,\\ |w^m|<\rho+1}} \Big|w^m - w^l\Big|^\alpha \sum_{i=1}^n \overline{B}_i^{-1}\Big(C_{12}(R,\rho)\mathrm{B}\big(\nabla w^l + \nabla\psi\big)\Big)\Big(|w^m_{x_i}| + |w^l_{x_i}|\Big)d\boldsymbol{x}$$

$$\le \nu^\alpha C_{13} \int\limits_{\substack{|w^l|<\rho+1,\\ |w^m|<\rho+1}} \Big(\mathrm{B}(\nabla w^l) + \mathrm{B}(\nabla w^m) + \mathrm{B}(\nabla\psi)\Big)d\boldsymbol{x}.$$

Taking into account (6.12), we obtain the estimate

$$|I_2| \le C_{14}(R,\rho)\nu^{\alpha}, \quad m,l \ge \rho+1. \tag{6.30}$$

Combining (6.28)–(6.30), we deduce

$$\int\limits_{E_{\nu,\theta,\rho}(R)} \gamma(\boldsymbol{x})d\boldsymbol{x} \le C_{15}(R,\rho)\nu^{\alpha}, \quad \nu \in (0,1].$$

For any $\delta > 0$, for fixed $R$ and $\rho$, by selecting $\nu$ we can obtain the estimate

$$C_{15}(R,\rho)\nu^{\alpha} < \delta.$$

Applying Lemma 4.8, for any $\varepsilon > 0$ we establish the inequality

$$\operatorname{mes} E_{\nu,\theta,\rho}(R) < \varepsilon, \quad m,l \ge m_1. \tag{6.31}$$

In addition, according to (6.21), we can select $m_2(\nu,R)$ such that

$$\operatorname{mes}\Big\{\Omega(R): \ \big|w^l - w^m\big| \ge \nu\Big\} < \varepsilon, \quad m,l \ge m_2. \tag{6.32}$$

Combining (6.25), (6.31), and (6.32), we deduce the inequality

$$\operatorname{mes}\Big\{\Omega(R): |\nabla(w^l - w^m)| \ge \theta\Big\} < 6\varepsilon, \quad m,l \ge m_0 = \max\{m_1,m_2\}.$$

This implies Hence the Cauchy property in measure of the sequence $\{\nabla w^m\}$ on the set $\Omega(R)$ for any $R > 0$; hence this implies (6.24) and the selective convergence

$$\nabla w^m \to \nabla w \quad \text{almost everywhere in } \Omega, \quad m \to \infty. \tag{6.33}$$

*Step* 5. We prove that

$$b^m(\boldsymbol{x},w^m) \to b_{\psi}(\boldsymbol{x},w) \quad \text{in } L_{1,\mathrm{loc}}(\Omega), \qquad m \to \infty, \tag{6.34}$$

$$b^m(\boldsymbol{x},w^m) \to b_{\psi}(\boldsymbol{x},w) \quad \text{almost everywhere in } \Omega, \quad m \to \infty. \tag{6.35}$$

From (6.11) we obtain for $h = k$:

$$\int\limits_{\{\Omega:|w^m|\ge 2k\}} \left(\big|b^m(\boldsymbol{x},w^m)\big| + \frac{\big|B_0'(w^m)\big|}{m}\right) d\boldsymbol{x}$$
$$\le \int\limits_{\{\Omega:|w^m|\ge k\}} \Big|A^m_{0\psi} - A^0_{0\psi}\Big| d\boldsymbol{x} + \int\limits_{\{\Omega:|w^m|\ge k\}} \big|A^0_{0\psi}\big| d\boldsymbol{x} + \frac{1}{k} \int\limits_{\{\Omega:k\le|w^m|<2k\}} \phi_{\psi} d\boldsymbol{x}.$$

Due to the inclusions $A^0_{0\psi}$, $\phi_{\psi} \in L_1(\Omega)$, the convergence of (6.1), and the absolute continuity of the integrals in the right-hand side of this inequality, taking into account (6.19), for any $\varepsilon > 0$ we can choose a sufficiently large $k$ such that

$$\int\limits_{\{\Omega:|w^m|\ge 2k\}} \left(\big|b^m(\boldsymbol{x},w^m)\big| + \frac{\big|B_0'(w^m)\big|}{m}\right) d\boldsymbol{x} < \varepsilon, \quad m = 1,2,\dots. \tag{6.36}$$

The continuity $b_{\psi}(\boldsymbol{x},s_0)$ in $s_0$ and the convergence $w^m \to w$ almost everywhere in $\Omega$ imply the convergence (6.35).

Now we establish the Cauchy property of the sequence $\{b^m(\boldsymbol{x},w^m)\}$ in the space $L_{1,\mathrm{loc}}(\Omega)$: for any compact set $Q \subset \Omega$

$$\int\limits_{Q} \Big|b^m(\boldsymbol{x},w^m) - b^l(\boldsymbol{x},w^l)\Big| d\boldsymbol{x} \to 0, \quad m,l \to \infty. \tag{6.37}$$

To do this, we introduce the notation

$$\Delta^{ml}(\boldsymbol{x}) = \Big| b^m(\boldsymbol{x}, w^m) - b^l(\boldsymbol{x}, w^l) \Big|$$

and write the relation

$$\int\limits_Q \Delta^{ml}(\boldsymbol{x})d\boldsymbol{x} = \int\limits_{\substack{\{Q:|w^m|\geq 2k,\\ |w^l|\geq 2k\}}} \Delta^{ml}(\boldsymbol{x})d\boldsymbol{x} + \int\limits_{\substack{\{Q:|w^m|<2k,\\ |w^l|<2k\}}} \Delta^{ml}(\boldsymbol{x})d\boldsymbol{x}$$
$$+ \int\limits_{\substack{\{Q:|w^m|<2k,\\ |w^l|\geq 2k\}}} \Delta^{ml}(\boldsymbol{x})d\boldsymbol{x} + \int\limits_{\substack{\{Q:|w^m|\geq 2k,\\ |w^l|<2k\}}} \Delta^{ml}(\boldsymbol{x})d\boldsymbol{x} = I_1 + I_2 + I_3 + I_4.$$

According to (6.36), for any $\varepsilon > 0$, by choosing $k$ the estimate $I_1 < 2\varepsilon$ is valid and uniform in $m$ and $l$.

Using (6.35) and (6.13) and the Lebesgue theorem, by choosing $m_0$ one can prove the inequality

$$I_2 \leq \int\limits_{\substack{\{Q:|w^m|<2k,\\ |w^l|<2k\}}} \Delta^{ml}(\boldsymbol{x})d\boldsymbol{x} < \varepsilon, \quad m, l > m_0.$$

We estimate the integral $I_3$. For the integration domain in $I_3$, the following embedding is valid:

$$\Big\{Q : |w^m| < 2k,\ |w^l| \geq 2k\Big\} \subset \Big\{Q : |w^m| \geq k,\ |w^l| \geq 2k\Big\} \cup \Big\{Q : |w^m| < k,\ |w^l| \geq 2k\Big\}.$$

According to (6.36), by choosing $k$ one can establish the estimates

$$I_{31} = \int\limits_{\substack{\{Q:|w^m|\geq k,\\ |w^l|\geq 2k\}}} \Big| b^m(\boldsymbol{x}, w^m) - b^l(\boldsymbol{x}, w^l) \Big| d\boldsymbol{x} < 2\varepsilon,$$

$$I_{32} = \int\limits_{\substack{\{Q:|w^m|<k,\\ |w^l|\geq 2k\}}} \Big| b^m(\boldsymbol{x}, w^m) - b^l(\boldsymbol{x}, w^l) \Big| d\boldsymbol{x} \leq \int\limits_{\substack{\{Q:|w^m|<k,\\ |w^l|\geq 2k\}}} G_{k+\|\psi\|_\infty}(\boldsymbol{x})d\boldsymbol{x} + \varepsilon,$$

which are uniform in $m$ and $l$. Since

$$\text{mes}\Big\{Q : |w^m| < k,\ |w^l| \geq 2k\Big\} \to \text{mes}\Big\{Q : |w| \leq k,\ |w| \geq 2k\Big\} = 0, \quad k, l \to \infty,$$

$G_{k+\|\psi\|_\infty}(\boldsymbol{x}) \in L_1(Q)$, and the integral is absolutely continuous, by selecting $m_0$ we can obtain the inequality

$$\int\limits_{\substack{\{Q:|w^m|<k,\\ |w^l|\geq 2k\}}} G_{k+\|\psi\|_\infty}(\boldsymbol{x})d\boldsymbol{x} < \varepsilon, \quad m, l \geq m_0.$$

Thus, $I_3 < 4\varepsilon$ for $m, l \geq m_0$. The integral $I_4$ can be estimated similarly.

Combining the estimates for $I_i$, $i = 1, 2, 3, 4$, we establish (6.37). Due to the completeness of the space $L_1(Q)$, there exists a function $v \in L_1(Q)$ such that

$$b^m(\boldsymbol{x}, w^m) \to v \quad \text{in } L_1(Q), \quad m \to \infty. \tag{6.38}$$

In addition, the convergence $b^m(\boldsymbol{x}, w^m) \to v$, $m \to \infty$, is selective almost everywhere in $\Omega$. Hence, in view of the convergence (6.35), we see that $v(\boldsymbol{x}) = b_\psi(\boldsymbol{x}, w)$ almost everywhere in $\Omega$. Thus, the convergence (6.34) is proved.

Next, we prove the convergence

$$\frac{B_0'(w^m)}{m} \to 0 \quad \text{almost everywhere in } \Omega, \quad m \to \infty, \tag{6.39}$$

$$\frac{B_0'(w^m)}{m} \to 0 \quad \text{in } L_{1,\mathrm{loc}}(\Omega), \qquad m \to \infty. \tag{6.40}$$

According to (6.36), for any $\varepsilon > 0$ we can choose $k$ such that

$$\int\limits_{\{\Omega: |w^m| \geq 2k\}} \frac{|B_0'(w^m)|/}{m} d\boldsymbol{x} < \varepsilon, \quad m = 1, 2, \dots.$$

In addition, by choosing $m_0$ one can obtain the inequality

$$\int\limits_{\{Q: |w^m| < 2k\}} \frac{|B_0'(w^m)|}{m} d\boldsymbol{x} \leq \frac{|B_0'(2k)|}{m} \operatorname{mes} Q < \varepsilon, \quad m \geq m_0.$$

From these estimates we obtain the convergence (6.40), which implies (6.39).

The estimate (6.15), in view of (6.35), according to the Fatou theorem, implies $b_\psi(\boldsymbol{x}, w) \in L_1(\Omega)$; this implies the validity of the condition ($1\psi$) of Definition $3.1\psi$.

*Step* 6. We show that $T_k w \in \mathring{H}^1_{\mathrm{B}}(\Omega)$ for any $k > 0$. Combining (6.18) and (2.7) for any fixed $k > 0$, we deduce the estimate

$$\|T_k w^m\|_{\mathring{H}^1_{\mathrm{B}}(\Omega)} = \|\nabla T_k w^m\|_{\mathrm{B}} \leq C_{17}(k), \quad m = 1, 2, \dots.$$

The reflexivity of the space $\mathring{H}^1_{\mathrm{B}}(\Omega)$ allows one to select a subsequence $T_k w^m \rightharpoonup v$ weakly convergent in $\mathring{H}^1_{\mathrm{B}}(\Omega)$, $m \to \infty$, where $v \in \mathring{H}^1_{\mathrm{B}}(\Omega)$. The continuity of the natural mapping $\mathring{H}^1_{\mathrm{B}}(\Omega) \to \mathrm{L}_{\mathrm{B}}(\Omega)$ implies the weak convergence

$$\nabla T_k w^m \rightharpoonup \nabla v \quad \text{in } \mathrm{L}_{\mathrm{B}}(\Omega), \quad m \to \infty. \tag{6.41}$$

Using the convergences (6.20) and (6.33), for any fixed $k > 0$ we obtain

$$\nabla T_k w^m \to \nabla T_k w \quad \text{almost everywhere in } \Omega, \quad m \to \infty.$$

Hence, applying Lemma 4.4, we have the weak convergence

$$\nabla T_k w^m \rightharpoonup \nabla T_k w \quad \text{in } \mathrm{L}_{\mathrm{B}}(\Omega), \quad m \to \infty. \tag{6.42}$$

The relations (6.41) and (6.42) imply the equality

$$v = T_k w \in \mathring{H}^1_{\mathrm{B}}(\Omega).$$

*Step* 7. To prove the inequality ($3.11\psi$), we take functions $T \in \mathcal{F}$ and $\xi \in C_0^\infty(\Omega)$ and apply the test function $v = T(w^m - \xi)$ in the identity (6.6). So we obtain

$$J = \Big[\boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big) \cdot \nabla T(w^m - \xi)\Big] \\ = -\bigg[\bigg(b^m(\boldsymbol{x}, w^m) + \frac{B_0'(w^m)}{m} + A^m_{0\psi}\bigg) T(w^m - \xi)\bigg] = -I. \tag{6.43}$$

The left integral can be written as follows:

$$J = \Big[\boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big) \cdot \nabla w^m T'(w^m - \xi) \\ - \boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big) \cdot \nabla \xi T'(w^m - \xi)\Big] = J_1 - J_2. \tag{6.44}$$

Due to the convergences $w^m \to w$, $T_m w^m \to w$, and $\nabla w^m \to \nabla w$ almost everywhere in $\Omega$ (see (6.20) and (6.33)), and due to the continuity of the function $\boldsymbol{a}_\psi(\boldsymbol{x}, s_0, \boldsymbol{s})$ with respect to $s_0$, $\boldsymbol{s}$, and $T'(r)$, we obtain

$$\boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big)T'(w^m - \xi) \to \boldsymbol{a}_\psi\big(\boldsymbol{x}, w, \nabla w\big)T'(w - \xi) \quad \text{almost everywhere in } \Omega, \quad m \to \infty.$$

Hence, by the Fatou lemma we obtain

$$\Big[\boldsymbol{a}_\psi\big(\boldsymbol{x}, w, \nabla w\big) \cdot \nabla w T'(w - \xi) d\Big] \le \lim_{m\to\infty} \inf J_1. \tag{6.45}$$

From (6.22) we obtain the boundedness of the sequence of norms:

$$\Big\|\overline{\mathrm{B}}\Big(\boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big)T'(w^m - \xi)\Big)\Big\|_1 \\ \le C_{18}\Big\|\overline{\mathrm{B}}\Big(\boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big)\Big)\chi\Big(|w^m| \le k + \|\xi\|_\infty\Big)\Big\|_1 \le C_{19}, \quad m = 1, 2, \dots.$$

Applying Lemma 4.4, we prove the weak convergence:

$$\boldsymbol{a}_\psi\big(\boldsymbol{x}, T_m w^m, \nabla w^m\big)T'(w^m - \xi) \rightharpoonup \boldsymbol{a}_\psi\big(\boldsymbol{x}, w, \nabla w\big)T'(w - \xi) \quad \text{in } L_{\overline{\mathrm{B}}}(\Omega), \quad m \to \infty.$$

Passing to the limit in $J_2$, we have

$$\lim_{m\to\infty} J_2 = \Big[\boldsymbol{a}_\psi\big(\boldsymbol{x}, w, \nabla w\big) \cdot \nabla \xi T'(w - \xi)\Big]. \tag{6.46}$$

The integral $I$ is also divided into two summands. The first integral

$$I_1 = \left[\left(b^m(\boldsymbol{x}, w^m) + \frac{B_0'(w^m)}{m}\right)T(w^m - \xi)\right]$$

is estimated as follows. Consider an increasing sequence $\{K^l\}$ of compact subsets of $\Omega$ such that $\bigcup\limits_{l=1}^{\infty} K^l = \Omega$. Let

$$\operatorname{supp}\xi \subset K^l, \ l \ge l_0, \quad v^m = w^m - \xi, \quad v = w - \xi, \quad c^m(\boldsymbol{x}, w^m) = b^m(\boldsymbol{x}, w^m) + \frac{B_0'(w^m)}{m}.$$

then taking into account (6.5), for $l \ge l_0$, we have

$$I_1 = \int\limits_{\Omega\setminus K^l} c^m(\boldsymbol{x}, w^m)T(w^m)d\boldsymbol{x} + \int\limits_{K^l} c^m(\boldsymbol{x}, w^m)T(v^m)d\boldsymbol{x} \ge \int\limits_{K^l} c^m(\boldsymbol{x}, w^m)T(v^m)d\boldsymbol{x} = \overline{I}_1.$$

Consider the integral

$$\widetilde{I}_1 = \int\limits_{K^l}\Big(b_\psi(\boldsymbol{x}, w)T(v) - c^m(\boldsymbol{x}, w^m)T(v^m)\Big)d\boldsymbol{x} \\ = \int\limits_{K^l}\Big(b_\psi(\boldsymbol{x}, w) - c^m(\boldsymbol{x}, w^m)\Big)T(v)d\boldsymbol{x} + \int\limits_{K^l} c^m(\boldsymbol{x}, w^m)\big(T(v) - T(v^m)\big)d\boldsymbol{x} = \widetilde{I}_{11} + \widetilde{I}_{12}.$$

In view of the convergences (6.34) and (6.40), we conclude that $\widetilde{I}_{11} \to 0$ as $m \to \infty$, and for $\widetilde{I}_{12}$ we obtain

$$\widetilde{I}_{12} = \int\limits_{\{K^l : |w^m| \ge L\}} c^m(\boldsymbol{x}, w^m)(T(v) - T(v^m))d\boldsymbol{x} \\ + \int\limits_{\{K^l : |w^m| < L\}} c^m(\boldsymbol{x}, w^m)(T(v) - T(v^m))d\boldsymbol{x} = \widetilde{I}_{121} + \widetilde{I}_{122}.$$

Due to (6.36), by choosing large $L$, we obtain the inequality $|\widetilde{I}_{121}| < \varepsilon$ (uniformly in $m$). For fixed $L$, in view of (6.13), using the Lebesgue theorem, we find that

$$|\widetilde{I}_{122}| < \varepsilon, \quad m \geq m_0.$$

So, $\widetilde{I}_1 \to 0$ for $m \to \infty$; therefore,

$$\int\limits_{K^l} b_\psi(\boldsymbol{x}, w)T(w - \xi)d\boldsymbol{x} = \lim_{m\to\infty} \overline{I}_1 \leq \lim_{m\to\infty} \inf I_1. \tag{6.47}$$

Passing to the limit as $l \to \infty$, we replace $K_l$ by $\Omega$.

Applying (6.1) and (6.2) and using the Lebesgue theorem, we pass to the limit as $m \to \infty$ in the second integral. We obtain

$$I_2 = \Big[A^m_{0\psi}T(w^m - \xi)d\Big] \to \Big[A^0_{0\psi}T(w - \xi)\Big]. \tag{6.48}$$

Combining (6.43)–(6.48), we deduce (3.11$\psi$). □

**Example 6.1.** Consider the equation

$$\sum_{i=1}^n \Big(g_i(u-\psi)B'_i(u_{x_i} - \psi_{x_i}) + f_i(\boldsymbol{x})\Big)_{x_i} - g_0(u-\psi)\varphi(\boldsymbol{x}) - f_0(\boldsymbol{x}) = 0 \tag{6.49}$$

with continuously differentiable $N$-functions $B_1(z), \dots, B_n(z)$ satisfying the $\Delta_2$-condition such that $B'_1(z), \dots, B'_n(z)$ are strictly monotonic for $z \geq 0$, and (2.11) is fulfilled. The functions $g_i(z)$, $z \in \mathbb{R}$, $i = 0, \dots, n$, are nondecreasing and continuous; in addition, $g_i(z)$, $i = 1, \dots, n$, are Lipschitz, positive, and bounded below. If $f_i(\boldsymbol{x}) \in L_{\overline{B}_i}(\Omega)$, $i = 1, 2, \dots, n$, $f_0(\boldsymbol{x}), \varphi(\boldsymbol{x}) \in L_1(\Omega)$, then for functions

$$a_i(\boldsymbol{x}, s_0, \boldsymbol{s}) = g_i(s_0 - \psi)B'_i\big(s_i - \psi_{x_i}\big) + f_i(\boldsymbol{x}),\ i = 1, 2, \dots, n,$$
$$a_0(\boldsymbol{x}, s_0) = g_0(s_0 - \psi)\varphi(\boldsymbol{x}) + f_0(\boldsymbol{x}),$$

the conditions (3.3)–(3.7), (3.9) are fulfilled. According to Theorem 3.1, there exists a solution of the problem (6.49), (1.2).

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