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On the local behavior of a class of inverse mappings

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Abstract. We study the families of mappings such that the inverse ones satisfy an inequality of the Poletskii type in the given domain. It is proved that those families are equicontinuous at the inner points, if the initial and mapped domains are bounded, and the majorant responsible for a distortion of the modulus is integrable. But if the initial domain is locally connected on its boundary, and if the boundary of the mapped domain is weakly flat, then the corresponding families of mappings are equicontinuous at the inner and boundary points.

Keywords. Inverse mappings, equicontinuity, mappings with bounded finite distortion, moduli, capacities.

1. Introduction

At the present time, the local behavior of quasiconformal mappings of the Euclidean space is well studied (see, e.g., [1, Theorem 19.2], [2, Theorem 3.17] and [3, Lemma 3.12, Corollary 3.22]). A certain number of works is devoted, in this case, to their behavior in the closure of a domain. We mention, for example, [4, Theorem 3.1] and [5, Theorem 3.1] (see also [6,7] and [8]). Let us ask: which is a local behavior of corresponding inverse mappings?

In the frame of the class of quasiconformal homeomorphisms, this question has no meaning. Indeed, the quasiconformality of a direct mapping f yields the quasiconformality of the mapping f^{-1} (in this case, the conformality constant for the mappings is the same (see, e.g., [1, Corollary 13.3] and [1, Theorem 34.3]). Thus, the study of the mappings inverse to quasiconformal ones gives no new information, and the posed question is removed.

The situation will be significantly changed, if we consider some more general class of homeomorphisms, which will be considered now. Let M denote the modulus of families of curves (see [1]), and let $dm(x)$ correspond to the Lebesgue measure in $\mathbb{R}^n$. Assume that the mapping $f : D \to \mathbb{R}^n$ is set in the domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, and it satisfies an inequality of the form

$$M(f(\Gamma)) \leqslant \int\limits_D Q(x) \cdot \rho^n(x)\, dm(x) \quad \forall\, \rho \in \operatorname{adm} \Gamma \tag{1.1}$$

where $Q : D \to [1, \infty]$ is some (given) fixed function (see, e.g., [9]). We recall that $\rho \in \operatorname{adm} \Gamma$, iff

$$\int\limits_\gamma \rho(x)|dx| \geqslant 1 \quad \forall\, \gamma \in \Gamma\,.$$

As for estimates of the form (1.1) in various classes of mappings, we refer, e.g., to [10, Theorems 4.6 and 6.10]. We note that, for an arbitrary function Q, we cannot replaced f by f^{-1} in (1.1); on this

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occasion, see Example 5.2 at the end of this work. Here, we will study the homeomorphisms g such that the inverse ones satisfy relation (1.1). In what follows, we will construct the example of a family of mappings under condition (1.1), which is not equicontinuous in the given domain. In this case, the family "inverse" to it is equicontinuous. Therefore, the analysis of the local behavior of such mappings has meaning.

It is worth to mention our previous works [11] and [12], where the analogous questions were considered. We note that the basic theorems of those works involve quite strong conditions imposed on the geometry of domains and the mappings. Therefore, they cannot be compared with results of the present work by the power of assertions. In particular, we reject the conditions normalization in the classes under study, which enriches essentially the results from the viewpoint of applications (see Example 5.1 below). As a rather unexpected discovery, we mention the obtained absence of any relationship between the equicontinuity of mappings inside a domain and the geometry of this domain. As for any domain, we require only its boundedness and the boundedness of its image at a mapping. We note that all previous results, including those in works [11] and [12], were related to some additional conditions imposed on domains and their boundaries.

The basic definitions and notations used below can be found in books [1] and [13] and, therefore, are omitted. Let E and $F \subset \overline{\mathbb{R}^n}$ be any sets. In what follows, by the symbol $\Gamma(E, F, D)$, we denote the family of all curves $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ that connect E and F in D, i.e. $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a, b)$. We recall that the domain $D \subset \mathbb{R}^n$ is called *locally connected at a point* $x_0 \in \partial D$, if, for any neighborhood U of the point x_0, there exists a neighborhood $V \subset U$ of the point x_0 such that $V \cap D$ is connected. The domain D is locally connected on ∂D, if D is locally connected at every point $x_0 \in \partial D$. The boundary of a domain D is called *weakly flat* at the point $x_0 \in \partial D$, if, for every $P > 0$ and for any neighborhood U of the point x_0, there exists a neighborhood $V \subset U$ of this point, and if this neighborhood is such that $M(\Gamma(E, F, D)) > P$ for any continua E and $F \subset D$ intersecting ∂U and ∂V. The boundary of a domain D is called weakly flat, if the corresponding property holds at every point of the boundary D.

Consider the domains $D, D' \subset \mathbb{R}^n$, $n \geqslant 2$, and any function $Q : \mathbb{R}^n \to [1, \infty]$ measurable by Lebesgue and such that $Q(x) \equiv 0$ for $x \notin D$. By $\mathfrak{R}_Q(D, D')$, we denote the family of all mappings $g : D' \to D$ such that $f = g^{-1}$ is a homeomorphism of the domain D on D' with condition (1.1). The following proposition is valid.

Theorem 1.1. *Assume that $\overline{D}$ and $\overline{D'}$ are compact sets in $\mathbb{R}^n$. If $Q \in L^1(D)$, then the family $\mathfrak{R}_Q(D, D')$ is equicontinuous in D'.*

Consider a number $\delta > 0$, the domains $D, D' \subset \mathbb{R}^n$, $n \geqslant 2$, a continuum $A \subset D$, and any function $Q : \mathbb{R}^n \to [1, \infty]$ measurable by Lebesgue and such that $Q(x) \equiv 0$ for $x \notin D$. By $\mathfrak{S}_{\delta,A,Q}(D, D')$, we denote the family of all mappings $g : D' \to D$ such that $f = g^{-1}$ is a homeomorphism of the domain D on D' with condition (1.1). In this case, $\operatorname{diam} f(A) \geqslant \delta$. The following proposition is valid.

Theorem 1.2. *Assume that the domain D is locally connected at all boundary points, $\overline{D}$ and $\overline{D'}$ are compact sets in $\mathbb{R}^n$, and the domain D' has a weakly flat boundary. Assume also that any connected component $\partial D'$ is a nondegenerate continuum. If $Q \in L^1(D)$, then every mapping $g \in \mathfrak{S}_{\delta,A,Q}(D, D')$ is extended by continuity to the mapping $\overline{g} : \overline{D'} \to \overline{D}$, $\overline{g}|_{D'} = g$. In this case, $\overline{g}(\overline{D'}) = \overline{D}$, and the family $\mathfrak{S}_{\delta,A,Q}(\overline{D}, \overline{D'})$, consisting of all extended mappings $\overline{g} : \overline{D'} \to \overline{D}$, is equicontinuous in $\overline{D'}$.*

Remark 1.1. The assertion of Theorem 1.1 was first established by us in the metric spaces under quite strong additional conditions imposed on the domains D and D', see [12, Theorem 2]. The main achievement of the present work is the assertion of this theorem *without any* conditions imposed on those domains, except for their boundedness. The version of Theorem 1.2 related to metric spaces was published in [12, Theorem 3] and was proved under the assumption that the domain D' is a QED-domain. The last condition is stronger that the condition of weak flatness of the boundary (see [13, Remark 3.14]). Thus, Theorem 1.2 in the Euclidean space is a stronger assertion, as compared with results in [12].

2. Auxiliary information

First of all, we will establish two elementary assertions playing a significant role in the proof of the main results. Let I be an open closed or semiopen interval in $\mathbb{R}$. For the curve $\gamma : I \to \mathbb{R}^n$, we set, as usual:

$$|\gamma| = \{x \in \mathbb{R}^n : \exists\, t \in [a,b] : \gamma(t) = x\}\,.$$

In this case, $|\gamma|$ is called a *support (image)* of γ. We say that the curve γ lies in a domain D, if $|\gamma| \subset D$. In addition, we say that the curves γ_1 and γ_2 are disjoint, if their supports are disjoint. The curve $\gamma : I \to \mathbb{R}^n$ is called a *Jordan arc*, if γ is a homeomorphism on I. The following assertion was proved in [12, Proposition 1]. However, we give its proof here for the sake of completeness.

Lemma 2.1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, which is locally connected on its boundary. Then any two pairs of different points $a \in D, b \in \overline{D}$, and $c \in D, d \in \overline{D}$ can be connected by the curves $\gamma_1 : [0,1] \to \overline{D}$ and $\gamma_2 : [0,1] \to \overline{D}$ that are nonoverlapping and are such that $\gamma_i(t) \in D$ for all $t \in (0,1)$, $i = 1,2$, $\gamma_1(0) = a$, $\gamma_1(1) = b$, $\gamma_2(0) = c$, and $\gamma_2(1) = d$.*

Proof. We note that the points of a domain, which are locally connected on the boundary, are attainable from inside of the domain by means of curves (see [13, Proposition 13.2]). In such case, if $n \geqslant 3$, we connect the points a and b by any Jordan arc γ_1 in the domain D not passing through the points c and d (this is possible in view of the local connectedness of D on the boundary and the transition from a curve to a broken line, if necessary). Then γ_1 does not divide the domain D as a set with topological dimension 1 (see [14, Corollary 1.5.IV]), which ensures the existence of the required curve γ_2. Thus, for $n \geqslant 3$, the assertion of Lemma 2.1 is established.

Let now $n = 2$. Then again the points c and d do not divide the domain D ([14, Corollary 1.5.IV]). In such case, it is also possible to connect the points a and b by a Jordan arc γ_1 in D not passing through the points c and d. In view of the Antoine theorem (see [15, Theorem 4.3, § 4]), the domain D can be mapped on some domain D^* by means of a flat homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ so that $\varphi(\gamma_1) = J$, and J is a segment in D^*. We note that the points of the boundary of the domain D^* are attainable from inside of D^* by means of curves. Thus, we can connect the points $\varphi(c)$ and $\varphi(d)$ in D^* by the Jordan curve $\alpha_2 : [0,1] \to \overline{D^*}$, which lies completely in D^*, except for, may be, its end point $\alpha_2(1) = \varphi(d)$.

It remains to show that curve α_2 can be chosen so that it does not intersect the segment J. Indeed, let α_2 intersect J, and let t_1 and t_2 be, respectively, the largest and least values of $t \in [0,1]$, for which $\alpha_2(t) \in |J|$. Let also

$$J = J(s) = \varphi(a) + (\varphi(b) - \varphi(a))s, \quad s \in [0,1]$$

be a parametrization of the segment J. Let $\widetilde{s_1}$ and $\widetilde{s_2} \in (0,1)$ be such that $J(\widetilde{s_1}) = \alpha_2(t_1)$ and $J(\widetilde{s_2}) = \alpha_2(t_2)$. We set $s_2 = \max\{\widetilde{s_1}, \widetilde{s_2}\}$. Let $e_1 = \varphi(b) - \varphi(a)$, and let e_2 be a unit vector orthogonal to e_1. Then the set

$$P_\varepsilon = \{x = \varphi(a) + x_1e_1 + x_2e_2,\ x_1 \in (-\varepsilon, s_2 + \varepsilon),\ x_2 \in (-\varepsilon, \varepsilon)\}\,,\ \varepsilon > 0\,,$$

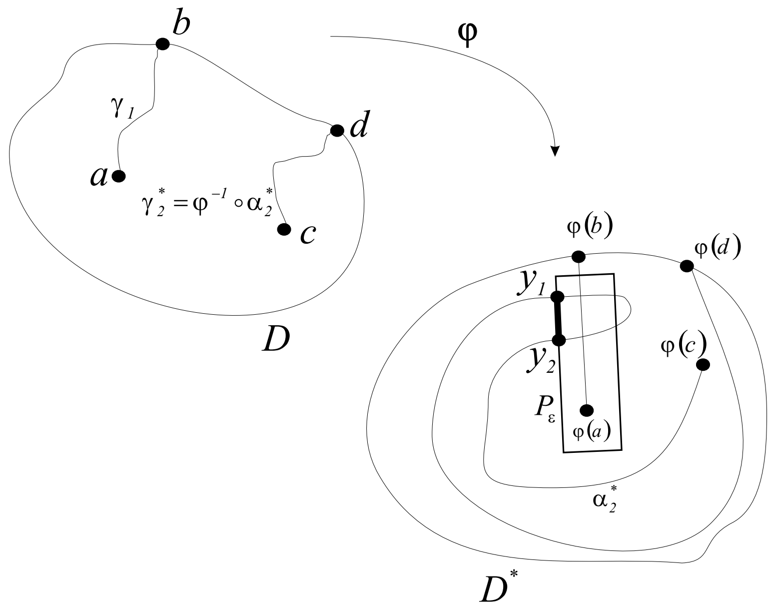

Figure 1: Possible connection of two pairs of points by curves in the domain

is a rectangle containing $|J_1|$, where J_1 is the contraction of J on the segment $[0, s_2]$ (see Fig. 1). We now choose $\varepsilon > 0$ so that $\varphi(c) \notin P_\varepsilon$, $\operatorname{dist}(P_\varepsilon, \partial D^*) > \varepsilon$. In view of [16, Theorem 1.I, Chapt. 5, § 46]), the curve α_2 intersects ∂P_ε for some $T_1 < t_1$ and $T_2 > t_2$. Let $\alpha_2(T_1) = y_1$ and $\alpha_2(T_2) = y_2$. Since $\partial P_\varepsilon \setminus \{z_0\}$, $z_0 := \varphi(a) + (s_2 + \varepsilon)e_1$, is a connected set, we can connect the points y_1 and y_2 by the curve $\alpha^*(t) : [T_1, T_2] \to \partial P_\varepsilon \setminus \{z_0\}$. Finally, we set

$$\alpha_2^*(t) \quad = \quad \begin{cases} \alpha_2(t), & t \in [0,1] \setminus [T_1, T_2], \\ \alpha^*(t), & t \in [T_1, T_2] \end{cases}$$

and $\gamma_2^* := \varphi^{-1} \circ \alpha_2^*$. Then γ_1 connects a and b in D, γ_2^* connects c and d in D. In this case, γ_1 and γ_2^* do not intersect each other, which was to be established. □

Above, we introduce the notion of a weakly flat boundary of a domain, not considering, in this case, the inner points. The following lemma contains the assertion about that the property of "weak flatness" takes always place at the indicated points.

Lemma 2.2. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and $x_0 \in D$. Then, for every $P > 0$ and for any neighborhood U of the point x_0, there exists a neighborhood $V \subset U$ of the same point such that $M(\Gamma(E, F, D)) > P$ for any continua E and $F \subset D$ intersecting ∂U and ∂V.*

Proof. Let U be any neighborhood of the point x_0. We choose $\varepsilon_0 > 0$ so that $\overline{B(x_0, \varepsilon_0)} \subset D \cap U$. Let c_n be a positive constant defined in relation (10.11) in [1], and let the number $\varepsilon \in (0, \varepsilon_0)$ be so small that $c_n \cdot \log \frac{\varepsilon_0}{\varepsilon} > P$. We set $V := B(x_0, \varepsilon)$. Let E, F be any continua intersecting ∂U and ∂V. Then E and F intersect also $S(x_0, \varepsilon_0)$ and ∂V (see [16, Theorem 1.I, Chapt. 5, § 46]). The required conclusion follows from [1, Sect. 10.12], since

$$M(\Gamma(E, F, D)) \geqslant c_n \cdot \log \frac{\varepsilon_0}{\varepsilon} > P\,.$$

□

3. Proof of Theorem 1.1

We prove Theorem 1.1 by contradiction. Assume that the family $\mathfrak{R}_Q(D, D')$ is not equicontinuous at some point $y_0 \in D'$. In other words, there exist $y_0 \in D'$ and $\varepsilon_0 > 0$ such that, for any $m \in \mathbb{N}$, there exist an element $y_m \in D'$, $|y_m - y_0| < 1/m$, and a homeomorphism $g_m \in \mathfrak{R}_Q(D, D')$, for which

$$|g_m(y_m) - g_m(y_0)| \geqslant \varepsilon_0 \,. \tag{3.1}$$

Through the points $g_m(y_m)$ and $g_m(y_0)$, we draw the straight line $r = r_m(t) = g_m(y_0) + (g_m(y_m) - g_m(y_0))t$, $-\infty < t < \infty$ (see Fig. 2). We note that the indicated straight line $r = r_m(t)$ must intersect

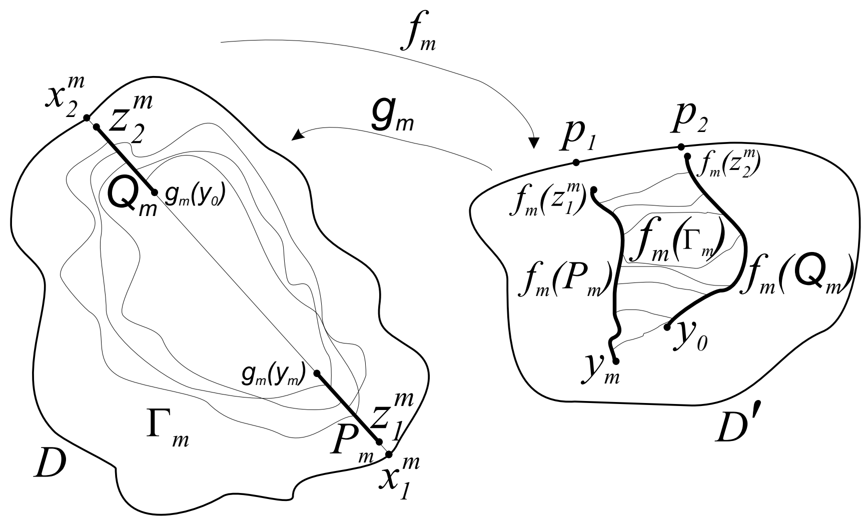

Figure 2: To the proof of Theorem 1.1

the boundary of the domain D for $t \geqslant 1$ in view of [16, Theorem 1.I, Chapt. 5, § 46]), since the domain D is bounded; thus, there exists $t_1^m \geqslant 1$ such that $r_m(t_1^m) = x_1^m \in \partial D$. Without loss of generality, we consider that $r_m(t) \in D$ for all $t \in [1, t_1^m)$. Then the segment $\gamma_1^m(t) = g_m(y_0) + (g_m(y_m) - g_m(y_0))t$, $t \in [1, t_1^m]$, belongs to D for all $t \in [1, t_1^m)$, $\gamma_1^m(t_1^m) = x_1^m \in \partial D$, and $\gamma_1^m(1) = g_m(y_m)$. In view of the analogous reasoning, there exist $t_2^m < 0$ and the segment $\gamma_2^m(t) = g_m(y_0) + (g_m(y_m) - g_m(y_0))t$, $t \in [t_2^m, 0]$, such that $\gamma_2^m(t_2^m) = x_2^m \in \partial D$, $\gamma_2^m(0) = g_m(y_0)$, and $\gamma_2^m(t)$ belongs to D for all $t \in (t_2^m, 0]$. We set $f_m := g_m^{-1}$ and note that f_m is a homeomorphism. For each fixed $m \in \mathbb{N}$, the limiting sets $C(f_m, x_1^m)$ and $C(f_m, x_2^m)$ of the mappings f_m at corresponding boundary points $x_1^m, x_2^m \in \partial D$ lie on $\partial D'$ (see [13, Proposition 13.5]). Hence, there exists a point $z_1^m \in D \cap |\gamma_1^m|$ such that $\operatorname{dist}(f_m(z_1^m), \partial D') < 1/m$. Since $\overline{D'}$ is a compact set, we can consider that the sequence $f_m(z_1^m) \to p_1 \in \partial D'$ as $m \to \infty$. Analogously, there exists a sequence $z_2^m \in D \cap |\gamma_2^m|$ such that $\operatorname{dist}(f_m(z_2^m), \partial D') < 1/m$ and $f_m(z_2^m) \to p_2 \in \partial D'$ as $m \to \infty$.

Let P_m be a part of the segment γ_1^m contained between the points $g_m(y_m)$ and z_1^m, and let Q_m be a part of the segment γ_2^m contained between the points $g_m(y_0)$ and z_2^m. By construction and in view of (3.1), we have $\operatorname{dist}(P_m, Q_m) \geqslant \varepsilon_0 > 0$. Let $\Gamma_m = \Gamma(P_m, Q_m, D)$. Then the function

$$\rho(x) = \begin{cases} \frac{1}{\varepsilon_0}, & x \in D, \\ 0, & x \notin D \end{cases}$$

is admissible for the family Γ_m. Indeed, for any (locally rectifiable) curve $\gamma \in \Gamma_m$, the relation $\int\limits_{\gamma} \rho(x)|dx| \geqslant \frac{l(\gamma)}{\varepsilon_0} \geqslant 1$ holds (where $l(\gamma)$ denotes the length of a curve γ). Since, by condition, the mappings f_m satisfy (1.1), we get

$$M(f_m(\Gamma_m)) \leqslant \frac{1}{\varepsilon_0^n} \int\limits_D Q(x)\, dm(x) := c < \infty\,, \tag{3.2}$$

since $Q \in L^1(D)$. On the other hand, $\operatorname{diam} f_m(P_m) \geqslant |y_m - f_m(z_1^m)| \geqslant (1/2) \cdot |y_0 - p_1| > 0$ and $\operatorname{diam} f_m(Q_m) \geqslant |y_0 - f_m(z_2^m)| \geqslant (1/2) \cdot |y_0 - p_2| > 0$ for large $m \in \mathbb{N}$. In addition,

$$\operatorname{dist}(f_m(P_m), f_m(Q_m)) \leqslant |y_m - y_0| \to 0, \quad m \to \infty\,.$$

Then, in view of Lemma 2.2,

$$M(f_m(\Gamma_m)) = M(\Gamma\left(f_m(P_m), f_m(Q_m), D'\right)) \to \infty\,, \quad m \to \infty\,,$$

which contradicts relation (3.2). This contradiction indicates that the assumption in (3.1) is improper, which completes the proof of the theorem. □

4. On the behavior of mappings in the closure of a domain

Consider the global behavior of mappings. The following assertion indicates that, for sufficiently good domains and mappings under condition (1.1), the image of a fixed continuum at those mappings cannot approach the boundary of the corresponding domain, as only the Euclidean diameter of the image of this continuum is bounded from below (see also [1, Theorems 21.13 and 21.14]).

Lemma 4.1. *Assume that the domain D is locally linearly connected on $\overline{D}$, $\overline{D}$ and $\overline{D'}$ are compact sets in $\mathbb{R}^n$, $n \geqslant 2$, D' has a weakly flat boundary, $Q \in L^1(D)$, and none of the connected components of the boundary $\partial D'$ degenerates into a point. Let $f_m : D \to D'$ be a sequence of homeomorphisms of the domain D on the domain D' with condition (1.1). Let also the exist a continuum $A \subset D$ and a number $\delta > 0$ such that* $\operatorname{diam} f_m(A) \geqslant \delta > 0$ *for all $m = 1, 2, \ldots$. Then there exists $\delta_1 > 0$ such that*

$$\operatorname{dist}(f_m(A), \partial D') > \delta_1 > 0 \quad \forall\, m \in \mathbb{N}\,.$$

Proof. Assume the contrary. Let, for each $k \in \mathbb{N}$, there exist $m = m_k : \operatorname{dist}(f_{m_k}(A), \partial D') < 1/k$. Without loss of generality, we consider that the sequence m_k is monotonically ascending. By condition, $\overline{D'}$ is a compact set. Therefore, $\partial D'$ is also the compact set as a closed subset of the compact set $\overline{D'}$. In addition, $f_{m_k}(A)$ is a compact set as the continuous image of the compact set A at the mapping f_{m_k}. Then there exist $x_k \in f_{m_k}(A)$ and $y_k \in \partial D'$ such that $\operatorname{dist}(f_{m_k}(A), \partial D') = |x_k - y_k| < 1/k$ (see Fig. 3). Since $\partial D'$ is a compact set, we can consider that $y_k \to y_0 \in \partial D'$, $k \to \infty$. Then we also have

$$x_k \to y_0 \in \partial D', \quad k \to \infty\,.$$

Let K_0 be a connected component of $\partial D'$ containing the point y_0. Then, obviously, K_0 is a continuum in $\mathbb{R}^n$. Since D' has a weakly flat boundary, the mapping $g_{m_k} := f_{m_k}^{-1}$ is extended to a continuous mapping $\overline{g}_{m_k} : \overline{D'} \to \overline{D}$ for every $k \in \mathbb{N}$ (see [13, Theorem 4.6]). Moreover, $\overline{g}_{m_k}$ is equicontinuous on $\overline{D'}$ as a mapping continuous on a compact set. Then, for any $\varepsilon > 0$, there exists $\delta_k = \delta_k(\varepsilon) < 1/k$ such that

$$|\overline{g}_{m_k}(x) - \overline{g}_{m_k}(x_0)| < \varepsilon \quad \forall\, x, x_0 \in \overline{D'}, \quad |x - x_0| < \delta_k\,, \quad \delta_k < 1/k\,. \tag{4.1}$$

Let $\varepsilon > 0$ be any number under the condition

$$\varepsilon < (1/2) \cdot \operatorname{dist}(\partial D, A)\,, \tag{4.2}$$

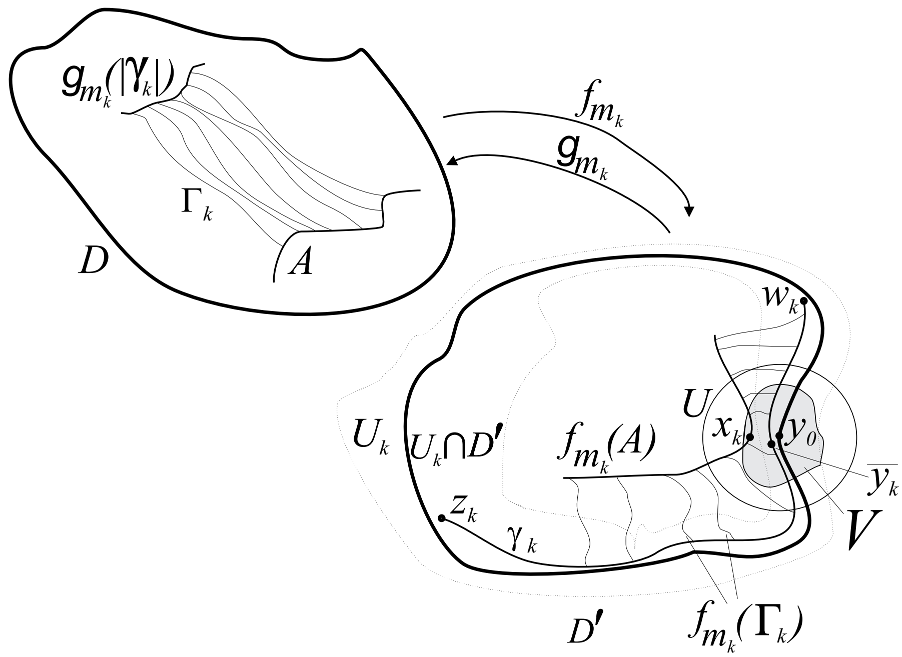

Figure 3: To the proof of Lemma 4.1

where A is the continuum from the condition of the lemma. For every fixed $k\in \mathbb{N}$, we consider the set

$$B_k:=\bigcup\limits_{x_0\in K_0} B(x_0, \delta_k)\,,\quad k\in \mathbb{N}\,.$$

We note that B_k is an open set containing K_0. In other words, B_k is some neighborhood of the continuum K_0. In view of [17, Lemma 2.2], there exists a neighborhood $U_k\subset B_k$ of the continuum K_0, such that $U_k\cap D^{\,\prime}$ is connected. Without loss of generality, we can consider that U_k is an open set. Then $U_k\cap D^{\,\prime}$ is also linearly connected (see [13, Proposition 13.1]). Let $\operatorname{diam} K_0=m_0$. Then there exist $z_0, w_0\in K_0$ such that $\operatorname{diam} K_0=|z_0-w_0|=m_0$. Hence, we can choose the sequences $\overline{y_k}\in U_k\cap D^{\,\prime}$, $z_k\in U_k\cap D^{\,\prime}$ and $w_k\in U_k\cap D^{\,\prime}$ so that $z_k\rightarrow z_0$, $\overline{y_k}\rightarrow y_0$ and $w_k\rightarrow w_0$ as $k\rightarrow\infty$. We can consider that

$$|z_k-w_k|>m_0/2, \quad \forall\, k\in \mathbb{N}\,. \tag{4.3}$$

Let us connect the points z_k, $\overline{y_k}$, and w_k successively by the curve γ_k in $U_k\cap D^{\,\prime}$ (it is possible, since $U_k\cap D^{\,\prime}$ is linearly connected). Let $|\gamma_k|$ be, as usual, the support (image) of the curve γ_k in $D^{\,\prime}$. Then $g_{m_k}(|\gamma_k|)$ is a compact set in D. Let $x\in |\gamma_k|$. Then there exists $x_0\in K_0: x\in B(x_0, \delta_k)$. We fix $\omega\in A\subset D$. Since $x\in|\gamma_k|$, x is an inner point of the domain $D^{\,\prime}$. Thus, we can write $g_{m_k}(x)$ instead $\overline{g}_{m_k}(x)$ for the indicated x. In this case in view of the triangle inequality for large $k\in \mathbb{N}$, relations (4.1) and (4.2) yield

$$|g_{m_k}(x)-\omega|\geqslant |\omega-\overline{g}_{m_k}(x_0)|-|\overline{g}_{m_k}(x_0)-g_{m_k}(x)|$$

$$\geqslant {\rm dist}\,(\partial D, A)-(1/2)\cdot{\rm dist}\,(\partial D, A)=(1/2)\cdot {\rm dist}\,(\partial D, A)>\varepsilon\,. \tag{4.4}$$

Passing in (4.4) to inf over all $x\in |\gamma_k|$ and all $\omega\in A$, we get

$$ {\rm dist}\,(g_{m_k}(|\gamma_k|), A)>\varepsilon, \quad \forall\,\, k=1,2,\ldots\,. \tag{4.5}$$

In view of (4.5), the length of any curve connecting $g_{m_k}(|\gamma_k|)$ and A in D is at least ε. We set $\Gamma_k := \Gamma(g_{m_k}(|\gamma_k|), A, D)$. Then the function $\rho(x)$, which is defined as $1/\varepsilon$ for $x \in D$ and is equal to 0 for $x \notin D$, is admissible for Γ_k, since $\int\limits_{\gamma} \rho(x)|dx| \geqslant \frac{l(\gamma)}{\varepsilon} \geqslant 1$ for $\gamma \in \Gamma_k$ (where $l(\gamma)$ denote the length of the curve γ). By the definition of mappings f_{m_k} in (1.1), we have

$$M(f_{m_k}(\Gamma_k)) \leqslant \frac{1}{\varepsilon^n} \int\limits_{D} Q(x)\, dm(x) = c = c(\varepsilon, Q) < \infty\,, \tag{4.6}$$

since, by condition, $Q \in L^1(D)$.

We now show the contradiction with (4.6) in view of the weak flatness of the boundary $\partial D'$. At the point $y_0 \in \partial D'$, we choose a ball $U := B(y_0, r_0)$, where $r_0 > 0$, $r_0 < \min\{\delta/4, m_0/4\}$, δ is the number from the condition of the lemma, and $\operatorname{diam} K_0 = m_0$. We note that $|\gamma_k| \cap U \neq \varnothing \neq |\gamma_k| \cap (D' \setminus U)$ for sufficiently large $k \in \mathbb{N}$, since $\operatorname{diam}|\gamma_k| \geqslant m_0/2 > m_0/4$ and $\overline{y_k} \in |\gamma_k|$, $\overline{y_k} \to y_0$ as $k \to \infty$. In view of the same reasoning, $f_{m_k}(A) \cap U \neq \varnothing \neq f_{m_k}(A) \cap (D' \setminus U)$. Since $|\gamma_k|$ and $f_{m_k}(A)$ are continua, we have

$$f_{m_k}(A) \cap \partial U \neq \varnothing, \quad |\gamma_k| \cap \partial U \neq \varnothing \tag{4.7}$$

(see [16, Theorem 1.I, Chapt. 5, § 46]). For a fixed $P > 0$, let $V \subset U$ be a neighborhood of the point y_0 corresponding to the definition of a weakly flat boundary. Let the neighborhood be such that, for any continua E and $F \subset D'$ under the condition $E \cap \partial U \neq \varnothing \neq E \cap \partial V$ and $F \cap \partial U \neq \varnothing \neq F \cap \partial V$, the inequality

$$M(\Gamma(E, F, D')) > P \tag{4.8}$$

holds. We note that, for sufficiently large $k \in \mathbb{N}$,

$$f_{m_k}(A) \cap \partial V \neq \varnothing, \quad |\gamma_k| \cap \partial V \neq \varnothing\,. \tag{4.9}$$

Indeed, $\overline{y_k} \in |\gamma_k|$, $x_k \in f_{m_k}(A)$, where $x_k, \overline{y_k} \to y_0 \in V$ as $k \to \infty$. Therefore, $|\gamma_k| \cap V \neq \varnothing \neq f_{m_k}(A) \cap V$ for large $k \in \mathbb{N}$. In addition, $\operatorname{diam} V \leqslant \operatorname{diam} U = 2r_0 < m_0/2$ and, since $\operatorname{diam}|\gamma_k| > m_0/2$ in view of (4.3), $|\gamma_k| \cap (D' \setminus V) \neq \varnothing$. Then $|\gamma_k| \cap \partial V \neq \varnothing$ (see [16, Theorem 1.I, Chapt. 5, § 46]). Analogously, $\operatorname{diam} V \leqslant \operatorname{diam} U = 2r_0 < \delta/2$ and, since $\operatorname{diam} f_{m_k}(A) > \delta$ by the condition of the lemma, $f_{m_k}(A) \cap (D' \setminus V) \neq \varnothing$. In view of [16, Theorem 1.I, Chapt. 5, § 46], we have $f_{m_k}(A) \cap \partial V \neq \varnothing$. The relations in (4.9) are established.

Thus, according to relations (4.7), (4.8), and (4.9), we get

$$M(\Gamma(f_{m_k}(A), |\gamma_k|, D')) > P\,. \tag{4.10}$$

We note that $\Gamma(f_{m_k}(A), |\gamma_k|, D') = f_{m_k}(\Gamma(A, g_{m_k}(|\gamma_k|), D)) = f_{m_k}(\Gamma_k)$, so that inequality (4.10) can be presented as

$$M(\Gamma(f_{m_k}(A), g_{m_k}(|\gamma_k|), D)) = M(f_{m_k}(\Gamma_k)) > P\,,$$

which contradicts inequality (4.6). This contradiction indicates that the assumption $\operatorname{dist}(f_{m_k}(A), \partial D') < 1/k$ is improper. The lemma is proved. □ □

Proof of Theorem 1.2. Since D' has a weakly flat boundary, every $g \in \mathfrak{G}_{\delta,A,Q}(D, D')$ is extended to a continuous mapping $\overline{g} : \overline{D'} \to \overline{D}$ (see [13, Theorem 4.6]).

Let us verify the equality $\overline{g}(\overline{D'}) = \overline{D}$. Indeed, by definition, $\overline{g}(\overline{D'}) \subset \overline{D}$. It remains to prove the inverse inclusion $\overline{D} \subset \overline{g}(\overline{D'})$. Let $x_0 \in \overline{D}$. Then we will show that $x_0 \in \overline{g}(\overline{D'})$. If $x_0 \in \overline{D}$, then $x_0 \in D$

or $x_0 \in \partial D$. If $x_0 \in D$, nothing should be proved, since, by condition, $\overline{g}(D') = D$. Let now $x_0 \in \partial D$. Then there exist $x_k \in D$ and $y_k \in D'$ such that $x_k = \overline{g}(y_k)$ and $x_k \to x_0$ as $k \to \infty$. Since $\overline{D'}$ is a compact set, we can consider that $y_k \to y_0 \in \overline{D'}$ as $k \to \infty$. Since $f = g^{-1}$ is a homeomorphism, $y_0 \in \partial D'$. Since $\overline{g}^{-1}$ is continuous in $\overline{D'}$, $\overline{g}(y_k) \to \overline{g}(y_0)$. However, in such case, $\overline{g}(y_0) = x_0$, since $\overline{g}(y_k) = x_k$, and $x_k \to x_0$ as $k \to \infty$. Hence, $x_0 \in \overline{g}(\overline{D'})$. The inclusion $\overline{D} \subset \overline{g}(\overline{D'})$ is proved, and, hence, $\overline{D} = \overline{g}(\overline{D'})$, which was to be proved.

The equicontinuity of the family $\mathfrak{S}_{\delta,A,Q}(\overline{D}, \overline{D'})$ at the inner points D' is a result of Theorem 1.1. It remains to show that this family is equicontinuous at boundary points. We carry on the proof by contradiction. Let there exist a point $z_0 \in \partial D'$, a number $\varepsilon_0 > 0$, the sequences $z_m \in \overline{D'}$, $z_m \to z_0$ as $m \to \infty$, and $\overline{g}_m \in \mathfrak{S}_{\delta,A,Q}(\overline{D}, \overline{D'})$ such that

$$|\overline{g}_m(z_m) - \overline{g}_m(z_0)| \geqslant \varepsilon_0, \quad m = 1, 2, \ldots . \tag{4.11}$$

We set $g_m := \overline{g}_m|_{D'}$. Since g_m is extended by continuity on the boundary D', we can consider that, $z_m \in D'$ and, hence, $\overline{g}_m(z_m) = g_m(z_m)$. In addition, there exists one more sequence $z'_m \in D'$, $z'_m \to z_0$ as $m \to \infty$, such that $|g_m(z'_m) - \overline{g}_m(z_0)| \to 0$ as $m \to \infty$. Since $\overline{D}$ is a compact set, we can consider that the sequences $g_m(z_m)$ and $\overline{g}_m(z_0)$ are convergent as $m \to \infty$. Let $g_m(z_m) \to \overline{x_1}$ and $\overline{g}_m(z_0) \to \overline{x_2}$ as $m \to \infty$. The continuity of the modulus in (4.11) implies that $\overline{x_1} \neq \overline{x_2}$. Moreover, since the homeomorphisms conserve a boundary, $\overline{x_2} \in \partial D$. Let x_1 and x_2 be any different points of the continuum A, none of the points coincides with $\overline{x_1}$. By Lemma 2.1, we can connect the points x_1 and $\overline{x_1}$ by the curve $\gamma_1 : [0, 1] \to \overline{D}$, and the points x_2 and $\overline{x_2}$ can be connected by the curve $\gamma_2 : [0, 1] \to \overline{D}$, so that $|\gamma_1| \cap |\gamma_2| = \varnothing$, $\gamma_i(t) \in D$ for all $t \in (0, 1)$, $i = 1, 2$, $\gamma_1(0) = x_1$, $\gamma_1(1) = \overline{x_1}$, $\gamma_2(0) = x_2$, and $\gamma_2(1) = \overline{x_2}$. Since D is locally connected on its boundary, there exist neighborhoods U_1 and U_2 of the points $\overline{x_1}$ and $\overline{x_2}$, whose closures are disjoint, and they are such that $W_i := D \cap U_i$ is a linearly connected set. By decreasing the neighborhoods U_i, if necessary, we can consider that $\overline{U_1} \cap |\gamma_2| = \varnothing = \overline{U_2} \cap |\gamma_1|$. Without loss of generality, we can consider that $g_m(z_m) \in W_1$ and $g_m(z'_m) \in W_2$ for all $m \in \mathbb{N}$. Let a_1 and a_2 be any points, which belong to $|\gamma_1| \cap W_1$ and $|\gamma_2| \cap W_2$. Let t_1, t_2 be such that $\gamma_1(t_1) = a_1$ and $\gamma_2(t_2) = a_2$. We now connect the point a_1 with the point $g_m(z_m)$ by the curve $\alpha_m : [t_1, 1] \to W_1$ such that $\alpha_m(t_1) = a_1$ and $\alpha_m(1) = g_m(z_m)$. Analogously, we connect the point a_2 with the point $g_m(z'_m)$ by the curve $\beta_m : [t_2, 1] \to W_2$ such that $\beta_m(t_2) = a_2$ and $\beta_m(1) = g_m(z'_m)$ (see Fig. 4). We now set

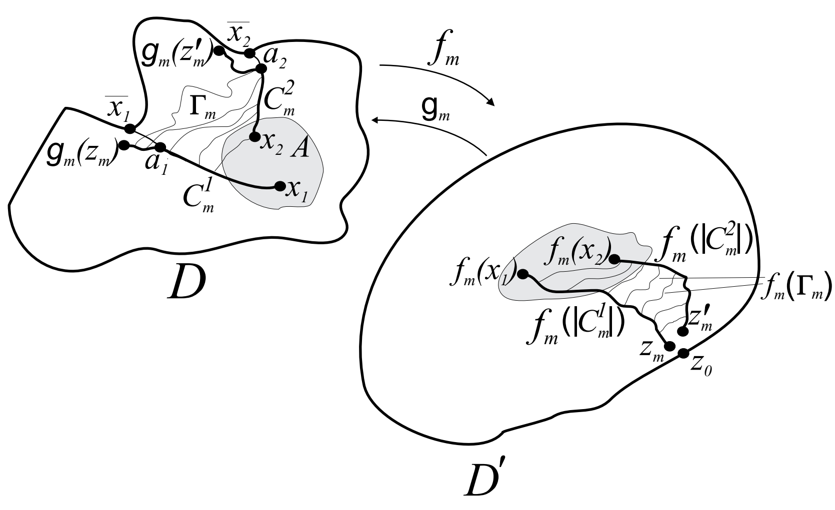

Figure 4: To the proof of Theorem 1.2

$$C_m^1(t)=\left\{\begin{array}{ll}\gamma_1(t), & t\in[0,t_1],\\ \alpha_m(t), & t\in[t_1,1],\end{array}\right.\qquad C_m^2(t)=\left\{\begin{array}{ll}\gamma_2(t), & t\in[0,t_2],\\ \beta_m(t), & t\in[t_2,1].\end{array}\right.$$

Let, as usual, $|C_m^1|$ and $|C_m^2|$ be the supports of the curves C_m^1 and C_m^2, respectively. We note that, by construction, $|C_m^1|$ and $|C_m^2|$ are two nonoverlapping continua in D. Moreover, $\mathrm{dist}\,(|C_m^1|,|C_m^2|)>l_0>0$ for all $m=1,2,\ldots$. We can take, for example,

$$l_0=\min\{\mathrm{dist}\,(|\gamma_1|,|\gamma_2|),\mathrm{dist}\,(|\gamma_1|,U_2),\mathrm{dist}\,(|\gamma_2|,U_1),\mathrm{dist}\,(U_1,U_2)\}.$$

Let now Γ_m be a family of curves connecting $|C_m^1|$ and $|C_m^2|$ in D. Then the function

$$\rho(x)=\left\{\begin{array}{ll}\frac{1}{l_0}, & x\in D\\ 0, & x\notin D\end{array}\right.$$

is admissible for the family Γ_m, since $\int\limits_{\gamma}\rho(x)|dx|\geqslant\frac{l(\gamma)}{l_0}\geqslant 1$ for $\gamma\in\Gamma_m$ (where $l(\gamma)$ denotes the length of the curve γ). By condition, the mappings f_m, $f_m=g_m^{-1}$, satisfy (1.1) for $Q\in L^1(D)$. From whence, we get

$$M(f_m(\Gamma_m))\leqslant\frac{1}{l_0^n}\int\limits_D Q(x)\,dm(x):=c=c(l_0,Q)<\infty\,. \tag{4.12}$$

On the other hand, by Lemma 4.1, there exists a number $\delta_1>0$ such that $\mathrm{dist}\,(f_m(A),\partial D')>\delta_1>0$, $m=1,2,\ldots$. This implies that

$$\mathrm{diam}\,f_m(|C_m^1|)\geqslant|z_m-f_m(x_1)|\geqslant(1/2)\cdot\mathrm{dist}\,(f_m(A),\partial D')>\delta_1/2\,,$$
$$\mathrm{diam}\,f_m(|C_m^2|)\geqslant|z_m'-f_m(x_2)|\geqslant(1/2)\cdot\mathrm{dist}\,(f_m(A),\partial D')>\delta_1/2 \tag{4.13}$$

for some $M_0\in\mathbb{N}$ and all $m\geqslant M_0$.

At the point $z_0\in\partial D'$, we choose the ball $U:=B(z_0,r_0)$ such that $r_0>0$ and $r_0<\delta_1/4$, where δ_1 is the number from the relations in (4.13). We note that $f_m(|C_m^1|)\cap U\neq\varnothing\neq f_m(|C_m^1|)\cap(D'\setminus U)$ for sufficiently large $m\in\mathbb{N}$, since $\mathrm{diam}\,f_m(|C_m^1|)\geqslant\delta_1/2$ and $z_m\in f_m(|C_m^1|)$, $z_m\to z_0$ as $m\to\infty$. In view of the same reasoning, $f_m(|C_m^2|)\cap U\neq\varnothing\neq f_m(|C_m^2|)\cap(D'\setminus U)$. Since $f_m(|C_m^1|)$ and $f_m(|C_m^2|)$ are continua, we have

$$f_m(|C_m^1|)\cap\partial U\neq\varnothing,\quad f_m(|C_m^2|)\cap\partial U\neq\varnothing\,, \tag{4.14}$$

see [16, Theorem 1.I, Chapt. 5, § 46]. For a fixed $P>0$, let $V\subset U$ be a neighborhood of the point z_0 corresponding to the definition of a weakly flat boundary. Let it be such that, for any continua E and $F\subset D'$ under the condition $E\cap\partial U\neq\varnothing\neq E\cap\partial V$ and $F\cap\partial U\neq\varnothing\neq F\cap\partial V$, the inequality

$$M(\Gamma(E,F,D'))>P \tag{4.15}$$

holds. We note that, for sufficiently large $m\in\mathbb{N}$,

$$f_m(|C_m^1|)\cap\partial V\neq\varnothing,\quad f_m(|C_m^2|)\cap\partial V\neq\varnothing\,. \tag{4.16}$$

Indeed, $z_m\in f_m(|C_m^1|)$, $z_m'\in f_m(|C_m^2|)$, where $z_m,z_m'\to z_0\in V$ as $m\to\infty$. Therefore, $f_m(|C_m^1|)\cap V\neq\varnothing\neq f_m(|C_m^2|)\cap V$ for large $m\in\mathbb{N}$. In addition, $\mathrm{diam}\,V\leqslant\mathrm{diam}\,U=2r_0<\delta_1/2$ and, since $\mathrm{diam}f_m(|C_m^1|)>\delta_1/2$ in view of (4.13), we have $f_m(|C_m^1|)\cap(D'\setminus V)\neq\varnothing$. Then $f_m(|C_m^1|)\cap\partial V\neq\varnothing$ (see [16, Theorem 1.I, Chapt. 5, § 46]). Analogously, $\mathrm{diam}\,V\leqslant\mathrm{diam}\,U=2r_0<\delta_1/2$ and, since $\mathrm{diam}\,f_m(|C_m^2|)>\delta_1/2$ in view of (4.13), we have $f_m(|C_m^2|)\cap(D'\setminus V)\neq\varnothing$. Then, by [16, Theorem 1.I, Chapt. 5, § 46], we get $f_m(|C_m^2|)\cap\partial V\neq\varnothing$. Thus, relation (4.16) is proved.

According to (4.15) with regard for (4.14) and (4.16), we get

$$M(f_m(\Gamma_m)) = M(\Gamma(f_m(|C_m^1|), f_m(|C_m^2|), D')) > P\,,$$

which contradicts inequality (4.12). This contradiction indicates that the input assumption made in (4.11) is improper. The theorem is proved. □

5. Some examples

We start from a simple example of mappings on the complex plane.

Example 5.1. As is known, the linear-fractional automorphisms of a unit circle $\mathbb{D} \subset \mathbb{C}$ onto itself are given by the formula $f(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$, $z \in \mathbb{D}$, $a \in \mathbb{C}$, $|a| < 1$, $\theta \in [0, 2\pi)$. The indicated mappings f are 1-homeomorphisms; all conditions of Theorem 1.2 are satisfied, except fot the condition $\operatorname{diam} f(A) \geqslant \delta$, which can be violated, generally speaking.

If, for example, $\theta = 0$ and $a = 1/n$, $n = 1, 2, \ldots$, then $f_n(z) = \frac{z-1/n}{1-z/n} = \frac{nz-1}{n-z}$. We set $A = [0, 1/2]$. Then $f_n(0) = -1/n \to 0$ and $f_n(1/2) = \frac{n-2}{2n-1} \to 1/2$, $n \to \infty$. From whence, it is seen that the sequence f_n satisfies the condition $\operatorname{diam} f_n(A) \geqslant \delta$, for example, for $\delta = 1/4$. By means of the direct calculations, we verify that $f_n^{-1}(z) = \frac{z+1/n}{1+z/n}$, and, hence, f_n^{-1} converges continuously to $f^{-1}(z) \equiv z$. Thus, the sequence $f_n^{-1}(z)$ is equicontinuous in $\overline{\mathbb{D}}$.

But if we set $f_n^{-1}(z) = \frac{z-(n-1)/n}{1-z(n-1)/n} = \frac{nz-n+1}{n-nz+1}$, then, as is easy to see, such sequence is locally continuously convergent to -1 inside $\mathbb{D}$; at the same time, $f_n^{-1}(1) = 1$. In view of this fact, we may conclude, by making direct calculations, that the sequence f_n^{-1} is not equicontinuous at 1. In this case, $f_n(z) = \frac{z+(n-1)/n}{1+z(n-1)/n}$, and condition $\operatorname{diam} f_n(A) \geqslant \delta$ cannot be satisfied for any $\delta > 0$ independent of n in view of Theorem 1.2.

From whence, it follows that, *under the conditions of Theorem 1.2, we cannot reject the additional requirement* $\operatorname{diam} f(A) \geqslant \delta$, *generally speaking.*

Example 5.2. Let $p \geqslant 1$ be so large that the number $n/p(n-1)$ is less than 1, and let, in addition, $\alpha \in (0, n/p(n-1))$ be any number. We define a sequence of mappings $f_m : \mathbb{B}^n \to B(0,2)$ of the ball $\mathbb{B}^n$ onto the ball $B(0,2)$ in the following way:

$$f_m(x) = \begin{cases} \frac{1+|x|^\alpha}{|x|} \cdot x\,, & 1/m \leqslant |x| \leqslant 1, \\ \frac{1+(1/m)^\alpha}{(1/m)} \cdot x\,, & 0 < |x| < 1/m\,. \end{cases}$$

We note that f_m satisfy (1.1) for $Q = \left(\frac{1+|x|^\alpha}{\alpha|x|^\alpha}\right)^{n-1} \in L^1(\mathbb{B}^n)$ (see [11, proof of Theorem 7.1]), and $B(0,2)$ has a weakly flat boundary (see [18, Lemma 4.3]). By construction, the mappings f_m fix the infinite number of points of a unit ball for all $m \geqslant 2$.

We now establish the equicontinuity of the mappings $g_m := f_m^{-1}$ in $\overline{B(0,2)}$ (for convenience, we use the notation g_m also for a continuous extension of g_m in $\overline{B(0,2)}$). It is easy to see that

$$g_m(y) := f_m^{-1}(y) = \begin{cases} \frac{y}{|y|}(|y|-1)^{1/\alpha}\,, & 1 + 1/m^\alpha \leqslant |y| < 2, \\ \frac{(1/m)}{1+(1/m)^\alpha} \cdot y\,, & 0 < |y| < 1 + 1/m^\alpha\,. \end{cases}$$

The mappings g_m map $B(0,2)$ onto $\mathbb{B}^n$. Let us fix $y_0 \in \overline{B(0,2)}$. Three following situations are possible:

1) $|y_0| < 1$. We choose $\delta_0 = \delta_0(y_0)$ so that $\overline{B(y_0, \delta_0)} \subset B(0,1)$. For a number $\varepsilon > 0$, we set $\delta_1 = \delta_1(\varepsilon, y_0) := \min\{\delta_0, \varepsilon\}$. In such case, for $y \in \overline{B(y_0, \delta_1)}$ and all $m = 1, 2, \ldots$, we have $|g_m(y) - g_m(y_0)| = \frac{(1/m)}{1+(1/m)^\alpha}|y - y_0| < |y - y_0| < \varepsilon$, which proves the equicontinuity of the family g_m at the point y_0.

2) $|y_0| > 1$. By the definition of the mappings g_m, there exist $m_0 = m_0(y_0) \in \mathbb{N}$ and $\delta_0 = \delta_0(y_0) > 0$ such that $g_m(y) = \frac{y}{|y|}(|y| - 1)^{1/\alpha}$ for all $\overline{B(y_0, \delta_0)} \cap \overline{B(0, 2)}$ and all $m \geqslant m_0$. Take $\varepsilon > 0$. By setting $g(y) = \frac{y}{|y|}(|y| - 1)^{1/\alpha}$, we note that $|g_m(y) - g_m(y_0)| = |g(y) - g(y_0)| < \varepsilon$ for $m \geqslant m_0$ and, for some $\overline{\delta} = \overline{\delta}(\varepsilon, y_0)$, $\overline{\delta} < \delta_0$, since the mapping $g(y) = \frac{y}{|y|}(|y| - 1)^{1/\alpha}$ is continuous in $\overline{B(0, 2)}$.

3) Finally, we consider the "boundary" case where $y_0 \in \mathbb{S}^{n-1} = \partial\mathbb{B}^n$. Let $\delta_0 = \delta_0(y_0)$ be such that $\overline{B(y_0, \delta_0)} \subset B(0, 2)$. By definition, we have $g_m(y_0) = \frac{(1/m)}{1+(1/m)^\alpha} \cdot y_0$, $m = 1, 2, \ldots$. We note that

$$|g_m(y) - g_m(y_0)| \leqslant$$

$$\leqslant \max\left\{\left|\frac{(1/m)}{1 + (1/m)^\alpha} \cdot y_0 - \frac{y}{|y|}(|y| - 1)^{1/\alpha}\right|, \frac{(1/m)}{1 + (1/m)^\alpha}|y - y_0|\right\}.$$

For a number $\varepsilon > 0$, we find the number $m_1 = m_1(\varepsilon) > 0$ such that $1/m < \varepsilon/2$. We set $\overline{\delta_0} = \overline{\delta_0}(\varepsilon, y_0) = \min\{1, \varepsilon/2, \delta_0\}$. Using the triangle inequality and the inequality $1/\alpha > 1$, we obtain $\left|\frac{y}{|y|}(|y| - 1)^{1/\alpha} - \frac{(1/m)}{1+(1/m)^\alpha} \cdot y_0\right| \leqslant (|y| - 1)^{1/\alpha} + 1/m < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for $m > m_1$ and $|y - y_0| < \overline{\delta_0}$. The last relation for $1 \leqslant m \leqslant m_1$ holds also for $|y - y_0| < \delta_m$ and some $\delta_m = \delta_m(\varepsilon, y_0) > 0$ in view of the continuity of the mappings g_m. Obviously, $\frac{(1/m)}{1+(1/m)^\alpha}|y - y_0| < \varepsilon$ for $|y - y_0| < \overline{\delta_0}$ and all $m = 1, 2, \ldots$. Finally, we have: $|g_m(y) - g_m(y_0)| < \varepsilon$ for all $m \in \mathbb{N}$ and $y \in B(y_0, \delta)$, where $\delta := \{\overline{\delta}_0, \delta_1, \ldots, \delta_{m_1}\}$. The equicontinuity of g_m in $\overline{B(0, 2)}$ is proved.

It is worth noting that the family $\mathfrak{G} = \{g_m\}_{m=1}^\infty$ is equicontinuous in $B(0, 2)$, whereas the family $\mathfrak{F} = \{f_m\}_{m=1}^\infty$ "inverse" to it does not (indeed, $|f_m(x_m) - f(0)| = 1 + 1/m \not\to 0$ as $m \to \infty$, where $|x_m| = 1/m$).

The family $\mathfrak{G}$ *contains the infinite number of mappings* $g_{m_k} := f_{m_k}^{-1}$, $f_{m_k} \in \mathfrak{F}$, *that do not satisfy relation (1.1).* Otherwise, the family $\mathfrak{F}$ "inverse" to $\mathfrak{G}$ would be equicontinuous in $\mathbb{B}^n$ according to Theorem 1.1.

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