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A NEW SUBCLASS OF THE CLASS OF NONSINGULAR $\mathcal{H}$-MATRICES AND RELATED INCLUSION SETS FOR EIGENVALUES AND SINGULAR VALUES

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The paper presents new nonsingularity conditions for $n \times n$ matrices, which involve a subset S of the index set $\{1, \dots, n\}$ and take into consideration the matrix sparsity pattern. It is shown that the matrices satisfying these conditions form a subclass of the class of nonsingular $\mathcal{H}$-matrices, which contains some known matrix classes such as the class of doubly strictly diagonally dominant (DSDD) matrices and the class of Dashnic–Zusmanovich type (DZT) matrices. The nonsingularity conditions established are used to obtain the corresponding eigenvalue inclusion sets, which, in their turn, are used in deriving new inclusion sets for the singular values of a square matrix, improving some recently suggested ones. Bibliography: 11 *titles.*

1. Introduction

The paper suggests new nonsingularity conditions for square matrices of order $n \geq 2$, which depend on a nonempty proper subset S of the index set $\langle n \rangle = \{1, \dots, n\}$ and take into account the matrix sparsity pattern. It is proved that the matrices satisfying these conditions form a subclass, referred to as $\{S\text{-SOB}\}$ (S-sparse Ostrowski–Brauer), of the class of nonsingular $\mathcal{H}$-matrices.

Recall that a matrix $A \in \mathbb{C}^{n \times n}$, $n \geq 2$, is a nonsingular $\mathcal{H}$-matrix if and only if its comparison matrix $\mathcal{M}(A) = (m_{ij})$, where

$$m_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases}$$

is a nonsingular $\mathcal{M}$-matrix.

Also we introduce the subclass of the so-called S-OB (S-Ostrowski–Brauer) matrices, which is contained in the class $\{S\text{-SOB}\}$.

The nonsingularity criteria suggested are then used, in a standard way, to obtain the corresponding eigenvalue inclusion sets.

Finally, based on a general result proved in [7], we derive a new inclusion set for the singular values of a square matrix, improving a recent result in [4].

The paper is organized as follows. In Sec. 2, a nonsingularity criterion is established. Based on this criterion, we introduce new matrix classes, referred to as $\{S\text{-SOB}\}$ and $\{S\text{-OB}\}$, and show that they are subclasses of the class of nonsingular $\mathcal{H}$-matrices. Also we list some elementary properties of matrices in these subclasses, mostly related to strict diagonal dominance. In Sec. 3, by using the nonsingularity criterion of Sec. 2, we obtain new eigenvalue inclusion sets for square matrices and show that they are contained in the Gerschgorin disks. Also we provide an improvement of the inclusion set for the singular values of a square matrix recently suggested in [4].

We conclude this introduction by specifying the notation used in the paper.

• For a subset $S \subseteq \langle n \rangle$, $|S|$ denotes the cardinality of S, and $\bar{S} = \langle n \rangle \setminus S$ is the complementary subset.

• I_n is the identity matrix of order n.

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• For a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$,

$$r_i(A) = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|, \quad i = 1, \ldots, n,$$

are the deleted absolute row sums of A; for $j \in \langle n\rangle$, $j \neq i$, we set

$$r_i^j(A) = r_i(A) - |a_{ij}|,$$

and if $S \subseteq \langle n\rangle$ is a nonempty subset of indices, then

$$r_i^S(A) = \sum_{\substack{j\in S\\ j\neq i}} |a_{ij}|, \quad i = 1, \ldots, n,$$

are the corresponding partial deleted absolute row sums.

2. Nonsingularity criteria

The main results of this section are Theorem 2.3, which establishes a new nonsingularity criterion, dependent on a subset of the index set, and Theorem 2.4, claiming that the matrices satisfying the hypotheses of Theorem 2.3 form a subclass of the class of nonsingular $\mathcal{H}$-matrices.

We start with establishing the following basic result.

Lemma 2.1. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, be a singular matrix, let $S \subset \langle n\rangle$ be a subset of the index set $\langle n\rangle$, $1 \leq |S| \leq n-1$, and let*

$$|x_p| = \max_{i\in\langle n\rangle} |x_i|. \tag{2.1}$$

If $p \in S$, then

either $r_p^{\bar{S}}(A) = 0$ and

$$|a_{pp}| - r_p^S(A) = |a_{pp}| - r_p(A) \leq 0 \tag{2.2}$$

or, for a certain $q \in \bar{S}$ such that $a_{pq} \neq 0$,

$$[|a_{pp}| - r_p^S(A)]\ |a_{qq}| \leq r_p^{\bar{S}}(A)\ r_q(A). \tag{2.3}$$

Proof. Let

$$Ax = 0, \tag{2.4}$$

where $x = (x_i) \in \mathbb{C}^n$ is a nonzero vector.

If $r_p^{\bar{S}}(A) = 0$, then from (2.4) and (2.1) it follows that

$$|a_{pp}|\ |x_p| \leq r_p(A)\ |x_p| = r_p^S(A)\ |x_p|,$$

which proves (2.2).

In the case where $r_p^{\bar{S}}(A) \neq 0$, $p \in S$, we choose $q \in \bar{S}$ in such a way that

$$|x_q| = \max_{j\in\bar{S}:\ a_{pj}\neq 0} |x_j|. \tag{2.5}$$

Using (2.4), (2.1), and (2.5), we derive

$$|a_{pp}|\,|x_p| \leq \sum_{\substack{i\in S\\ i\neq p}} |a_{pi}|\,|x_i| + \sum_{\substack{j\in\bar{S}:\\ a_{pj}\neq 0}} |a_{pj}|\,|x_j| \leq r_p^S(A)\,|x_p| + r_p^{\bar{S}}(A)\,|x_q|.$$

Therefore,

$$[|a_{pp}| - r_p^S(A)]\ |x_p| \leq r_p^{\bar{S}}(A)\,|x_q|. \tag{2.6}$$

Here, two cases are possible. If $x_q = 0$, then, by (2.6),

$$|a_{pp}| - r_p^S(A) \le 0,$$

and inequality (2.3) holds trivially. If $x_q \ne 0$, then using (2.4) and (2.1), we derive

$$|a_{qq}|\,|x_q| \le \sum_{i\ne q} |a_{qi}|\,|x_i| \le r_q(A)\,|x_p|. \tag{2.7}$$

Since both x_p and x_q are nonzero, inequality (2.3) readily follows from (2.6) and (2.7).

This completes the proof of the lemma. □

Consider two special cases of Lemma 2.1.

First let $S = \{p\}$, where p is chosen in accordance with (2.1). In this case, we obviously have

$$r_p^S(A) = 0 \quad \text{and} \quad r_p^{\bar S}(A) = r_p(A),$$

whence inequality (2.2) and the condition $r_p^{\bar S}(A) = 0$ imply that the pth row of A is zero, whereas inequality (2.3) reads as

$$|a_{pp}|\,|a_{qq}| \le r_p(A)\, r_q(A). \tag{2.8}$$

Thus, in the case under consideration, Lemma 2.1 reduces to the following known assertion.

Corollary 2.1. *If a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, is free of zero rows and singular, then inequality* (2.3) *holds for a pair of indices $p, q \in \langle n\rangle$, $p \ne q$, such that $a_{pq} \ne 0$.*

Obviously, Corollary 2.1 amounts to the following sparse version of the classical Ostrowski–Brauer nonsingularity criterion (see [9, 2]).

Theorem 2.1 ([6])**.** *If a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, is free of zero rows and satisfies the condition*

$$|a_{pp}|\,|a_{qq}| > r_p(A)\, r_q(A) \quad \textit{for all} \quad p \ne q \quad \textit{such that} \quad a_{pq} \ne 0, \tag{2.9}$$

then A is nonsingular.

Matrices satisfying condition (2.9) for all $p \ne q$ (without the requirement that $a_{pq} \ne 0$) are sometimes referred to as doubly strictly diagonally dominant (DSDD) matrices (see, e.g., [3, 11]).

Now consider yet another specific choice of the set S. Let again p be chosen in accordance with (2.1) and let q be an arbitrary index distinct from p. Set $S = \langle n\rangle \setminus \{q\}$. In this case, $p \in S$, inequality (2.3) reads as

$$[|a_{pp}| - r_p^q(A)]\ |a_{qq}| \le |a_{pq}|\ r_q(A), \tag{2.10}$$

and, by Lemma 2.1, for a singular matrix A, inequality (2.10) holds whenever $a_{pq} \ne 0$. If $a_{pq} = 0$, then (2.10) holds by virtue of (2.2). Thus, we arrive at the following result.

Corollary 2.2. *If a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, is singular, then inequality* (2.10) *holds for a certain $p \in \langle n\rangle$ and every $q \in \langle n\rangle$, $q \ne p$.*

Corollary 2.2 amounts to the following nonsingularity criterion.

Theorem 2.2 ([11])**.** *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$. If for every $p \in \langle n\rangle$ the inequality*

$$[|a_{pp}| - r_p^q(A)]\ |a_{qq}| > |a_{pq}|\ r_q(A) \tag{2.11}$$

holds with a certain $q = q(p) \ne p$, then A is nonsingular.

Matrices that satisfy the above condition were introduced in [11] and called Dashnic–Zusmanovich type (DZT) matrices. For most recent results on DZT matrices, see [8].

In the case where S is an arbitrary nonempty proper subset of the index set, from Lemma 2.1 we immediately obtain the following result, generalizing both Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, and let $S \subset \langle n\rangle$, $1 \leq |S| \leq n-1$. Assume that the following conditions are fulfilled:*
(i) $|a_{pp}| > r_p^S(A)$ *for all* $p \in S$;
(ii) $|a_{qq}| > r_q^{\bar{S}}(A)$ *for all* $q \in \bar{S}$;
(iii) *for all* $p \in S$ *and all* $q \in \bar{S}$ *such that* $a_{pq} \neq 0$,

$$[|a_{pp}| - r_p^S(A)]\, |a_{qq}| > r_p^{\bar{S}}(A)\, r_q(A); \tag{2.12}$$

(iv) *for all* $p \in S$ *and all* $q \in \bar{S}$ *such that* $a_{qp} \neq 0$,

$$[|a_{qq}| - r_q^{\bar{S}}(A)]\, |a_{pp}| > r_q^S(A)\, r_p(A). \tag{2.13}$$

Then the matrix A is nonsingular.

Theorem 2.3 is stated with account for the sparsity pattern of the matrix A. If the matrix sparsity is ignored and we require that conditions (2.12) and (2.13) be fulfilled for all $p \in S$ and all $q \in \bar{S}$, then conditions (i) and (ii) become exuberant, and we obtain the following simplified statement.

Corollary 2.3. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$, and let $S \subset \langle n\rangle$, $1 \leq |S| \leq n-1$. If inequalities* (2.12) *and* (2.13) *hold for all $p \in S$ and all $q \in \bar{S}$, then A is nonsingular.*

As is known [3, 11], the matrices satisfying the assumptions of Theorems 2.1 and 2.2 form subclasses of the class of nonsingular $\mathcal{H}$-matrices. Therefore, it is reasonable to conjecture that the matrices satisfying the hypotheses of Theorem 2.3 also form a subclass of nonsingular $\mathcal{H}$-matrices. This conjecture is confirmed by the following theorem.

Theorem 2.4. *If a matrix $A \in \mathbb{C}^{n\times n}, n \geq 2$, satisfies the assumptions of Theorem* 2.3*, then A is a nonsingular $\mathcal{H}$-matrix.*

Proof. Observe that a matrix A satisfies conditions (i)–(iv) of Theorem 2.3 if and only if its comparison matrix $\mathcal{M}$(A) satisfies them. Therefore, by Theorem 2.3, $\mathcal{M}$(A) is nonsingular, and we must demonstrate that it is an $\mathcal{M}$-matrix. To this end, by [1, Condition D_{15} of Theorem 6.2.3], it is sufficient to prove that the shifted matrix $\mathcal{M}(A) + \varepsilon I_n$ is nonsingular for every $\varepsilon \geq 0$. Obviously, if conditions (i)–(iv) hold for $\mathcal{M}$(A), then they hold for $\mathcal{M}(A) + \varepsilon I_n$ a fortiori. Thus, $\mathcal{M}(A) + \varepsilon I_n$ is nonsingular for all $\varepsilon \geq 0$ by Theorem 2.3, whence A is a nonsingular $\mathcal{H}$-matrix. □

In what follows, for a subset S of the index set, the classes of matrices satisfying the conditions of Theorem 2.3 and Corollary 2.3 will be referred to as $\{S$-SOB$\}$ (S-Sparse Ostrowski–Brauer) and $\{S$-OB$\}$ (S-Ostrowski–Brauer), respectively.

We conclude this section with a list of some elementary properties of matrices from the classes $\{S$-SOB$\}$ and $\{S$-OB$\}$, most of which are related to strict diagonal dominance.

1. The following inclusions are valid:

$$\{\text{SDD}\} \subsetneqq \{S\text{-OB}\} \subsetneqq \{S\text{-SOB}\} \subsetneqq \mathcal{H}.$$

2. The classes $\{S$-SOB$\}$ and $\{S$-OB$\}$ are invariant with respect to left multiplication by nonsingular diagonal matrices.

3. If $A \in \{S\text{-SOB}\}$ or $A \in \{S\text{-OB}\}$, then the two principal submatrices $A[S] = (a_{ij})_{i,j\in S}$ and $A[\bar{S}] = (a_{ij})_{i,j\in \bar{S}}$ of A both are strictly diagonally dominant.

4. If A is an S-SOB matrix, $p \in S$, $q \in \bar{S}$, and $a_{pq} \neq 0$ or $a_{qp} \neq 0$, then, by virtue of (2.12) or (2.13), at least one of the rows p and q is strictly diagonally dominant.

5. If $A \in \{S\text{-OB}\}$ and, for a certain $p \in S$, the pth row is not strictly diagonally dominant, then, by either (2.12) or (2.13), all the rows with numbers $q \in \bar{S}$ are strictly diagonally dominant. Thus, if $A \notin \{\text{SDD}\}$, then all its rows that are not strictly diagonally dominant are simultaneously contained in either S or $\bar{S}$.

3. Applications

In this section, we present new inclusion sets for the eigenvalues of a square matrix and the corresponding inclusion sets for the singular values.

3.1. Eigenvalue inclusion sets. Applying Lemma 2.1 to the singular matrix $\lambda I_n - A$, where $\lambda \in \operatorname{Spec}A$ is an eigenvalue of A, we conclude that for any subset $S \subset \langle n \rangle$, $1 \le |S| \le n-1$, there exist some $p \in S$ and $q \in \bar{S}$ such that at least one of the following conditions is fulfilled:

(i)
$$r_p^{\bar{S}}(A) = 0 \quad \text{and} \quad |\lambda - a_{pp}| \le r_p^S(A);$$

(ii)
$$r_q^{S}(A) = 0 \quad \text{and} \quad |\lambda - a_{qq}| \le r_q^{\bar{S}}(A);$$

(iii)
$$a_{pq} \neq 0 \quad \text{and} \quad \left[|\lambda - a_{pp}| - r_p^S(A)\right] \; |\lambda - a_{qq}| \le r_p^{\bar{S}}(A)\, r_q(A);$$

(iv)
$$a_{qp} \neq 0 \quad \text{and} \quad \left[|\lambda - a_{qq}| - r_q^{\bar{S}}(A)\right] \; |\lambda - a_{pp}| \le r_q^{S}(A)\, r_p(A).$$

This leads us to the following eigenvalue inclusion sets, dependent on S.

Theorem 3.1. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \ge 2$, and let $S \subset \langle n \rangle$, $1 \le |S| \le n-1$, be an arbitrary subset of the index set $\langle n \rangle$. Then*

$$\begin{aligned} \operatorname{Spec}A \subseteq \Omega(A,S) \equiv & \bigcup_{\substack{p\in S:\\ r_p^{\bar{S}}(A)=0}} \left\{z \in \mathbb{C}:\ |z - a_{pp}| \le r_p^S(A)\right\} \\ & \cup \bigcup_{\substack{q\in \bar{S}:\\ r_q^{S}(A)=0}} \left\{z \in \mathbb{C}:\ |z - a_{qq}| \le r_q^{\bar{S}}(A)\right\} \\ & \cup \bigcup_{\substack{p\in S, q\in \bar{S}:\\ a_{pq}\neq 0}} \Omega_{pq}^S(A) \cup \bigcup_{\substack{p\in S, q\in \bar{S}:\\ a_{qp}\neq 0}} \Omega_{qp}^{\bar{S}}(A), \end{aligned} \tag{3.1}$$

where

$$\Omega_{pq}^S(A) = \left\{z \in \mathbb{C}:\ [|z - a_{pp}| - r_p^S(A)]\, |z - a_{qq}| \le r_p^{\bar{S}}(A)\, r_q(A)\right\}, \quad p \in S,\ q \in \bar{S}, \tag{3.2}$$

and

$$\Omega_{qp}^{\bar{S}}(A) = \left\{z \in \mathbb{C}:\ [|z - a_{qq}| - r_q^{\bar{S}}(A)]\, |z - a_{pp}| \le r_q^{S}(A)\, r_p(A)\right\}, \quad p \in S,\ q \in \bar{S}. \tag{3.3}$$

If, in Theorem 3.1, the sparsity considerations are ignored, then we obtain the following result, which is simpler but less sharp.

Corollary 3.1. *Under the assumptions of Theorem* 3.1,

$$\mathrm{Spec}A \subseteq \bigcup_{p\in S,\; q\in \bar S} \Omega_{pq}(A,S), \tag{3.4}$$

where

$$\Omega_{pq}(A,S) \equiv \Omega_{pq}^{S}(A) \cup \Omega_{qp}^{\bar S}(A), \quad p\in S,\; q\in \bar S.$$

Proof. We must show that the set $\Omega(A,S)$ is contained in the right-hand side of (3.4). Indeed, if $r_p^{\bar S}(A)=0$, then, for every $q\in\bar S$, the set

$$\Omega_{pq}^{S}(A) = \big\{ z\in\mathbb{C}:\ [|z-a_{pp}| - r_p^{S}(A)]\,|z-a_{qq}| \le 0 \big\}$$

obviously contains the set

$$\big\{ z\in\mathbb{C}:\ |z-a_{pp}| \le r_p^{S}(A) \big\}.$$

Similarly, if $r_q^{S}(A)=0$, then, for every $p\in S$, the set

$$\big\{ z\in\mathbb{C}:\ |z-a_{qq}| \le r_q^{\bar S}(A) \big\}$$

is trivially contained in the set

$$\Omega_{qp}^{\bar S}(A) = \big\{ z\in\mathbb{C}:\ [|z-a_{qq}| - r_q^{\bar S}(A)]\,|z-a_{qq}| \le 0 \big\}. \qquad \square$$

As is readily seen, for all $p\in S$ and all $q\in\bar S$,

$$\Omega_{pq}(A,S) \subseteq \Gamma_p(A)\cup\Gamma_q(A), \tag{3.5}$$

where

$$\Gamma_i(A) = \{ z\in\mathbb{C}:\ |z-a_{ii}| \le r_i(A)\}, \quad i=1,\dots,n,$$

are the Gerschgorin disks for the matrix A.

Indeed, let $z\in\Omega_{pq}(A,S)$. If $z\in\Gamma_p(A)$, then, certainly, $z\in\Gamma_p(A)\cup\Gamma_q(A)$. In the case where $z\notin\Gamma_p(A)$, we have

$$|z-a_{pp}| > r_p(A) \quad \text{and} \quad |z-a_{pp}| - r_p^{S}(A) > r_p^{\bar S}(A),$$

and from (3.2) and (3.3) it follows that

$$|z-a_{qq}| \le r_q(A),$$

i.e., $z\in\Gamma_q(A)$. This proves (3.5).

Since, as shown in the proof of Corollary 3.1, $\Omega(A,S)\subseteq \bigcup_{p\in S,\, q\in\bar S}\Omega_{pq}(A,S)$, from (3.5) we immediately obtain that

$$\Omega(A,S) \subseteq \bigcup_{p\in S,\, q\in\bar S} \Omega_{pq}(A,S) \subseteq \Gamma(A) \equiv \bigcup_{i\in\langle n\rangle} \Gamma_i(A), \tag{3.6}$$

i.e., the new eigenvalue inclusion sets of Theorem 3.1 and Corollary 3.1 are contained in the Gerschgorin set $\Gamma(A)$.

3.2. Inclusion sets for the singular values of a square matrix. The approach to deriving inclusion sets for singular values developed in [7] allows one to obtain them by applying known eigenvalue inclusion sets to a certain matrix of double size associated with a given matrix. This approach is based on the following lemma.

Lemma 3.1 ([7]). *Let $A = (a_{ij}) = D_A + B \in \mathbb{C}^{n\times n}$, where $D_A = \operatorname{diag}(a_{11}, \dots, a_{nn})$. If $\sigma \in \Sigma(A)$ is a singular value of A, then the matrix*

$$C = (c_{ij}) = C(\sigma, A) = \begin{bmatrix} \sigma^2 I_n - |D_A|^2 & 0 \\ 0 & \sigma^2 I_n - |D_A|^2 \end{bmatrix} - \begin{bmatrix} D_A B^* & \sigma B \\ \sigma B^* & D_A^* B \end{bmatrix} \tag{3.7}$$

is singular. Furthermore, if $\sigma \geq 0$ and $\sigma \neq |a_{ii}|$, $i = 1, \dots, n$, then C is singular if and only if $\sigma \in \Sigma(A)$.

Lemma 3.1 says that zero is an eigenvalue of the matrix $C = C(\sigma, A)$. Consequently, zero is contained in any eigenvalue inclusion set for C. In particular, by virtue of Corollary 3.1,

$$0 \in \bigcup_{p\in S, q\in \bar S} \left[\Omega^S_{pq}(C) \cup \Omega^{\bar S}_{qp}(C)\right], \tag{3.8}$$

where S is an arbitrary nonempty proper subset of the index set $\{1, 2, \dots 2n\}$, and the sets $\Omega^S_{pq}(C)$ and $\Omega^{\bar S}_{qp}(C)$ are defined in accordance with (3.2) and (3.3). Take $S = \langle n\rangle$. Then the sets $\Omega^S_{pq}(C)$ and $\Omega^{\bar S}_{qp}(C)$ are as follows:

$$\begin{aligned} \Omega^S_{pq}(C) = \big\{z \in \mathbb{C} :\ &\big[\big|z - (\sigma^2 - |a_{pp}|^2)\big| - |a_{pp}|\, r_p(A^*)\big] \cdot \big|z - (\sigma^2 - |a_{qq}|^2)\big| \\ &\leq \sigma\, r_p(A)\ [\sigma\, r_q(A^*) + |a_{qq}|\, r_q(A)]\big\}, \quad p, q \in \langle n\rangle; \end{aligned} \tag{3.9}$$

$$\begin{aligned} \Omega^{\bar S}_{qp}(C) = \big\{z \in \mathbb{C} :\ &\big[\big|z - (\sigma^2 - |a_{qq}|^2)\big| - |a_{qq}|\, r_q(A)\big] \cdot \big|z - (\sigma^2 - |a_{pp}|^2)\big| \\ &\leq \sigma\, r_q(A^*)\ [\sigma\, r_p(A) + |a_{pp}|\, r_p(A^*)]\big\}, \quad p, q \in \langle n\rangle. \end{aligned} \tag{3.10}$$

The inclusion $0 \in \Omega^S_{pq}(C)$ amounts to the inequality

$$\begin{aligned} &\big[|\sigma^2 - |a_{pp}|^2| - |a_{pp}|\, r_p(A^*)\big] \cdot |\sigma^2 - |a_{qq}|^2| \\ &\qquad \leq \sigma\, r_p(A)\, [\sigma\, r_q(A^*) + |a_{qq}|\, r_q(A)], \end{aligned} \tag{3.11}$$

whereas the inclusion $0 \in \Omega^{\bar S}_{qp}(C)$ is equivalent to the inequality

$$\begin{aligned} &\big[\sigma^2 - |a_{qq}|^2| - |a_{qq}|\, r_q(A)\big] \cdot |\sigma^2 - |a_{pp}|^2| \\ &\qquad \leq \sigma\, r_q(A^*)\, [\sigma\, r_p(A)] + |a_{pp}|\, r_p(A^*)]\,. \end{aligned} \tag{3.12}$$

Combining (3.8) with (3.11) and (3.12), we arrive at the following inclusion set for the singular values of a square matrix, first suggested in [4].

Theorem 3.2 ([4]). *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, where $n \geq 2$. Then*

$$\Sigma(A) \subseteq \Delta(A) \equiv \bigcup_{p,q\in\langle n\rangle} [\Delta_{pq}(A) \cup \Delta_{qp}(A^*)], \tag{3.13}$$

where

$$\begin{aligned} \Delta_{pq}(A) = \big\{z \geq 0 :\ &\big[|z^2 - |a_{pp}|^2| - |a_{pp}|\, r_p(A^*)\big] \cdot |z^2 - |a_{qq}|^2| \\ &\leq z\, r_p(A)\, [z\, r_q(A^*) + |a_{qq}|\, r_q(A)]\big\}, \quad p, q \in \langle n\rangle. \end{aligned} \tag{3.14}$$

As is readily verified, for all $p, q \in \langle n \rangle$, we have

$$\begin{aligned}\Delta_{pq}(A) \cup \Delta_{qp}(A^*) &\subseteq \big\{z \ge 0 :\ |z^2 - |a_{pp}|^2| \le |a_{pp}|\ r_p(A^*) + z\ r_p(A)\big\} \\ &\cup \big\{z \ge 0 :\ |z^2 - |a_{qq}|^2| \le |a_{qq}|\ r_q(A) + z\ r_q(A^*)\big\} \\ &\subseteq \bigcup_{i \in \langle n \rangle} \big\{z \ge 0 :\ |z^2 - |a_{ii}|^2| \le \varphi_i(z, A)\big\},\end{aligned}$$

where

$$\varphi_i(z, A) = \max\{|a_{ii}|\ r_i(A^*) + z\ r_i(A),\ |a_{ii}|\ r_i(A) + z\ r_i(A^*)\}, \quad i = 1, \dots, n.$$

This shows that

$$\Delta(A) \subseteq \bigcup_{i \in \langle n \rangle} \big\{z \ge 0 :\ |z^2 - |a_{ii}|^2| \le \varphi_i(z, A)\big\},$$

whence Theorem 3.2 provides an improvement of Theorem 2.1 in [7], claiming that

$$\Sigma(A) \subseteq \bigcup_{i \in \langle n \rangle} \big\{z \ge 0 :\ |z^2 - |a_{ii}|^2| \le \varphi_i(z, A)\big\} \tag{3.15}$$

and corresponding to combining Lemma 3.1 with Gerschgorin's theorem. Note that the inclusion (3.15) was first obtained, though not explicitly stated, in [10]. Quite recently, it has appeared as Theorem 2 in the papers [4] and [5], both citing [10] but not referring to the result in question. It should also be mentioned that up to ignoring some misprints, Theorem 3 in [5] is a weaker version, disregarding the matrix sparsity pattern, of Corollary 2.4 in [7].

Note that the inclusion set provided by Theorem 3.2 actually applies not only to a given matrix A but to all the equimodular matrices $B = (b_{ij})$ for which $|b_{ij}| = |a_{ij}|$, $1 \le i, j \le n$.

Arguing as above but applying Theorem 3.1 rather than Corollary 3.1, we come to the following refinement of Theorem 3.2, which takes into consideration the matrix sparsity pattern.

Theorem 3.3. *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, where $n \ge 2$. Then*

$$\begin{aligned}\Sigma(A) \subseteq &\bigcup_{\substack{p \in \langle n \rangle: \\ r_p(A)=0}} \big\{z \ge 0 :\ \big|z^2 - |a_{pp}|^2\big| \le |a_{pp}|\ r_p(A^*)\big\} \\ &\cup \bigcup_{\substack{q \in \langle n \rangle: \\ r_q(A^*)=0}} \big\{z \ge 0 :\ \big|z^2 - |a_{qq}|^2\big| \le |a_{qq}|\ r_q(A)\big\} \\ &\cup \left[\bigcup_{\substack{p,q \in \langle n \rangle: \\ a_{pq} \ne 0}} \Delta_{pq}(A)\right] \cup \left[\bigcup_{\substack{p,q \in \langle n \rangle: \\ a_{qp} \ne 0}} \Delta_{qp}(A^*)\right],\end{aligned} \tag{3.16}$$

where the sets $\Delta_{pq}(A)$ are defined in accordance with (3.14).

Observe that for a matrix A having no diagonal rows and, in particular, for an arbitrary irreducible matrix A, (3.16) takes the following simpler form:

$$\Sigma(A) \subseteq \left[\bigcup_{\substack{p,q \in \langle n \rangle: \\ a_{pq} \ne 0}} \Delta_{pq}(A)\right] \cup \left[\bigcup_{\substack{p,q \in \langle n \rangle: \\ a_{qp} \ne 0}} \Delta_{qp}(A^*)\right], \tag{3.17}$$

which is the sparse version of (3.13).

We conclude this paper by mentioning that if one chooses the set S distinct from $\langle n \rangle$, say, $S = \{k\}$, where $k \in \langle n \rangle$, then, by applying Lemmas 2.1 and Theorem 3.1 or Corollary 3.1, one can obtain other inclusion sets for the matrix singular values.

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