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RELATIONSHIP GRAPHS OF REAL CAYLEY–DICKSON ALGEBRAS

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The paper studies the anticommutativity condition for elements of arbitrary real Cayley–Dickson algebras. As a consequence, the anticommutativity graphs on equivalence classes of such algebras are classified. Under some additional assumptions on the algebras considered, an expression for the centralizer of an element in terms of its orthogonalizer is obtained. Conditions sufficient for this interrelation to hold are provided. Also examples of real Cayley–Dickson algebras in which the centralizer and orthogonalizer of an element are not interrelated in this way are considered. Bibliography: 28 titles.

1. Introduction

Studies in the area of graphs determined by relations in algebraic systems have originated from group theory, see, e.g., [7]. Rings and algebras were first studied in this way in 1988 by Beck [11], where the zero divisor graph of a commutative ring was introduced. In Beck's definition, the vertex set of the graph coincides with the set of all elements of the ring. Then Anderson and Livingston [6] gave another definition, which excludes zero and all zero divisors of the ring. Mulay [26] considered a new zero divisor graph, whose vertex set consists of the equivalence classes of the zero divisors of the ring. As to the zero divisor graphs of noncommutative rings, they were introduced by Redmond [27] (namely, $\Gamma_Z(\mathcal{R})$ and $\bar{\Gamma}_Z(\mathcal{R})$). In [12], Bozic and Petrovic studied the diameters of the zero divisor graphs of matrix rings over commutative rings and their relationship with the diameters of the zero divisor graphs of the ground rings.

In addition to the zero divisor graphs, commutativity graphs of matrix rings and some other rings were also intensively studied, see [3] and the references therein. In particular, in [2, 4, 16], different authors explored the connectivity and diameters of the commutativity graphs of matrix rings, and also the way they are determined by the ground rings. In [9, 10, 19], the orthogonality graphs ($\Gamma_O(\mathcal{R})$) of matrix rings were studied.

In this paper, we consider relationship graphs of non-associative algebras and focus on their combinatorial characteristics, such as the diameter, clique number, and description of cliques. The paper is organized as follows. Section 2 contains main definitions and notation concerning relationship graphs and some basic facts of graph theory. The Cayley–Dickson process is described in detail in Sec. 3. In Sec. 5, we establish the anticommutativity condition for the elements of an arbitrary real Cayley–Dickson algebra and present some auxiliary lemmas. Theorem 6.3 in Sec. 6 provides a classification of the anticommutativity graphs on equivalence classes of real Cayley–Dickson algebras. Some particular cases of this theorem for quaternions, split-complex numbers, and split-quaternions are considered in Sec. 7. An expression for the centralizer in terms of the orthogonalizer and some sufficient conditions for this expression to hold are provided in Sec. 8, where we also demonstrate that these conditions are essential.

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2. Definitions and Notation

Let $\mathbb{F}$ be an arbitrary field and let $(\mathcal{A}, +, \cdot)$ be an algebra with the identity $1_{\mathcal{A}}$ over the field $\mathbb{F}$. The algebra $\mathcal{A}$ is not assumed to be commutative and associative. Given $a, b \in \mathcal{A}$, we say that

- a and b *commute* if $ab = ba$;
- a and b *anticommute* if $ab + ba = 0$;
- a and b *are orthogonal* if $ab = ba = 0$;
- a is a *left zero divisor* if $a \neq 0$ and there exists a nonzero $b \in \mathcal{A}$ such that $ab = 0$;
- a is a *right zero divisor* if $a \neq 0$ and there exists a nonzero $b \in \mathcal{A}$ such that $ba = 0$;
- a is a *two-sided zero divisor* if it is both a left and a right zero divisor;
- a is a *zero divisor* if it is a left or a right zero divisor.

Definition 2.1. *The center* of an algebra $\mathcal{A}$ is the set $C_{\mathcal{A}} = \{a \in \mathcal{A} \mid ab = ba \text{ for all } b \in \mathcal{A}\}$. *The centralizer* of a subset $S \subset \mathcal{A}$ is $C_{\mathcal{A}}(S) = \{a \in \mathcal{A} \mid as = sa \text{ for all } s \in S\}$, i.e., the set of all elements in $\mathcal{A}$ that commute with every element of S. For an arbitrary $a \in \mathcal{A}$, we denote $C_{\mathcal{A}}(\{a\})$ by $C_{\mathcal{A}}(a)$.

By $Z^*(\mathcal{A})$ we denote the set of zero divisors of $\mathcal{A}$, and $Z^{**}(\mathcal{A})$ is the set of two-sided zero divisors of $\mathcal{A}$. The set $AC^*(\mathcal{A}) = \{a \in \mathcal{A} \setminus \{0\} \mid \exists\, b \in \mathcal{A} \setminus \{0\} \colon ab + ba = 0\}$ is the set of all nonzero elements $a \in \mathcal{A}$ such that a anticommutes with a certain nonzero $b \in \mathcal{A}$.

Definition 2.2. *The anticentralizer* of a subset $S \subset \mathcal{A}$ is the set $\mathrm{Anc}_{\mathcal{A}}(S) = \{a \in \mathcal{A} | as + sa = 0 \text{ for all } s \in S\}$, i.e., the set of all elements in $\mathcal{A}$ that anticommute with every element of S. For an arbitrary $a \in \mathcal{A}$, we denote $\mathrm{Anc}_{\mathcal{A}}(\{a\})$ by $\mathrm{Anc}_{\mathcal{A}}(a)$.

Definition 2.3. *The left annihilator* of a subset $S \subset \mathcal{A}$ is the set

$$\mathrm{l.Ann}_{\mathcal{A}}(S) = \{a \in \mathcal{A} | \, as = 0 \text{ for all } s \in S\}.$$

Similarly, *the right annihilator* of S is

$$\mathrm{r.Ann}_{\mathcal{A}}(S) = \{a \in \mathcal{A} \mid sa = 0 \text{ for all } s \in S\}.$$

For an arbitrary $a \in \mathcal{A}$, we denote $\mathrm{l.Ann}_{\mathcal{A}}(\{a\})$ by $\mathrm{l.Ann}_{\mathcal{A}}(a)$ and $\mathrm{r.Ann}_{\mathcal{A}}(\{a\})$ by $\mathrm{r.Ann}_{\mathcal{A}}(a)$.

Definition 2.4. *The orthogonalizer* of a subset $S \subset \mathcal{A}$ is the set $O_{\mathcal{A}}(S) = \{a \in \mathcal{A} | \, as = sa = 0 \text{ for all } s \in S\}$, i.e., the set of all elements in $\mathcal{A}$ that are orthogonal to every element of S. For an arbitrary $a \in \mathcal{A}$, we denote $O\mathcal{A}(\{a\})$ by $O\mathcal{A}(a)$.

Now we introduce some equivalence relations that will be used below.

Definition 2.5.

(1) *Let $a, b \in \mathcal{A} \setminus C_{\mathcal{A}}$. We say that a and b are* C-equivalent *($a \sim_C b$)* if $C_{\mathcal{A}}(a) = C_{\mathcal{A}}(b)$. The equivalence class of a is denoted by $[a]_C$.

(2) *Let $a, b \in AC^*(\mathcal{A})$. We say that a and b are* AC-equivalent *($a \sim_{AC} b$) if* $\mathrm{Anc}_{\mathcal{A}}(a) = \mathrm{Anc}_{\mathcal{A}}(b)$. *The equivalence class of a is denoted by $[a]_{AC}$.*

(3) *Let $a, b \in Z^*(\mathcal{A})$. We say that a and b are* Z-equivalent *($a \sim_Z b$) if* $\mathrm{l.Ann}_{\mathcal{A}}(a) = \mathrm{l.Ann}_{\mathcal{A}}(b)$ *and* $\mathrm{r.Ann}_{\mathcal{A}}(a) = \mathrm{r.Ann}_{\mathcal{A}}(b)$. *The equivalence class of a is denoted by $[a]_Z$.*

(4) *Let $a, b \in Z^{**}(\mathcal{A})$. We say that a and b are* O-equivalent *($a \sim_O b$) if $O_{\mathcal{A}}(a) = O_{\mathcal{A}}(b)$. The equivalence class of a is denoted by $[a]_O$.*

Remark 2.6. Let $S \subset \mathcal{A}$. It is readily seen that $C_{\mathcal{A}}$, $C_{\mathcal{A}}(S)$, $\mathrm{Anc}_{\mathcal{A}}(S)$, $\mathrm{l.Ann}_{\mathcal{A}}(S)$, $\mathrm{r.Ann}_{\mathcal{A}}(S)$, and $O_{\mathcal{A}}(S)$ are vector spaces over $\mathbb{F}$.

Now we can introduce some relationship graphs that will be studied in this paper.

Definition 2.7. For an algebra $\mathcal{A}$, we define the following structures:

(1) *The commutativity graph* $\Gamma_C(\mathcal{A})$: its vertex set is $\mathcal{A} \setminus C_{\mathcal{A}}$, and distinct vertices a and b are adjacent if and only if $ab = ba$.
(2) *The commutativity graph on the equivalence classes* $\Gamma_C^E(\mathcal{A})$: its vertex set is $\{[a]_C \big| a \in \mathcal{A} \setminus C_{\mathcal{A}}\}$, and distinct vertices $[a]_C$ and $[b]_C$ are adjacent if and only if $ab = ba$.
(3) *The anticommutativity graph* $\Gamma_{AC}(\mathcal{A})$: its vertex set is $AC^*(\mathcal{A})$, and distinct vertices a and b are adjacent if and only if $ab + ba = 0$.
(4) *The anticommutativity graph on the equivalence classes* $\Gamma_{AC}^E(\mathcal{A})$: its vertex set is $\{[a]_{AC} \big| a \in AC^*(\mathcal{A})\}$, and distinct vertices $[a]_{AC}$ and $[b]_{AC}$ are adjacent if and only if $ab + ba = 0$.
(5) *The orthogonality graph* $\Gamma_O(\mathcal{A})$: its vertex set is $Z^{**}(\mathcal{A})$, and distinct vertices a and b are adjacent if and only if $ab = ba = 0$.
(6) *The orthogonality graph on the equivalence classes* $\Gamma_O^E(\mathcal{A})$: its vertex set is $\{[a]_O \big| a \in Z^{**}(\mathcal{A})\}$, and distinct vertices $[a]_O$ and $[b]_O$ are adjacent if and only if $ab = ba = 0$.
(7) *The directed zero divisor graph* $\Gamma_Z(\mathcal{A})$: its vertex set is $Z^*(\mathcal{A})$, and distinct vertices a and b are connected by an arc going from a to b if and only if $ab = 0$.
(8) *The directed zero divisor graph on the equivalence classes* $\Gamma_Z^E(\mathcal{A})$: its vertex set is $\{[a]_Z \big| a \in Z^*(\mathcal{A})\}$, and distinct vertices $[a]_Z$ and $[b]_Z$ are connected by an arc going from $[a]_Z$ to $[b]_Z$ if and only if $ab = 0$.
(9) *The undirected zero divisor graph* $\bar{\Gamma}_Z(\mathcal{A})$: its vertex set is $Z^*(\mathcal{A})$, and distinct vertices a and b are adjacent if and only if $ab = 0$ or $ba = 0$.
(10) *The undirected zero divisor graph on the equivalence classes* $\bar{\Gamma}_Z^E(\mathcal{A})$: its vertex set is $\{[a]_Z \big| a \in Z^*(\mathcal{A})\}$, and distinct vertices $[a]_Z$ and $[b]_Z$ are adjacent if and only if $ab = 0$ or $ba = 0$.

Proposition 2.8. *The graphs* $\Gamma_C^E(\mathcal{A})$, $\Gamma_{AC}^E(\mathcal{A})$, $\Gamma_O^E(\mathcal{A})$, $\Gamma_Z^E(\mathcal{A})$, *and* $\bar{\Gamma}_Z^E(\mathcal{A})$ *are well defined.*

Proof. As is readily verified, adjacency in these graphs is independent of representatives of the equivalence classes. □

We will need the following definitions of graph theory.

Definition 2.9. Let Γ be an undirected graph.

- Γ is said to be *connected* if for every pair of vertices $\{x, y\}$ there exists a path going from x to y, that is, x and y are connected. Otherwise Γ is said to be disconnected.
- *A connected component* of Γ is a maximal connected subgraph of Γ.
- *The distance* $d(x, y)$ between two vertices x and y in Γ is the number of edges in a shortest path connecting them. If there is no path connecting x and y, i.e., they belong to different connected components, then $d(x, y) = \infty$.
- *The diameter* $d(\Gamma)$ of Γ is defined as $\sup\limits_{x,y\in\Gamma} d(x, y)$.
- *A clique* C in Γ is a subset of vertices of Γ such that every two distinct vertices in C are adjacent.
- A clique C is said to be *maximal* if for any clique $\tilde{C}$ such that $C \subset \tilde{C}$ we have $C = \tilde{C}$.

3. Construction of Cayley–Dickson algebras

In this section, mainly based on [22, 28], we recall a classical construction of non-associative algebras, which is called the Cayley–Dickson process.

Definition 3.1 ([22, p. 139, Definition 1.5.1])**.** Let $(\mathcal{A}, +, \cdot)$ be an algebra over a field $\mathbb{F}$. *An involution* $a \mapsto \bar{a}$ in $\mathcal{A}$ is an endomorphism of the vector space $\mathcal{A}$ such that for all $a, b \in \mathcal{A}$ we have $\bar{\bar{a}} = a$ and $\overline{ab} = \bar{b}\bar{a}$.

Definition 3.2. Let $(\mathcal{A}, +, \cdot)$ be an algebra over a field $\mathbb{F}$ with the identity $1_{\mathcal{A}}$ and an involution $a \mapsto \bar{a}$. This involution is said to be *regular* if it satisfies the conditions $a + \bar{a} \in \mathbb{F}1_{\mathcal{A}}$ and $a\bar{a} \in \mathbb{F}1_{\mathcal{A}}$ for any $a \in \mathcal{A}$.

Henceforth, we assume that $\mathcal{A}$ is an algebra over a field $\mathbb{F}$ with a regular involution $a \mapsto \bar{a}$. Consider the following definitions, which are analogous to those for complex numbers.

Definition 3.3. *The real part* of an element $a \in \mathcal{A}$ is $\mathfrak{Re}(a) = \frac{a+\bar{a}}{2}$; *the imaginary part* of a is $\mathfrak{Im}(a) = \frac{a-\bar{a}}{2}$; *the norm* of a is $n(a) = a\bar{a} = \bar{a}a$. An element a is said to be *pure* if $\mathfrak{Re}(a) = 0$.

The norm of a is often defined as $\sqrt{a\bar{a}}$; however, in this paper, we use the norm $n(a) = a\bar{a}$. Note that most of the results can easily be extended to the norm modified in this way.

The next proposition describes some properties of the above notions. Its proof is based on properties of endomorphisms of vector spaces.

Proposition 3.4. *Let $\mathbb{F} = \mathbb{R}$, $a, b \in \mathcal{A}$, $r \in \mathbb{R}$. Then the following equalities hold:*

$$\begin{aligned} \mathfrak{Re}(a+b) &= \mathfrak{Re}(a) + \mathfrak{Re}(b), & \mathfrak{Re}(\mathfrak{Re}(a)) &= \mathfrak{Re}(a), \\ \mathfrak{Im}(a+b) &= \mathfrak{Im}(a) + \mathfrak{Im}(b), & \mathfrak{Re}(\mathfrak{Im}(a)) &= 0, \\ \mathfrak{Re}(ra) &= r\mathfrak{Re}(a), & \mathfrak{Im}(\mathfrak{Re}(a)) &= 0, \\ \mathfrak{Im}(ra) &= r\mathfrak{Im}(a), & \mathfrak{Im}(\mathfrak{Im}(a)) &= \mathfrak{Im}(a). \end{aligned}$$

Now we turn to the Cayley–Dickson process itself.

Definition 3.5 ([28])**.** *The algebra $\mathcal{A}\{\gamma\}$* produced by the Cayley–Dickson process applied to $\mathcal{A}$ with the parameter $\gamma \in \mathbb{F}$ is defined as the set of ordered pairs of elements of $\mathcal{A}$ with the operations

$$\begin{aligned} \alpha(a, b) &= (\alpha a, \alpha b), \\ (a, b) + (c, d) &= (a + c, b + d), \\ (a, b)(c, d) &= (ac + \gamma\bar{d}b, da + b\bar{c}) \end{aligned}$$

and the involution

$$\overline{(a, b)} = (\bar{a}, -b), \qquad a, b, c, d \in \mathcal{A},\ \alpha \in \mathbb{F}.$$

Proposition 3.6 ([28])**.** *The algebra $\mathcal{A}\{\gamma\}$ is an algebra over $\mathbb{F}$ with the identity $1_{\mathcal{A}\{\gamma\}} = (1_{\mathcal{A}}, 0)$ and a regular involution.*

Proposition 3.7 ([28])**.** *Let $\mathcal{A}$ be an n-dimensional algebra and let $\{e_i\}_{i=1,\dots,n}$ be a basis in $\mathcal{A}$. Then $\mathcal{A}\{\gamma\}$ is a $2n$-dimensional algebra, and $\{(e_i, 0), (0, e_i)\}_{i=1,\dots,n}$ is a basis in $\mathcal{A}\{\gamma\}$.*

Thus, starting with a one-dimensional algebra and successively applying the Cayley–Dickson process to it, at the nth step we obtain a 2^n-dimensional algebra.

Lemma 3.8 ([28])**.** *Let $a, b \in \mathcal{A}$, $(a, b) \in \mathcal{A}\{\gamma\}$. Then*

$$\begin{aligned} \mathfrak{Re}((a, b)) &= \mathfrak{Re}(a), \\ \mathfrak{Im}((a, b)) &= (\mathfrak{Im}(a), b), \\ n((a, b)) &= n(a) - \gamma n(b). \end{aligned}$$

Proof. Consider the following strings of equalities:

$$\mathfrak{Re}((a,b)) = \frac{(a,b) + \overline{(a,b)}}{2} = \frac{(a,b) + (\bar{a}, -b)}{2} = \frac{(a + \bar{a}, 0)}{2} = \frac{(2\mathfrak{Re}(a), 0)}{2} = \mathfrak{Re}(a)1_{\mathcal{A}\{\gamma\}} = \mathfrak{Re}(a);$$

$$\mathfrak{Im}((a,b)) = \frac{(a,b) - \overline{(a,b)}}{2} = \frac{(a,b) - (\bar{a}, -b)}{2} = \frac{(a - \bar{a}, 2b)}{2} = \frac{(2\mathfrak{Im}(a), 2b)}{2} = (\mathfrak{Im}(a), b);$$

$$n((a,b)) = (a,b)(\overline{a,b}) = (a,b)(\bar{a},-b) = (a\bar{a}-\gamma\bar{b}b, -ba+b\bar{\bar{a}}) = (n(a)-\gamma n(b),0)$$
$$= (n(a)-\gamma n(b))(1_{\mathcal{A}},0) = (n(a)-\gamma n(b))1_{\mathcal{A}\{\gamma\}} = n(a)-\gamma n(b). \qquad \square$$

Definition 3.9. Let $b, c \in \mathcal{A}$, $a = (b,c) \in \mathcal{A}\{\gamma\}$. An element a is said to be *doubly pure* if $\mathfrak{Re}(b) = \mathfrak{Re}(c) = 0$.

Proposition 3.10. *If $a \in \mathcal{A}\{\gamma\}$ is doubly pure, then a is pure.*

Proof. If $a = (b,c)$, $\mathfrak{Re}(b) = \mathfrak{Re}(c) = 0$, then $\mathfrak{Re}(a) = \mathfrak{Re}((b,c)) = \mathfrak{Re}(b) = 0$. $\square$

Notation 3.11. $\mathcal{A}\{\gamma_1,\dots,\gamma_n\}$ denotes $(\dots(\mathcal{A}\{\gamma_1\})\dots)\{\gamma_n\}$.

The question whether this construction is correct naturally arises. In particular, does the order of application of the Cayley–Dickson process with different values of γ influence the result we obtain? The following lemma contains an observation concerning this issue.

Lemma 3.12. *If $\mathcal{A}$ is a commutative algebra, then for any $\gamma_1,\gamma_2 \in \mathbb{F}$ the algebras $\mathcal{A}\{\gamma_1,\gamma_2\}$ and $\mathcal{A}\{\gamma_2,\gamma_1\}$ are isomorphic.*

Proof. Consider the mapping $\phi : \mathcal{A}\{\gamma_1,\gamma_2\} \longrightarrow \mathcal{A}\{\gamma_2,\gamma_1\}$ such that for all $a,b,c,d \in \mathcal{A}$,

$$\phi : ((a,b)_{\gamma_1},(c,d)_{\gamma_1})_{\gamma_2} \mapsto ((a,c)_{\gamma_2},(-b,d)_{\gamma_2})_{\gamma_1}.$$

Obviously, ϕ is bijective. Show that ϕ preserves multiplication, whence ϕ is a homomorphism. Indeed, we have

$$\begin{aligned}
&\Big((a,b)_{\gamma_1},(c,d)_{\gamma_1}\Big)_{\gamma_2}\Big((a',b')_{\gamma_1},(c',d')_{\gamma_1}\Big)_{\gamma_2}\\
&= \Big((a,b)_{\gamma_1}(a',b')_{\gamma_1} + \gamma_2(\overline{c',d'})_{\gamma_1}(c,d)_{\gamma_1}, (c',d')_{\gamma_1}(a,b)_{\gamma_1} + (c,d)_{\gamma_1}(\overline{a',b'})_{\gamma_1}\Big)_{\gamma_2}\\
&= \Big((a,b)_{\gamma_1}(a',b')_{\gamma_1} + \gamma_2(\bar{c'},-d')_{\gamma_1}(c,d)_{\gamma_1}, (c',d')_{\gamma_1}(a,b)_{\gamma_1} + (c,d)_{\gamma_1}(\bar{a'},-b')_{\gamma_1}\Big)_{\gamma_2}\\
&= \Big((aa'+\gamma_1\bar{b'}b, b'a+b\bar{a'})_{\gamma_1} + \gamma_2(\bar{c'}c-\gamma_1\bar{d}d', d\bar{c'}-d'\bar{c})_{\gamma_1},\\
&\qquad (c'a+\gamma_1\bar{b}d', bc'+d'\bar{a})_{\gamma_1} + (c\bar{a'}-\gamma_1\bar{b'}d, -b'c+da')_{\gamma_1}\Big)_{\gamma_2}\\
&= \Big(\big(aa'+\gamma_1\bar{b'}b+\gamma_2\bar{c'}c-\gamma_1\gamma_2\bar{d}d', (b'a+b\bar{a'})+\gamma_2(d\bar{c'}-d'\bar{c})\big)_{\gamma_1},\\
&\qquad \big((c'a+c\bar{a'})+\gamma_1(\bar{b}d'-\bar{b'}d), (bc'-b'c)+(d'\bar{a}+da')\big)_{\gamma_1}\Big)_{\gamma_2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\phi\Big(\big((a,b)_{\gamma_1},(c,d)_{\gamma_1}\big)_{\gamma_2}\Big)\phi\Big(\big((a',b')_{\gamma_1},(c',d')_{\gamma_1}\big)_{\gamma_2}\Big)\\
&= \Big((a,c)_{\gamma_2},(-b,d)_{\gamma_2}\Big)_{\gamma_1}\Big((a',c')_{\gamma_2},(-b',d')_{\gamma_2}\Big)_{\gamma_1}\\
&= \Big(\big(aa'+\gamma_2\bar{c'}c+\gamma_1\bar{b'}b-\gamma_2\gamma_1\bar{d}d', (c'a+c\bar{a'})+\gamma_1(-d\bar{b'}+d'\bar{b})\big)_{\gamma_2},\\
&\qquad \big((-b'a-b\bar{a'})+\gamma_2(\bar{c}d'-\bar{c'}d), (-cb'+c'b)+(d'\bar{a}+da')\big)_{\gamma_2}\Big)_{\gamma_1}\\
&= \Big(\big(aa'+\gamma_1\bar{b'}b+\gamma_2\bar{c'}c-\gamma_1\gamma_2\bar{d}d', (c'a+c\bar{a'})+\gamma_1(d'\bar{b}-d\bar{b'})\big)_{\gamma_2},\\
&\qquad \big(-((b'a+b\bar{a'})+\gamma_2(\bar{c'}d-\bar{c}d')), (bc'-b'c)+(d'\bar{a}+da')\big)_{\gamma_2}\Big)_{\gamma_1}\\
&= \Big(\big(aa'+\gamma_1\bar{b'}b+\gamma_2\bar{c'}c-\gamma_1\gamma_2\bar{d}d', (c'a+c\bar{a'})+\gamma_1(\bar{b}d'-\bar{b'}d)\big)_{\gamma_2},
\end{aligned}$$

$$\Big(-((b'a+b\bar{a'})+\gamma_2(d\bar{c'}-d'\bar{c})),(bc'-b'c)+(d'\bar{a}+da')\Big)_{\gamma_2}\Big)_{\gamma_1}$$
$$=\phi\Big(\big((aa'+\gamma_1\bar{b'}b+\gamma_2\bar{c'}c-\gamma_1\gamma_2\bar{d}d',(b'a+b\bar{a'})+\gamma_2(d\bar{c'}-d'\bar{c})\big)_{\gamma_1},$$
$$\big((c'a+c\bar{a'})+\gamma_1(\bar{b}d'-\bar{b'}d),(bc'-b'c)+(d'\bar{a}+da')\big)_{\gamma_1}\big)_{\gamma_2}\Big)$$
$$=\phi\Big(\big((a,b)_{\gamma_1},(c,d)_{\gamma_1}\big)_{\gamma_2}\big((a',b')_{\gamma_1},(c',d')_{\gamma_1}\big)_{\gamma_2}\Big).$$
□

Definition 3.13. For every integer $n \geq 0$ and arbitrary reals $\gamma_0, \dots, \gamma_{n-1}$, we inductively define the algebras $\mathcal{A}_n = \mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ as follows:

1) $\mathcal{A}_0 = \mathbb{R}$, and $e_0^{(0)} = 1$ is its only basis element.
2) If $\mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is already available, then $\mathcal{A}_{n+1}\{\gamma_0, \dots, \gamma_n\} = (\mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\})\{\gamma_n\}$. Its basis elements are $e_0^{(n+1)}, \dots, e_{2^{n+1}-1}^{(n+1)}$, which are defined by

$$e_i^{(n+1)} = \begin{cases} (e_i^{(n)}, 0), & 0 \leq i \leq 2^n - 1, \\ (0, e_{i-2^n}^{(n)}), & 2^n \leq i \leq 2^{n+1} - 1. \end{cases}$$

Lemma 3.14 ([28]). *For every integer $n \geq 0$, the structure $\mathcal{A}_n$ constructed in Definition 3.13 is a 2^n-dimensional algebra over $\mathbb{R}$ with the identity $e_0^{(n)}$ and a regular involution.*

Proof. This assertion follows from Propositions 3.6 and 3.7 by induction on n. □

We will use the following notation: $1 = 1^{(n)} = e_0^{(n)}$ and $r = r1^{(n)}$ for $r \in \mathbb{R}$. The superscript is omitted if it is clear from the context. From the definition of $\mathcal{A}_n$ it follows that real numbers commute with all its elements, and we obtain the following result.

Corollary 3.15. *For every integer $n \geq 0$, we have $\mathbb{R} \subset C_{\mathcal{A}_n}$.*

Proposition 3.16 ([22, p. 161, Exercise 2.5.1]). *Let $\gamma' = \alpha^2\gamma$, with $\alpha \neq 0$. Then $\mathcal{A}\{\gamma\}$ and $\mathcal{A}\{\gamma'\}$ are isomorphic.*

From Proposition 3.16 it follows that in studying real Cayley–Dickson algebras $\mathcal{A}_n = \mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$, it is sufficient to consider $\gamma_k \in \{0, \pm1\}$, $k = 0, \dots, n-1$, because for other values of γ_k the resulting algebras are isomorphic to these ones.

Assume that $0^0 = 1$.

Notation 3.17. Let $\mathcal{A}_n$ be a real Cayley–Dickson algebra. For every $m = 0, \dots, 2^n - 1$, we set

$$\delta_m^{(n)} = \prod_{l=0}^{n-1} (-\gamma_l)^{c_{m,l}},$$

where the indices $c_{m,l} \in \{0, 1\}$ are uniquely determined by the condition $m = \sum_{l=0}^{n-1} c_{m,l} 2^l$.

Proposition 3.18. *For any $m = 0, \dots, 2^n - 1$, the value of $\delta_m^{(n)}$ is determined uniquely.*

Proof. This assertion follows from the uniqueness of the binary representation of a nonnegative integer. □

Remark 3.19. For any $\gamma_0, \dots, \gamma_{n-1}$, we have $\delta_0^{(n)} = 1$.

In the sequel, we will need the following lemma.

Lemma 3.20. *Let* $a = a_0 + a_1 e_1^{(n)} + \cdots + a_{2^n-1} e_{2^n-1}^{(n)} \in \mathcal{A}_n$. *Then*

$$\begin{aligned} \bar{a} &= a_0 - a_1 e_1^{(n)} - \cdots - a_{2^n-1} e_{2^n-1}^{(n)}; \\ \mathfrak{Re}(a) &= a_0; \\ \mathfrak{Im}(a) &= a_1 e_1^{(n)} + \cdots + a_{2^n-1} e_{2^n-1}^{(n)}; \\ n(a) &= \sum_{m=0}^{2^n-1} \delta_m^{(n)} a_m^2. \end{aligned}$$

Proof. The equalities are obtained from Lemma 3.8 by direct calculations. □

4. Some examples of real Cayley–Dickson algebras

Definition 4.1. An algebra $\mathcal{A}_n\{\gamma_0, \ldots, \gamma_{n-1}\}$ is said to belong to *the main Cayley–Dickson sequence* if $\gamma_i = -1$ for every $i = 0, \ldots, n-1$.

Example 4.2. Examples of real Cayley–Dickson algebras of the main sequence are provided by the complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$), octonions ($\mathbb{O}$), and sedenions ($\mathbb{S}$). We refer the reader to [8] for the definitions of $\mathbb{H}$ and $\mathbb{O}$, and to [15] for that of $\mathbb{S}$.

Definition 4.3. An algebra $\mathcal{A}_n\{\gamma_0, \ldots, \gamma_{n-1}\}$ is called a *real Cayley–Dickson split algebra* if $\gamma_i = -1$ for all $i = 0, \ldots, n-2$ and $\gamma_{n-1} = 1$.

Example 4.4. Examples of real Cayley–Dickson split algebras are provided by the split-complex numbers ($\hat{\mathbb{C}}$), split-quaternions (coquaternions; $\hat{\mathbb{H}}$), and split-octonions (hyperbolic octonions; $\hat{\mathbb{O}}$), all of them being defined in [13].

It is known that $\hat{\mathbb{H}}$ is isomorphic to the algebra $M_2(\mathbb{R})$ of 2×2 matrices over $\mathbb{R}$, see [22, p. 66], and $\hat{\mathbb{O}}$ is isomorphic to the Zorn vector-matrix algebra, see [22, p. 158]. The constructive definitions of $\hat{\mathbb{C}}$ and $\hat{\mathbb{H}}$ are given below. We will need them in Sec. 7.

Definition 4.5 ([13])**.** The algebra $\hat{\mathbb{C}}$ is the algebra of elements of the form $a + b\ell$, where $a, b \in \mathbb{R}$, $\ell^2 = 1$, with the involution $\overline{a + b\ell} = a - b\ell$.

Definition 4.6 ([13])**.** The algebra $\hat{\mathbb{H}}$ is a four-dimensional algebra over $\mathbb{R}$; its basis elements are $1, i, \ell, \ell i$. The involution in $\hat{\mathbb{H}}$ is defined by the formula $\overline{a_0 + a_1 i + a_2 \ell + a_3 \ell i} = a_0 - a_1 i - a_2 \ell - a_3 \ell i$, and multiplication is defined by Table 1.

Table 1. Multiplication table of the unit split-quaternions.

$\times$	1	i	ℓ	ℓi
1	1	i	ℓ	ℓi
i	i	-1	$-\ell i$	ℓ
ℓ	ℓ	ℓi	1	i
ℓi	ℓi	$-\ell$	$-i$	1

Proposition 4.7 ([23], [22, pp. 64–66])**.** *The following isomorphisms hold:*

$$\begin{aligned} &\mathbb{C} \cong \mathcal{A}_1\{-1\}; && \hat{\mathbb{C}} \cong \mathcal{A}_1\{1\}; \\ &\mathbb{H} \cong \mathcal{A}_2\{-1, -1\}; && \hat{\mathbb{H}} \cong \mathcal{A}_2\{-1, 1\}; \\ &\mathbb{O} \cong \mathcal{A}_3\{-1, -1, -1\}; && \hat{\mathbb{O}} \cong \mathcal{A}_3\{-1, -1, 1\}; \\ &\mathbb{S} \cong \mathcal{A}_4\{-1, -1, -1, -1\}. && \end{aligned}$$

5. Anticommutativity in real Cayley–Dickson algebras

Everywhere below, we assume that $\mathcal{A}_n = \mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$, where $\gamma_i \in \{-1, 0, 1\}$, $i = 0, \dots, n-1$, is an arbitrary real Cayley–Dickson algebra.

Lemma 5.1. *Let $a \in \mathcal{A}_n$. Then the following conditions are equivalent:*
1) *$a \in O_{\mathcal{A}_n}(a)$, i.e., $a^2 = 0$;*
2) *$a \in \mathrm{Anc}_{\mathcal{A}_n}(a)$;*
3) *$n(a) = 0$ and $\mathfrak{Re}(a) = 0$.*

Proof. 1) $\Leftrightarrow$ 2)**::** The conditions $a^2 = 0$ and $2a^2 = 0$ are equivalent in $\mathcal{A}_n$.

1) $\Rightarrow$ 3)**::** Let $a^2 = 0$. Then $n(a) = a\bar{a} = a(2\mathfrak{Re}(a) - a) = 2\mathfrak{Re}(a)a - a^2 = 2\mathfrak{Re}(a)a \in \mathbb{R}$. Hence either $\mathfrak{Re}(a) = 0$ and then $n(a) = 0$, or $a \in \mathbb{R}$ and then $a = 0$.

3) $\Rightarrow$ 1)**::** Let $n(a)=0$ and $\mathfrak{Re}(a)=0$. Then $\bar{a}=-a$ and $a^2=-a\bar{a}=-n(a) = 0$. □

Lemma 5.2. *Let $a \in \mathcal{A}_n$ and $\mathfrak{Re}(a) = 0$. Then $a^2 = -n(a) \in \mathbb{R}$.*

Proof. Since $\mathfrak{Re}(a) = 0$, it follows that $\bar{a} = -a$ and $a^2 = -a\bar{a} = -n(a) \in \mathbb{R}$. □

Notation 5.3. Given elements $a = \sum_{m=0}^{2^n-1} a_m e_m^{(n)}, b = \sum_{m=0}^{2^n-1} b_m e_m^{(n)} \in \mathcal{A}_n$, we denote

$$\Lambda(a, b) = \sum_{m=0}^{2^n-1} \delta_m^{(n)} a_m b_m,$$

where $\delta_m^{(n)}$ is defined in Notation 3.17.

Proposition 5.4. *$\Lambda(a, b)$ is a real-valued symmetric bilinear form, that is, for all $a, b \in \mathcal{A}$, $\alpha \in \mathbb{R}$,*

$$\begin{aligned}
\Lambda(a_1 + a_2, b) &= \Lambda(a_1, b) + \Lambda(a_2, b);\\
\Lambda(\alpha a, b) &= \alpha\Lambda(a, b);\\
\Lambda(a, b) &= \Lambda(b, a);\\
\Lambda(a, b) &\in \mathbb{R}.
\end{aligned}$$

Proof. All the properties can be verified directly. □

Proposition 5.5. *For any $a \in \mathcal{A}$, we have $n(a) = \Lambda(a, a)$.*

Proof. This follows from Lemma 3.20. □

Proposition 5.6. *Let $a, b \in \mathcal{A}_n$. Then $\bar{a}b + \bar{b}a = 2\Lambda(a, b) \in \mathbb{R}$.*

Proof. Using Propositions 5.4 and 5.5, we derive

$$\begin{aligned}
\bar{a}b + \bar{b}a &= (\bar{a} + \bar{b})(a + b) - \bar{a}a - \bar{b}b = \overline{(a + b)}(a + b) - \bar{a}a - \bar{b}b\\
&= n(a + b) - n(a) - n(b) = \Lambda(a + b, a + b) - \Lambda(a, a) - \Lambda(b, b)\\
&= \big(\Lambda(a, a) + \Lambda(a, b) + \Lambda(b, a) + \Lambda(b, b)\big) - \Lambda(a, a) - \Lambda(b, b) = 2\Lambda(a, b).
\end{aligned}$$

□

Corollary 5.7. *Let $a, b \in \mathcal{A}_n$, $\mathfrak{Re}(a) = \mathfrak{Re}(b) = 0$. Then $ab + ba = -2\Lambda(a, b) \in \mathbb{R}$.*

Proof. This is a special case of Proposition 5.6 for $\bar{a} = -a$, $\bar{b} = -b$. □

Lemma 5.8. *Let $a \in \mathcal{A}_n$, $a \neq 0$.*
(1) *If $\mathfrak{Re}(a) \neq 0$, $n(a) \neq 0$, then $\mathrm{Anc}_{\mathcal{A}_n}(a) = 0$.*
(2) *If $\mathfrak{Re}(a) \neq 0$, $n(a) = 0$, then $\mathrm{Anc}_{\mathcal{A}_n}(a) = \mathbb{R}\bar{a}$.*
(3) *If $\mathfrak{Re}(a) = 0$, then $b \in \mathrm{Anc}_{\mathcal{A}_n}(a)$ if and only if $\mathfrak{Re}(b) = 0$ and $\Lambda(a, b) = 0$.*

Proof. Consider the anticommutativity condition for some nonzero elements $a, b \in \mathcal{A}_n$:

$$0 = ab + ba = (\mathfrak{Re}(a) + \mathfrak{Im}(a))(\mathfrak{Re}(b) + \mathfrak{Im}(b)) + (\mathfrak{Re}(b) + \mathfrak{Im}(b))(\mathfrak{Re}(a) + \mathfrak{Im}(a))$$
$$= (2\mathfrak{Re}(a)\mathfrak{Re}(b) + \mathfrak{Im}(a)\mathfrak{Im}(b) + \mathfrak{Im}(b)\mathfrak{Im}(a)) + 2(\mathfrak{Re}(a)\mathfrak{Im}(b) + \mathfrak{Re}(b)\mathfrak{Im}(a)).$$

Note that

$$A_1 = 2\mathfrak{Re}(a)\mathfrak{Re}(b) + \mathfrak{Im}(a)\mathfrak{Im}(b) + \mathfrak{Im}(b)\mathfrak{Im}(a) = \mathfrak{Re}(ab + ba),$$

whereas

$$A_2 = 2(\mathfrak{Re}(a)\mathfrak{Im}(b) + \mathfrak{Re}(b)\mathfrak{Im}(a)) = \mathfrak{Im}(ab + ba).$$

The equality $A_1 + A_2 = 0$ implies $A_1 = A_2 = 0$.

First assume that $\mathfrak{Re}(a) \neq 0$. In this case,

$$\mathfrak{Im}(b) = -\frac{\mathfrak{Re}(b)}{\mathfrak{Re}(a)}\mathfrak{Im}(a),$$

$$b = \mathfrak{Re}(b) + \mathfrak{Im}(b) = \frac{\mathfrak{Re}(b)}{\mathfrak{Re}(a)}(\mathfrak{Re}(a) - \mathfrak{Im}(a)) = \frac{\mathfrak{Re}(b)}{\mathfrak{Re}(a)}\bar{a}. \tag{5.1}$$

Thus,

$$0 = ab + ba = \frac{\mathfrak{Re}(b)}{\mathfrak{Re}(a)}(a\bar{a} + \bar{a}a) = 2\frac{\mathfrak{Re}(b)}{\mathfrak{Re}(a)}n(a). \tag{5.2}$$

If $n(a) \neq 0$, then equality (5.2) implies $\mathfrak{Re}(b) = 0$, and, by equality (5.1), we obtain $b = 0$. Thus, (1) is proved.

If $n(a) = 0$, then (2) follows from equality (5.1).

The case where $\mathfrak{Re}(b) \neq 0$ is considered similarly and leads us to the conditions of either (1) or (2).

If $\mathfrak{Re}(a) = \mathfrak{Re}(b) = 0$, then, by Corollary 5.7, we obtain (3). □

Let q denote the number of zero elements among $\gamma_0, \dots, \gamma_{n-1}$. Now we introduce the following subsets of the set of indices $\{1, \dots, 2^n - 1\}$:

$$M_+ = \left\{1 \leq m \leq 2^n - 1 \;\middle|\; \delta_m^{(n)} > 0\right\},$$
$$M_- = \left\{1 \leq m \leq 2^n - 1 \;\middle|\; \delta_m^{(n)} < 0\right\},$$
$$M_0 = \left\{1 \leq m \leq 2^n - 1 \;\middle|\; \delta_m^{(n)} = 0\right\},$$
$$M_\pm = M_+ \cup M_-.$$

Proposition 5.9. *In the above notation,* $M_+ \cup M_- \cup M_0 = \{1, \dots, 2^n - 1\}$ *and* $M_+ \cap M_- = M_+ \cap M_0 = M_0 \cap M_- = \varnothing$.

Proof. The relations immediately follows from the definitions of M_+, M_-, M_0. □

Proposition 5.10. *In the above notation, the following assertions hold*:

(1) $|M_\pm| = 2^{n-q} - 1$, $|M_0| = 2^n - 2^{n-q}$*;*
(2) *if* $\gamma_0, \dots, \gamma_{n-1} \leq 0$, *then* $|M_+| = 2^{n-q} - 1$, $|M_-| = 0$;
(3) *if for a certain* $i \in \{0, \dots, n-1\}$ *the condition* $\gamma_i > 0$ *is fulfilled, then* $|M_+| = 2^{n-q-1} - 1$, $|M_-| = 2^{n-q-1}$.

Proof. The assertions follow from the definitions of M_+, M_-, M_0, Notation 3.17, and Proposition 5.9. □

We will need the following linear subspaces of $\mathcal{A}_n$:

$$\mathcal{A}_n^+ = \bigoplus_{m\in M_+} \langle e_m^{(n)} \rangle, \quad \mathcal{A}_n^- = \bigoplus_{m\in M_-} \langle e_m^{(n)} \rangle, \quad \mathcal{A}_n^0 = \bigoplus_{m\in M_0} \langle e_m^{(n)} \rangle,$$

$$\mathcal{A}_n^\pm = \mathcal{A}_n^+ \oplus \mathcal{A}_n^-, \quad \mathcal{A}_n' = \mathcal{A}_n^\pm \oplus \mathcal{A}_n^0.$$

By definition, $\mathcal{A}_n' = \mathfrak{Im}(\mathcal{A}_n)$, that is, this is the set of pure elements of $\mathcal{A}_n$.

Let $a \in \mathcal{A}_n$, $a = \sum\limits_{m=0}^{2^n-1} a_m e_m^{(n)}$. Obviously, $\mathfrak{Im}(a) \in \mathcal{A}_n'$. We will also need the following notation:

$$a_+ = \sum_{m\in M_+} a_m e_m^{(n)} \in \mathcal{A}_n^+, \quad a_- = \sum_{m\in M_-} a_m e_m^{(n)} \in \mathcal{A}_n^-,$$

$$a_0 = \sum_{m\in M_0} a_m e_m^{(n)} \in \mathcal{A}_n^0, \quad a_\pm = \mathfrak{Im}(a) - a_0 \in \mathcal{A}_n^\pm.$$

Below, we present some immediate consequences of the above definitions, which will be used in the proof of the main result.

Corollary 5.11. *Let* $a \in \mathcal{A}_n' \setminus \{0\}$.

(1) *If* $a \in \mathcal{A}_n^0$, *then* $\mathrm{Anc}_{\mathcal{A}_n}(a) = \mathcal{A}_n'$ *and* $\dim(\mathrm{Anc}_{\mathcal{A}_n}(a)) = 2^n - 1$.
(2) *If* $a \notin \mathcal{A}_n^0$, *then* $\mathrm{Anc}_{\mathcal{A}_n}(a) \subsetneq \mathcal{A}_n'$ *and* $\dim(\mathrm{Anc}_{\mathcal{A}_n}(a)) = 2^n - 2$.

Proof. Since $\mathfrak{Re}(a) = 0$, from Lemma 5.8 it follows that $b \in \mathrm{Anc}_{\mathcal{A}_n}(a)$ if and only if $\mathfrak{Re}(b) = 0$ and $\Lambda(a, b) = 0$.

(1) If $a \in \mathcal{A}_n^0$, then the condition $\Lambda(a, b) = 0$ is fulfilled automatically, whence the only essential condition is $b \in \mathcal{A}_n'$.
(2) If $a \notin \mathcal{A}_n^0$, then the equation $\Lambda(a, b) = 0$ determines a proper (2^n-2)-dimensional subspace of $\mathcal{A}_n'$. ☐

Corollary 5.12. *Let* $a \in \mathcal{A}_n'$, $n(a) \neq 0$. *Then*

(1) $\mathcal{A}_n' = \mathbb{R}a \oplus \mathrm{Anc}_{\mathcal{A}_n}(a)$;
(2) $\mathcal{A}_n = C_{\mathcal{A}_n}(a) + \mathrm{Anc}_{\mathcal{A}_n}(a)$.

Proof. (1) By Lemma 5.1, $n(a) \neq 0$ implies $a \notin \mathrm{Anc}_{\mathcal{A}_n}(a)$. Moreover, from Corollary 5.11 it follows that $\mathrm{Anc}_{\mathcal{A}_n}(a) \subset \mathcal{A}_n'$ and $\dim(\mathrm{Anc}_{\mathcal{A}_n}(a)) \geq 2^n - 2$. Therefore, $\mathbb{R}a \oplus \mathrm{Anc}_{\mathcal{A}_n}(a) \subset \mathcal{A}_n'$, $\dim(\mathbb{R}a \oplus \mathrm{Anc}_{\mathcal{A}_n}(a)) \geq 2^n - 1 = \dim(\mathcal{A}_n')$, whence $\mathcal{A}_n' = \mathbb{R}a \oplus \mathrm{Anc}_{\mathcal{A}_n}(a)$.

(2) Since $1, a \in C_{\mathcal{A}_n}(a)$, we have

$$\mathcal{A}_n = \mathbb{R} \oplus \mathcal{A}_n' = \mathbb{R} \oplus (\mathbb{R}a \oplus \mathrm{Anc}_{\mathcal{A}_n}(a)) = (\mathbb{R} \oplus \mathbb{R}a) \oplus \mathrm{Anc}_{\mathcal{A}_n}(a) \subset C_{\mathcal{A}_n}(a) + \mathrm{Anc}_{\mathcal{A}_n}(a).$$

This completes the proof. ☐

Remark 5.13. The sum in the equality $\mathcal{A}_n = C_{\mathcal{A}_n}(a) + \mathrm{Anc}_{\mathcal{A}_n}(a)$ can be replaced with the direct sum if and only if $O_{\mathcal{A}_n}(a) = \{0\}$.

Proof. $C_{\mathcal{A}_n}(a) \cap \mathrm{Anc}_{\mathcal{A}_n}(a) = O_{\mathcal{A}_n}(a)$. ☐

Example 5.14. As an example of the case where $O_{\mathcal{A}_n}(a) \neq \{0\}$, consider an arbitrary zero divisor a in $\mathcal{A}_n = \mathbb{S}$.

Lemma 5.15. *For* $\mathcal{A}_n$, *the following assertions hold*:

(1) *An element* $a \in AC^*(\mathcal{A}_n)$ *such that* $\mathfrak{Re}(a) = 0$ *exists if and only if either* $n \geq 2$ *or* $n = 1$ *and* $\gamma_0 = 0$.
(2) *An element* $a \in \mathcal{A}_n$ *such that* $\mathfrak{Re}(a) \neq 0$ *and* $n(a) = 0$ *exists if and only if* $\mathcal{A}_n^- \neq 0$, *that is, at least one of the numbers* $\gamma_0, \ldots, \gamma_{n-1}$ *is positive.*

(3) *$AC^*(\mathcal{A}_n) = \varnothing$ if and only if either $n = 0$ or $n = 1$ and $\gamma_0 < 0$.*

Proof. (1) It is known that $a \in AC^*(\mathcal{A}_n)$ and $\mathfrak{Re}(a) = 0$ if and only if the equation $\Lambda(a, b) = 0$ in the variable $b = \sum\limits_{m=1}^{2^n-1} b_m e_m^{(n)}$ has at least one nonzero solution.

• If $n \geq 2$, then any $a \in \mathcal{A}'_n \setminus \{0\}$ satisfies this condition.

• If $n = 1$, then $a = a_1 e_1^{(1)}$, and the equation in the variable $b = b_1 e_1^{(1)}$ takes the form $-\gamma_0 a_1 b_1 = 0$. Since $a \neq 0$, we have $a_1 \neq 0$. Hence this equation has a nonzero solution if and only if $\gamma_0 = 0$.

(2) If $\gamma_0, \dots, \gamma_{n-1} \leq 0$, $a = \sum\limits_{m=0}^{2^n-1} a_m e_m^{(n)}$, then the coefficients at all terms of $n(a) = \sum\limits_{m=0}^{2^n-1} \delta_m^{(n)} a_m^2$ are nonzero. Moreover, $\delta_0^{(n)} = 1$. Thus, $\mathfrak{Re}(a) \neq 0$ implies $n(a) > 0$.

If there exists a certain $i \in \{0, \dots, n-1\}$ such that $\gamma_i > 0$, then one can take $a = \sqrt{\gamma_i} + e_{2^i}^{(n)}$.

(3) Assume that either $n = 0$ or $n = 1$ and $\gamma_0 < 0$. Then, obviously, $AC^*(\mathcal{A}_n) = \varnothing$. Now assume that either $n \geq 2$ or $n = 1$ and $\gamma_0 \geq 0$. Prove that $AC^*(\mathcal{A}_n) \neq \varnothing$.

• Consider the case where either $n \geq 2$ or $n = 1$ and $\gamma_0 = 0$. Then, as has been shown above, there exists $a \in AC^*(\mathcal{A}_n)$ such that $\mathfrak{Re}(a) = 0$.

• Consider the case where $n = 1$ and $\gamma_0 > 0$. As we have proved above, there exists $b \in \mathcal{A}_n$ such that $\mathfrak{Re}(b) \neq 0$ and $n(b) = 0$, whence $b \in AC^*(\mathcal{A}_n)$. □

Corollary 5.16. *Let $AC^*(\mathcal{A}_n) \neq \varnothing$, $a \in \mathcal{A}_n$.*

(1) *If $\mathcal{A}_n^- \neq 0$ or $\mathcal{A}_n^0 = 0$, then a anticommutes with every $b \in AC^*(\mathcal{A}_n)$ if and only if $a = 0$.*
(2) *If $\mathcal{A}_n^- = 0$ and $\mathcal{A}_n^0 \neq 0$, then a anticommutes with every $b \in AC^*(\mathcal{A}_n)$ if and only if $a \in \mathcal{A}_n^0$.*

Proof. Obviously, if $a = 0$, then $ab + ba = 0$ for any $b \in AC^*(\mathcal{A}_n)$. We now check the existence of a nonzero a satisfying this condition. In view of the condition $AC^*(\mathcal{A}_n) \neq \varnothing$, we have $a \in AC^*(\mathcal{A}_n)$.

(1)
- Let $\mathcal{A}_n^- \neq 0$. Then there exists $b \in \mathcal{A}_n$ such that $\mathfrak{Re}(b) \neq 0$ and $n(b) = 0$. Then $b, \bar{b} \in AC^*(\mathcal{A}_n)$, but $\mathrm{Anc}_{\mathcal{A}_n}(b) \cap \mathrm{Anc}_{\mathcal{A}_n}(\bar{b}) = 0$, and the desired assertion follows.
- Now let $\mathcal{A}_n^- = 0$ and $\mathcal{A}_n^0 = 0$. Since $AC^*(\mathcal{A}_n) \neq \varnothing$, we have $n \geq 2$. Then $AC^*(\mathcal{A}_n)$ consists of nonzero elements of $\mathcal{A}'_n = \mathcal{A}_n^+$. For an arbitrary element $a \in AC^*(\mathcal{A}_n)$ we have $\mathrm{Anc}_{\mathcal{A}_n}(a) \subsetneq \mathcal{A}'_n$. Hence there exists $b \in AC^*(\mathcal{A}_n) \setminus \mathrm{Anc}_{\mathcal{A}_n}(a)$.

(2) If $\mathcal{A}_n^- = 0$ and $\mathcal{A}_n^0 \neq 0$, then $AC^*(\mathcal{A}_n) = \mathcal{A}'_n \setminus \{0\}$. Now let $a \in AC^*(\mathcal{A}_n)$. If $a \in \mathcal{A}_n^0$, then $\mathrm{Anc}_{\mathcal{A}_n}(a) = \mathcal{A}'_n$, and, consequently, a anticommutes with all elements of $AC^*(\mathcal{A}_n)$. If $a \notin \mathcal{A}_n^0$, then $\mathrm{Anc}_{\mathcal{A}_n}(a) \subsetneq \mathcal{A}'_n$, whence a cannot satisfy the desired condition. □

Lemma 5.17. *Let $a, b \in \mathcal{A}'_n$.*

(1) *Assume that $\gamma_0, \dots, \gamma_{n-1} \leq 0$. Then a and b anticommute if and only if $a_\pm$ and $b_\pm$ are orthogonal as elements of a $(2^{n-q} - 1)$-dimensional Euclidean space.*
(2) *Assume that there exists $i \in \{0, \dots, n-1\}$ such that $\gamma_i > 0$. Then a and b anticommute if and only if $a_\pm$ and $b_\pm$ are orthogonal as elements of a pseudo-Euclidean space with signature $(2^{n-q-1} - 1, 2^{n-q-1})$.*

Proof. Both assertions follow from Lemma 5.8. □

Remark 5.18. The problem of finding cliques consisting of pure elements in $\Gamma_{AC}^E(\mathcal{A}_n)$ has been reduced to the problem of finding orthogonal systems either in a $(2^{n-q} - 1)$-dimensional Euclidean space or in a pseudo-Euclidean space with signature $(2^{n-q-1} - 1, 2^{n-q-1})$.

The following fact is well known.

Lemma 5.19 ([18, p. 282]). *Let E be a pseudo-Euclidean space and let U be a subspace of E. By $U^{\perp}$ denote the set of elements that are orthogonal to every element of U. Then $U^{\perp}$ also is a subspace of E, and the following equality holds:*

$$\dim(U) + \dim(U^{\perp}) = \dim(E).$$

Corollary 5.20. *Let $S \subset \mathcal{A}_n^{\pm}$. Then*

$$\mathrm{rk}(S) + \dim(\mathrm{Anc}_{\mathcal{A}_n}(S) \cap \mathcal{A}_n^{\pm}) = 2^{n-q} - 1,$$

where $\mathrm{rk}(S)$ is the cardinality of a maximal linearly independent subsystem of S.

Proof. The desired relation immediately follows from Lemmas 5.17 and 5.19. □

Lemma 5.21. *Let $a \in AC^*(\mathcal{A}_n)$.*

(1) *If $\mathfrak{Re}(a) = 0$, then $a \sim_{AC} b$ if and only if $b \neq 0$ and $b \in (\mathbb{R} \setminus \{0\})a + \mathcal{A}_n^0$, i.e., $b = \alpha a + x$ for some $0 \neq \alpha \in \mathbb{R}$ and $x \in \mathcal{A}_n^0$.*

(2) *If $\mathfrak{Re}(a) \neq 0$ and $n(a) = 0$, then $a \sim_{AC} b$ if and only if $b \in (\mathbb{R} \setminus \{0\})a$.*

Proof. Now we use Lemma 5.8.

(1) If $\mathfrak{Re}(a) = 0$, then $\mathrm{Anc}_{\mathcal{A}_n}(a) \subset \mathcal{A}_n'$, whence $a \sim_{AC} b$ implies $\mathfrak{Re}(b) = 0$. Thus, as it follows from Lemma 5.8, $a \sim_{AC} b$ if and only if the equations $\Lambda(a, d) = 0$ and $\Lambda(b, d) = 0$ in the variable $d = \sum\limits_{m=1}^{2^n-1} d_m e_m^{(n)}$ determine the same solution set. From the definition of the sets M_0 and $M_{\pm}$ it follows that $M_0 \cup M_{\pm} = \{1, \ldots, 2^n - 1\}$ and, in addition, $\delta_m^{(n)} = 0$ for $m \in M_0$, and $\delta_m^{(n)} \neq 0$ for $m \in M_{\pm}$. Thus,

$$\Lambda(a, d) = \sum_{m=1}^{2^n-1} \delta_m^{(n)} a_m d_m = \sum_{m \in M_{\pm}} \delta_m^{(n)} a_m d_m.$$

A similar formula holds for $\Lambda(b, d)$. Consequently, $a \sim_{AC} b$ if and only if $b \neq 0$, $\mathfrak{Re}(b) = 0$, and $b_{\pm} = \alpha a_{\pm}$ for a certain $0 \neq \alpha \in \mathbb{R}$. As is readily seen, this condition is fulfilled if and only if $b \neq 0$ and $b \in (\mathbb{R} \setminus \{0\})a + \mathcal{A}_n^0$.

(2) If $\mathfrak{Re}(a) \neq 0$ and $n(a) = 0$, then $\mathrm{Anc}_{\mathcal{A}_n}(a) = \mathbb{R}\bar{a}$ and $\mathrm{Anc}_{\mathcal{A}_n}(a) \cap \mathcal{A}_n' = 0$. Thus, $a \sim_{AC} b$ implies $\mathfrak{Re}(b) \neq 0$, $n(b) = 0$, $\mathrm{Anc}_{\mathcal{A}_n}(b) = \mathbb{R}\bar{b}$. Moreover, $\mathbb{R}\bar{a} = \mathbb{R}\bar{b}$ if and only if $b \in (\mathbb{R} \setminus \{0\})a$. □

Remark 5.22. From Lemma 5.21 it follows that for any $a \in \mathcal{A}_n' \setminus \mathcal{A}_n^0$ we have $a \sim_{AC} a_{\pm}$. However, if $a \in \mathcal{A}_n^0 \setminus \{0\}$, then $a_{\pm} = 0$, $\mathrm{Anc}_{\mathcal{A}_n}(a) = \mathcal{A}_n'$, and $\mathrm{Anc}_{\mathcal{A}_n}(a_{\pm}) = \mathcal{A}_n$.

Notation 5.23. Let $\mathcal{A}_n^0 \neq 0$. Then $0'$ denotes a chosen element of $\mathcal{A}_n^0 \setminus \{0\}$. Also we set $\overline{\mathcal{A}_n^{\pm}} = (\mathcal{A}_n^{\pm} \setminus \{0\}) \cup \{0'\}$.

Proposition 5.24. *For any $a \in \mathcal{A}_n^0 \setminus \{0\}$ we have $a \sim_{AC} 0'$.*

Proof. Since $a, 0' \in \mathcal{A}_n^0 \setminus \{0\}$, we have $\mathrm{Anc}_{\mathcal{A}_n}(a) = \mathrm{Anc}_{\mathcal{A}_n}(0') = \mathcal{A}_n'$, whence $a \sim_{AC} 0'$. □

Corollary 5.25. (1) *Let $\mathcal{A}_n^0 = 0$. Then for any $a \in \mathcal{A}_n' \setminus \{0\}$ we can choose a representative of the equivalence class $[a]_{AC}$ that lies in $\mathcal{A}_n^{\pm} \setminus \{0\}$.*

(2) *Let $\mathcal{A}_n^0 \neq 0$. Then for any $a \in \mathcal{A}_n' \setminus \{0\}$ we can choose a representative of the equivalence class $[a]_{AC}$ that lies in $\overline{\mathcal{A}_n^{\pm}}$.*

6. Anticommutativity graphs of real Cayley–Dickson algebras

Notation 6.1. In order to describe the connected components of the graph $\Gamma_{AC}^E(\mathcal{A}_n)$, we will need the following types of induced subgraphs of $\Gamma_{AC}^E(\mathcal{A}_n)$ on the vertex sets listed below:

(C_1) :: $\{[a], [\bar{a}]\}$, where $\mathfrak{Re}(a) \neq 0$ and $n(a) = 0$;
(C_2) :: $\{[a] \mid a \in \mathcal{A}_n^{\pm} \setminus \{0\}\}$;
(C_3) :: $\{[a] \mid a \in \overline{\mathcal{A}_n^{\pm}}\}$.

Notation 6.2. In order to describe the cliques of the graph $\Gamma_{AC}^{E}(\mathcal{A}_n)$, we will need the following types of subsets of the vertex set of $\Gamma_{AC}^{E}(\mathcal{A}_n)$:

(Q_1) :: $\{[a], [\bar{a}]\}$, where $\mathfrak{Re}(a) \neq 0$ and $n(a) = 0$;

(Q_2^k) :: $\{[r_1 a_1 + \cdots + r_k a_k] \mid (r_1, \ldots, r_k) \in \mathbb{R}^k \setminus \{0\}\} \cup \{[b_1], \ldots, [b_{2^n - 2k - 1}]\}$, where
- $0 \leq k \leq 2^{n-1} - 1$ (if $k = 0$, then this set takes the form $\{[b_1], \ldots, [b_{2^n - 1}]\}$);
- $a_1, \ldots, a_k, b_1, \ldots, b_{2^n - 2k - 1} \in \mathcal{A}_n^{\pm}$ anticommute pairwise and form a linearly independent system;
- $n(a_j) = 0$ for all $j = 1, \ldots, k$;
- $n(b_j) \neq 0$ for all $j = 1, \ldots, 2^n - 2k - 1$;

(Q_3^k) :: $\{[0']\} \cup \{[r_1 a_1 + \cdots + r_k a_k] \mid (r_1, \ldots, r_k) \in \mathbb{R}^k \setminus \{0\}\} \cup \{[b_1], \ldots, [b_{2^{n-q} - 2k - 1}]\}$, where
- $0 \leq k \leq 2^{n-q-1} - 1$ (if $k = 0$ this set takes the form $\{[0'], [b_1], \ldots, [b_{2^{n-q} - 1}]\}$),
- $a_1, \ldots, a_k, b_1, \ldots, b_{2^{n-q} - 2k - 1} \in \mathcal{A}_n^{\pm}$ anticommute pairwise and form a linearly independent system;
- $n(a_j) = 0$ for all $j = 1, \ldots, k$;
- $n(b_j) \neq 0$ for all $j = 1, \ldots, 2^{n-q} - 2k - 1$.

We now proceed to the classification of anticommutativity graphs of the algebras $\mathcal{A}_n$.

Theorem 6.3. *Let $\gamma_0, \ldots, \gamma_{n-1} \in \{-1, 0, 1\}$, $\mathcal{A}_n = \mathcal{A}_n\{\gamma_0, \ldots, \gamma_{n-1}\}$.*

(A) *Let either $n = 0$ or $n = 1$ and $\gamma_0 < 0$. Then the vertex set of $\Gamma_{AC}^{E}(\mathcal{A}_n)$ is the empty set.*

(B) *If $n = 1$ and $\gamma_0 > 0$, then $\Gamma_{AC}^{E}(\mathcal{A}_n)$ is a complete graph on two vertices.*

(C) *If $n \geq 1$ and $q = n$, then the vertex set of $\Gamma_{AC}^{E}(\mathcal{A}_n)$ is a singleton.*

(D) *If $n \geq 2$, $q = 0$, and $\mathcal{A}_n^{-} \neq 0$, then the vertex set of $\Gamma_{AC}^{E}(\mathcal{A}_n)$ is the set of equivalence classes of the elements of $\{a \in \mathcal{A}_n \mid \mathfrak{Re}(a) \neq 0, n(a) = 0\}$ and $\mathcal{A}_n^{\pm} \setminus \{0\}$.*

The connected components of this graph are of the form C_1 and C_2. The diameter of every connected component of the form C_1 equals 1, whereas the diameter of every connected component of the form C_2 equals 2.

The maximal cliques in $\Gamma_{AC}^{E}(\mathcal{A}_n)$ are of the form Q_1 and Q_2^k, $0 \leq k \leq 2^{n-1} - 1$.

(E) *If $n \geq 2$, $q = 0$, and $\mathcal{A}_n^{-} = 0$, then the vertex set of $\Gamma_{AC}^{E}(\mathcal{A}_n)$ is the set of equivalence classes of the elements of $\mathcal{A}_n^{\pm} \setminus \{0\}$.*

This graph is connected, and its diameter is equal to 2 (that is, $\Gamma_{AC}^{E}(\mathcal{A}_n)$ consists of the only connected component, namely, C_2).

The maximal cliques in $\Gamma_{AC}^{E}(\mathcal{A}_n)$ are of the form Q_2^0.

(F) *If $n \geq 2$, $0 < q < n$, and $\mathcal{A}_n^{-} \neq 0$, then the vertex set of $\Gamma_{AC}^{E}(\mathcal{A}_n)$ is the set of equivalence classes of the elements of $\{a \in \mathcal{A}_n \mid \mathfrak{Re}(a) \neq 0, n(a) = 0\}$ and of $\overline{\mathcal{A}_n^{\pm}}$.*

The connected components of this graph are of the form C_1 and C_3. The diameter of every connected component of the form C_1 equals 1, and the diameter of every connected component of the form C_3 equals 1 if $q = n - 1$, and 2 if $q < n - 1$.

The maximal cliques in $\Gamma_{AC}^{E}(\mathcal{A}_n)$ are of the form Q_1 and Q_3^k, $0 \leq k \leq 2^{n-q-1} - 1$.

(G) *If $n \geq 2$, $0 < q < n$, and $\mathcal{A}_n^{-} = 0$, then the vertex set of $\Gamma_{AC}^{E}(\mathcal{A}_n)$ is the set of equivalence classes of the elements of $\overline{\mathcal{A}_n^{\pm}}$.*

This graph is connected; its diameter is equal to 1 if $q = n - 1$ and to 2 if $q < n - 1$ (that is, $\Gamma_{AC}^{E}(\mathcal{A}_n)$ consists of the only connected component C_3).

The maximal cliques in $\Gamma_{AC}^{E}(\mathcal{A}_n)$ are of the form Q_3^0.

Proof. (1) Case (A) immediately follows from Lemma 5.15.

(2) In Case (B), we have $AC^*(\mathcal{A}_n) \cap \mathcal{A}'_n = \varnothing$ by Lemma 5.15. Then, as it follows from Lemma 5.8, we have $AC^*(\mathcal{A}_n) = \{a \in \mathcal{A}_n \,|\, \mathfrak{Re}(a) \neq 0, n(a) = 0\} = (\mathbb{R} \setminus \{0\})(\sqrt{\gamma_0} + e_1^{(1)}) \cup (\mathbb{R} \setminus \{0\})(\sqrt{\gamma_0} - e_1^{(1)})$. Thus, the vertex set of $\Gamma^E_{AC}(\mathcal{A}_n)$ consists of $\big[\sqrt{\gamma_0} + e_1^{(1)}\big]_{AC}$ and $\big[\sqrt{\gamma_0} - e_1^{(1)}\big]_{AC}$, and these two vertices are adjacent.

(3) In cases (C), (E), and (G), from Lemma 5.15 it follows that $AC^*(\mathcal{A}_n) \subset \mathcal{A}'_n$. Therefore, by Lemma 5.8, $AC^*(\mathcal{A}_n) = \mathcal{A}'_n \setminus \{0\}$.

(4) In case (C), we have $\mathcal{A}_n^{\pm} = 0$, whence $AC^*(\mathcal{A}_n) = \mathcal{A}_n^0 \setminus \{0\}$. Thus, the vertex set of $\Gamma^E_{AC}(\mathcal{A}_n)$ contains the only one element $[0']_{AC}$.

(5) Cases (F) and (G) can be inferred from cases (D) and (E), respectively, by adding the element $[0']_{AC}$ and changing $\dim(\mathcal{A}_n^{\pm})$ from $2^n - 1$ to $2^{n-q} - 1$. It should also be noted that if $n - q = 1$ in cases (F) and (G), then $\dim(\mathcal{A}_n^{\pm}) = 1$, whence the connected component C_3 consists of two elements, and its diameter is equal to 1. If $n - q \geq 2$, then $\dim(\mathcal{A}_n^{\pm}) \geq 3$; therefore, in C_3 there are at least two nonadjacent elements.

(6) Thus, it only remains to consider cases (D) and (E), where $n \geq 2$ and $q = 0$. The distinction between them is discussed in Lemma 5.17.

(7) First consider case (D). From Lemma 5.8 it is clear that the equivalence classes of the elements $\{a \in \mathcal{A}_n \mid \mathfrak{Re}(a) \neq 0, n(a) = 0\}$ and those of the elements in $\mathcal{A}_n^{\pm} \setminus \{0\}$ lie in different connected components.

(a) From Lemma 5.8 it follows that the equivalence classes of the elements of

$$\{a \in \mathcal{A}_n \,|\, \mathfrak{Re}(a) \neq 0, n(a) = 0\}$$

are contained in connected components of the form C_1, and their maximal cliques coincide with the connected components themselves and are of the form Q_1.

(b) Now we show that the subgraph of $\Gamma^E_{AC}(\mathcal{A}_n)$ on the vertex set $\big\{[a]_{AC} \big| a \in \mathcal{A}_n^{\pm} \setminus \{0\}\big\}$ is connected and its diameter equals 2. We use Lemma 5.17. The subspace of the elements of $\mathcal{A}_n^{\pm}$ that are orthogonal (in the sense of a pseudo-Euclidean space) to a given $a \in \mathcal{A}_n^{\pm} \setminus \{0\}$ is of dimension $2^n - 2$. Since $\dim(\mathcal{A}_n^{\pm}) = 2^n - 1$, the intersection of the orthogonal subspaces of any two elements from $\mathcal{A}_n^{\pm}$ is of dimension at least $2^n - 3$ and hence nonzero. It follows that C_2 is connected and $d(C_2) \leq 2$. Moreover, since $\dim(\mathcal{A}_n^{\pm}) \geq 3$, there exist two vertices that are not adjacent, and we also have $d(C_2) \geq 2$.

(c) Now consider the maximal cliques in the connected component C_2.

Note that if $r_1, \ldots, r_k \in \mathbb{R}$ and distinct vertices $[a_1], \ldots, [a_k], [r_1 a_1 + \cdots + r_k a_k]$ form a clique, then from $r_j \neq 0$ it follows that a_j anticommutes with itself. Then, by Lemma 5.1, $n(a_j) = 0$ and $\mathfrak{Re}(a_j) = 0$. Hence $n(r_1 a_1 + \cdots + r_k a_k) = 0$.

Let Q be a maximal clique in C_2. For every equivalence class in Q we choose its representative from $\mathcal{A}_n^{\pm}$. Let S denote the set of these representatives, whereas S_0 is the subset of S consisting of elements with zero norm. $\mathrm{Lin}(S)$ and $\mathrm{Lin}(S_0)$ are linear spaces generated by S and S_0, respectively; $\mathrm{Lin}(S_0) \subset \mathrm{Lin}(S) \subset \mathcal{A}_n^{\pm}$. Let

$$m = \mathrm{rk}(S) = \dim(\mathrm{Lin}(S)), \quad k = \mathrm{rk}(S_0) = \dim(\mathrm{Lin}(S_0)), \quad m \geq k.$$

Obviously, $S_0 \subset \mathrm{Anc}_{\mathcal{A}_n}(S) \cap \mathcal{A}_n^{\pm}$, whence $\dim(\mathrm{Anc}_{\mathcal{A}_n}(S) \cap \mathcal{A}_n^{\pm}) \geq k$. Moreover, from the above remark it follows that

$$\mathrm{Lin}(S) \cap (\mathrm{Anc}_{\mathcal{A}_n}(S) \cap \mathcal{A}_n^{\pm}) = \mathrm{Lin}(S_0).$$

By using Corollary 5.20, we obtain that $m + \dim(\mathrm{Anc}_{\mathcal{A}_n}(S) \cap \mathcal{A}_n^{\pm}) = 2^n - 1$, whence $2k \leq k + m \leq 2^n - 1$. Consequently, $k \leq 2^{n-1} - 1$. Furthermore, if $k + m < 2^n - 1$, then there is an element $b \in (\mathrm{Anc}_{\mathcal{A}_n}(S) \cap \mathcal{A}_n^{\pm}) \setminus \mathrm{Lin}(S)$, and $Q \cup \big\{[b]_{AC}\big\}$ also is a clique. Since the clique Q is maximal by assumption, we have $k+m=2^n-1$, and thus Q is of the form Q_2^k.

Now we present examples of cliques of the form Q_2^k for all possible values of k, $0 \le k \le 2^{n-1}-1$:

$$\begin{aligned} a_j &= e_{2j-1}^{(n)} + e_{2j}^{(n)}, \quad & j &= 1, \dots, k, \\ b_j &= e_{2k+j}^{(n)}, & j &= 1, \dots, 2^n - 2k - 1. \end{aligned}$$

(8) In case (E), all elements of $AC^*(\mathcal{A}_n)$ are pure and have nonzero norms. The anticommutativity relation in this case can be expressed in terms of orthogonality in a (2^n-1)-dimensional Euclidean space. In a similar way, we prove that the subgraph of $\Gamma_{AC}^E(\mathcal{A}_n)$ of the form C_2 (which is, in fact, $\Gamma_{AC}^E(\mathcal{A}_n)$ itself) is connected, and its diameter equals 2. The explicit form of the cliques (Q_2^0) follows from the fact that any orthogonal system in a Euclidean space can be extended to an orthogonal basis. □

7. Some examples of anticommutativity graphs

In the cases of quaternions, split-complex numbers, and split-quaternions, Lemma 5.8 and Theorem 6.3 take the following forms.

Lemma 7.1. *Let $a \in \mathbb{H}$, $a \neq 0$. Then $\mathrm{Anc}_{\mathbb{H}}(a) \neq 0$ if and only if $\mathfrak{Re}(a) = 0$.*

If $\mathfrak{Re}(a) = 0$, then $\dim(\mathrm{Anc}_{\mathbb{H}}(a)) = 2$, and $b \in \mathrm{Anc}_{\mathbb{H}}(a)$ if and only if $\mathfrak{Re}(b) = 0$ and $a_1b_1 + a_2b_2 + a_3b_3 = 0$, where $a = a_1i + a_2j + a_3k$, $b = b_1i + b_2j + b_3k$.

Proof. This is a special case of Lemma 5.8, corresponding to $n=2$, $\gamma_0=-1$, and $\gamma_1=-1$. □

Theorem 7.2. *The vertex set of $\Gamma_{AC}^E(\mathbb{H})$ is the set of equivalence classes of the nonzero elements of $\mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$. The graph $\Gamma_{AC}^E(\mathbb{H})$ is connected, and its diameter equals 2.*

The cliques correspond to the orthogonal systems in $\mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ free of zero elements (with Euclidean inner product in $\mathbb{R}^3$). Every maximal clique has three vertices.

Proof. This is a special case of Theorem 6.3, corresponding to $n=2$, $\gamma_0=-1$, $\gamma_1=-1$. □

Lemma 7.3. *Every nontrivial pair of anticommuting elements in $\hat{\mathbb{C}}$ consists of $a + a\ell$ and $b - b\ell$, where $a, b \in \mathbb{R} \setminus \{0\}$.*

Proof. This is a special case of Lemma 5.8, corresponding to $n=1$, $\gamma_0=1$. □

Theorem 7.4. *$\Gamma_{AC}^E(\hat{\mathbb{C}})$ is a complete graph on the vertex set $\{[1+\ell]_{AC}, [1-\ell]_{AC}\}$.*

Proof. This is a special case of Theorem 6.3, corresponding to $n=1$, $\gamma_0=1$. □

Lemma 7.5. *Let $a \in \hat{\mathbb{H}}$, $a \neq 0$.*

1. *If $\mathfrak{Re}(a) = 0$, then $\dim(\mathrm{Anc}_{\hat{\mathbb{H}}}(a)) = 2$, and $b \in \mathrm{Anc}_{\hat{\mathbb{H}}}(a)$ if and only if $\mathfrak{Re}(b) = 0$ and $a_1b_1 - a_2b_2 - a_3b_3 = 0$, where $a = a_1i + a_2\ell + a_3\ell i$, $b = b_1i + b_2\ell + b_3\ell i$.*
2. *If $\mathfrak{Re}(a) \neq 0$ and $n(a) = 0$, then $\dim(\mathrm{Anc}_{\hat{\mathbb{H}}}(a)) = 1$, $\mathrm{Anc}_{\hat{\mathbb{H}}}(a) = \mathbb{R}\bar{a}$. Any nonzero $b \in \mathrm{Anc}_{\hat{\mathbb{H}}}(a)$ satisfies the condition $\mathfrak{Re}(b) \neq 0$.*
3. *If $\mathfrak{Re}(a) \neq 0$ and $n(a) \neq 0$, then $\mathrm{Anc}_{\hat{\mathbb{H}}}(a) = 0$.*

Proof. This is a special case of Lemma 5.8, corresponding to $n=2$, $\gamma_0=-1$, $\gamma_1=1$. □

Theorem 7.6. *The vertex set of $\Gamma_{AC}^E(M_2(\mathbb{R}))$ is the set of equivalence classes of the nonzero elements of $M_2(\mathbb{R})$ that have either zero trace or zero norm. The vertex sets of the connected components of this graph have one of the following two forms:*

1. *the equivalence classes of all nonzero elements with zero trace; the diameter of such a connected component is equal to 2;*

(2) $\{[A]_{AC}, [\bar{A}]_{AC}\}$, *where* $\operatorname{tr}(A) \neq 0$ *and* $\det(A) = 0$.

The maximal cliques of $\Gamma^E_{AC}(M_2(\mathbb{R}))$ *are as follows:*

(i) $\{[A]_{AC}, [\bar{A}]_{AC}\}$, *where* $\operatorname{tr}(A) \neq 0$, $\det(A) = 0$;

(ii) $\{[A]_{AC}, [B]_{AC}, [C]_{AC}\}$, *where*
- A, B, C *are linearly independent and anticommute pairwise,*
- $\operatorname{tr}(A) = \operatorname{tr}(B) = \operatorname{tr}(C) = 0$,
- $\det(A), \det(B), \det(C) \neq 0$;

(iii) $\{[A]_{AC}, [B]_{AC}\}$, *where*
- A *and* B *anticommute,*
- $\operatorname{tr}(A) = \operatorname{tr}(B) = 0$,
- $\det(A) = 0$, $\det(B) \neq 0$.

Proof. This is a special case of Theorem 6.3, corresponding to $n = 2$, $\gamma_0 = -1$, and $\gamma_1 = 1$. □

8. Relationship between the centralizer and orthogonalizer

Definition 8.1. The associator of three elements $a, b, c \in \mathcal{A}$ is the element $[a, b, c] = (ab)c - a(bc)$.

Proposition 8.2. *The associator is a trilinear function of its arguments.*

Proof. This assertion follows directly from the definition of an algebra over a field. □

Definition 8.3. An algebra $\mathcal{A}$ is said to be *flexible* if $(ab)a = a(ba)$ for all $a, b \in \mathcal{A}$.

Lemma 8.4 ([28, Theorem 1])**.** *For all* $n \in \mathbb{N} \cup \{0\}$ *and all* $\gamma_0, \dots, \gamma_{n-1} \in \mathbb{R}$, *the algebra* $\mathcal{A}_n$ *is flexible.*

Corollary 8.5. *For all* $a, b, c \in \mathcal{A}_n$ *we have* $[a, b, c] = -[c, b, a]$.

Proof.

$$\begin{aligned} 0 &= [a + c, b, a + c] = [a, b, a] + [a, b, c] + [c, b, a] + [c, b, c] \\ &= 0 + [a, b, c] + [c, b, a] + 0 = [a, b, c] + [c, b, a]. \end{aligned}$$ □

Definition 8.6. An algebra $\mathcal{A}$ is said to be *alternative* if the relations $a^2b = a(ab)$ and $ba^2 = (ba)a$ hold for all $a, b \in \mathcal{A}$.

Lemma 8.7 ([5, p. 172])**.** *The algebra* $\mathcal{A}\{\gamma\}$ *is alternative if and only if* $\mathcal{A}$ *is associative.*

Corollary 8.8 ([28, p. 436])**.** *The algebra* $\mathcal{A}_n$ *is alternative if and only if* $n \leq 3$.

Corollary 8.9. *For* $n \leq 3$ *the associator in* $\mathcal{A}_n$ *is skew-symmetric, that is, it changes its sign as its arguments are transposed.*

Proof. The proof is similar to that of Corollary 8.5. □

Lemma 8.10. *Let* $a \in \mathcal{A}_n$, $\mathfrak{Im}(a) \neq 0$. *Then*

$$C_{\mathcal{A}_n}(a) = \mathbb{R} \oplus O_{\mathcal{A}_n}(\mathfrak{Im}(a)) \oplus V,$$

where $\dim(V) \leq 1$.

Proof. Obviously, $C_{\mathcal{A}_n}(a) = C_{\mathcal{A}_n}(\mathfrak{Im}(a))$, and thus we need to show that

$$\mathfrak{Im}(C_{\mathcal{A}_n}(\mathfrak{Im}(a))) = O_{\mathcal{A}_n}(\mathfrak{Im}(a)) \oplus V,$$

where $\dim(V) \leq 1$. From Lemma 5.8 it follows that $\operatorname{Anc}_{\mathcal{A}_n}(\mathfrak{Im}(a)) \subset \mathcal{A}'_n$. We have

$$O_{\mathcal{A}_n}(\mathfrak{Im}(a)) = C_{\mathcal{A}_n}(\mathfrak{Im}(a)) \cap \operatorname{Anc}_{\mathcal{A}_n}(\mathfrak{Im}(a)) = \mathfrak{Im}(C_{\mathcal{A}_n}(\mathfrak{Im}(a))) \cap \operatorname{Anc}_{\mathcal{A}_n}(\mathfrak{Im}(a)).$$

Moreover, if $b \in \mathfrak{Im}(C_{\mathcal{A}_n}(\mathfrak{Im}(a)))$ (and thus $\mathfrak{Re}(b) = 0$), then the condition $b \in \mathrm{Anc}_{\mathcal{A}_n}(\mathfrak{Im}(a))$ is given by a linear equation (maybe trivial). Therefore,

$$\dim(\mathfrak{Im}(C_{\mathcal{A}_n}(\mathfrak{Im}(a)))) - \dim(O_{\mathcal{A}_n}(\mathfrak{Im}(a))) \leq 1. \qquad \square$$

Lemma 8.11. *Let $a \in \mathcal{A}_n$, $\mathfrak{Im}(a) \neq 0$.*

(1) *If $n(\mathfrak{Im}(a)) = 0$ and, in addition, either $\mathfrak{Im}(a) \in \mathcal{A}_n^0$ or $n \leq 3$, then $C_{\mathcal{A}_n}(a) = \mathbb{R} \oplus O_{\mathcal{A}_n}(\mathfrak{Im}(a))$;*
(2) *if $n(\mathfrak{Im}(a)) \neq 0$, then $C_{\mathcal{A}_n}(a) = \mathbb{R} \oplus \mathbb{R}a \oplus O_{\mathcal{A}_n}(\mathfrak{Im}(a))$.*

Proof. Obviously, for any $a \in \mathcal{A}_n$ we have $C_{\mathcal{A}_n}(a) \supseteq \mathbb{R} + \mathbb{R}a + O_{\mathcal{A}_n}(\mathfrak{Im}(a))$. As it follows from Lemma 5.1, the conditions $n(\mathfrak{Im}(a)) = 0$ and $\mathfrak{Im}(a) \in O_{\mathcal{A}_n}(\mathfrak{Im}(a))$ are equivalent. By the lemma assumption, $\mathfrak{Im}(a) \neq 0$. Therefore,

(1) if $n(\mathfrak{Im}(a)) = 0$, this relation takes the form $C_{\mathcal{A}_n}(a) \supseteq \mathbb{R} \oplus O_{\mathcal{A}_n}(\mathfrak{Im}(a))$;
(2) if $n(\mathfrak{Im}(a)) \neq 0$, it is of the form $C_{\mathcal{A}_n}(a) \supseteq \mathbb{R} \oplus \mathbb{R}a \oplus O_{\mathcal{A}_n}(\mathfrak{Im}(a))$.

Now we show that in the above-mentioned cases, the reverse inclusion also holds. Let $b \in C_{\mathcal{A}_n}(a)$, then $\mathfrak{Im}(b) \in C_{\mathcal{A}_n}(\mathfrak{Im}(a))$.

(1) First assume that $n(\mathfrak{Im}(a)) = 0$, i.e., $(\mathfrak{Im}(a))^2 = 0$.
  - If $\mathfrak{Im}(a) \in \mathcal{A}_n^0$, then from Lemma 5.8 it follows that $\mathfrak{Im}(b) \in \mathrm{Anc}_{\mathcal{A}_n}(\mathfrak{Im}(a))$, whence
  $$\mathfrak{Im}(b) \in \mathrm{Anc}_{\mathcal{A}_n}(\mathfrak{Im}(a)) \cap C_{\mathcal{A}_n}(\mathfrak{Im}(a)) = O_{\mathcal{A}_n}(\mathfrak{Im}(a)).$$
  - If $n \leq 3$, then the algebra $\mathcal{A}_n$ is alternative by Corollary 8.8. Note that
  $$\overline{\mathfrak{Im}(a)\mathfrak{Im}(b)} = \overline{\mathfrak{Im}(b)} \cdot \overline{\mathfrak{Im}(a)} = \mathfrak{Im}(b)\mathfrak{Im}(a) = \mathfrak{Im}(a)\mathfrak{Im}(b),$$
  that is, $\mathfrak{Im}(a)\mathfrak{Im}(b) = r \in \mathbb{R}$. Therefore,
  $$0 = (\mathfrak{Im}(a))^2\mathfrak{Im}(b) = \mathfrak{Im}(a)(\mathfrak{Im}(a)\mathfrak{Im}(b)) = r\mathfrak{Im}(a),$$
  and we have $r = 0$, i.e., $\mathfrak{Im}(b) \in O_{\mathcal{A}_n}(\mathfrak{Im}(a))$.

(2) Consider the case where $n(\mathfrak{Im}(a)) \neq 0$, i.e., $(\mathfrak{Im}(a))^2 \neq 0$. By Corollary 5.12, there exists a unique decomposition
$$\mathfrak{Im}(b) = k\mathfrak{Im}(a) + d,$$
where $d \in \mathrm{Anc}_{\mathcal{A}_n}(\mathfrak{Im}(a))$. Note that
$$d = \mathfrak{Im}(b) - k\mathfrak{Im}(a) \in C_{\mathcal{A}_n}(\mathfrak{Im}(a)),$$
$$d \in \mathrm{Anc}_{\mathcal{A}_n}(\mathfrak{Im}(a)) \cap C_{\mathcal{A}_n}(\mathfrak{Im}(a)) = O_{\mathcal{A}_n}(\mathfrak{Im}(a))$$
$$\mathfrak{Im}(b) \in \mathbb{R}\mathfrak{Im}(a) \oplus O_{\mathcal{A}_n}(\mathfrak{Im}(a)). \qquad \square$$

Example 8.12. If $\mathcal{A}_n$ belongs to the main sequence, then any $a \in \mathcal{A}_n$, $\mathfrak{Im}(a) \neq 0$, satisfies the assumption of Lemma 8.11.

Lemma 8.13 ([1], [23, Lemma 1.2])**.** *Let $\mathcal{A}$ be an arbitrary algebra. Then for arbitrary $x, y, z, w \in \mathcal{A}$ the following relation holds*:
$$x[y, z, w] + [x, y, z]w = [xy, z, w] - [x, yz, w] + [x, y, zw].$$

Lemma 8.14 ([28, Lemma 2])**.** *For any $x, y, z \in \mathcal{A}_n$ we have $\mathfrak{Re}([x, y, z]) = 0$.*

Proposition 8.15. *Let $a \in \mathcal{A}_4\{-1, -1, -1, 1\} = \mathbb{O}\{1\} = \hat{\mathbb{S}}$, $n(\mathfrak{Im}(a)) = 0$. Then $C_{\hat{\mathbb{S}}}(a) = \mathbb{R} \oplus O_{\hat{\mathbb{S}}}(\mathfrak{Im}(a))$.*

Proof. Let $b \in C_{\hat{\mathbb{S}}}(a) \setminus (\mathbb{R} \oplus O_{\hat{\mathbb{S}}}(\mathfrak{Im}(a)))$. Then $\mathfrak{Im}(b) \in C_{\hat{\mathbb{S}}}(a)$, and therefore
$$\overline{\mathfrak{Im}(a)\mathfrak{Im}(b)} = \overline{\mathfrak{Im}(b)} \cdot \overline{\mathfrak{Im}(a)} = \mathfrak{Im}(b)\mathfrak{Im}(a) = \mathfrak{Im}(a)\mathfrak{Im}(b),$$

that is, $\mathfrak{Im}(a)\mathfrak{Im}(b) = r \in \mathbb{R}$. Since $\mathfrak{Im}(b) \notin O_{\hat{\mathbb{S}}}(\mathfrak{Im}(a))$, we have $r \neq 0$. Assume, without loss of generality, that $r = 1$.

Denote $\mathfrak{Im}(a) = (a_1, a_2)$, $\mathfrak{Im}(b) = (b_1, b_2)$, where $a_1, a_2, b_1, b_2 \in \mathbb{O}$, $\mathfrak{Re}(a_1) = \mathfrak{Re}(b_1) = 0$. Then

$$0 = (a_1, a_2)^2 = (a_1^2 + \bar{a_2}a_2, a_2a_1 + a_2\bar{a_1}) = (a_1^2 + n(a_2), 0),$$
$$(1, 0) = (a_1, a_2)(b_1, b_2) = (a_1b_1 + \bar{b_2}a_2, b_2a_1 + a_2\bar{b_1}) = (a_1b_1 + \bar{b_2}a_2, b_2a_1 - a_2b_1).$$

By using the equalities $b_2a_1 = a_2b_1$ and $n(a_2) = -a_1^2 \in \mathbb{R}$ and also the alternativity of $\mathbb{O}$, we obtain

$$n(a_2)b_2 = -b_2a_1^2 = -(b_2a_1)a_1 = -(a_2b_1)a_1,$$
$$n(a_2)\bar{b_2} = \overline{n(a_2)b_2} = \overline{-(a_2b_1)a_1} = -\bar{a_1}(\bar{b_1}\bar{a_2}) = -a_1(b_1\bar{a_2}).$$

Now we multiply the equality $1 = a_1b_1 + \bar{b_2}a_2$ by $\bar{a_2}$ on the right and substitute the expression for $n(a_2)\bar{b_2}$. In this way, we obtain

$$\bar{a_2} = (a_1b_1)\bar{a_2} + (\bar{b_2}a_2)\bar{a_2} = (a_1b_1)\bar{a_2} + \bar{b_2}(a_2\bar{a_2}) = (a_1b_1)\bar{a_2} + n(a_2)\bar{b_2}$$
$$= (a_1b_1)\bar{a_2} - a_1(b_1\bar{a_2}) = [a_1, b_1, \bar{a_2}].$$

From Lemma 8.14 it follows that $\mathfrak{Re}(\bar{a_2}) = 0$, whence $\bar{a_2} = -a_2$ and $a_2 = [a_1, b_1, a_2]$. Recall that the associator in $\mathbb{O}$ is skew-symmetric. Then, by applying Lemma 8.14 to $x = w = a_2, y = a_1, z = b_1$, we obtain

$$-2n(a_2) = 2a_2^2 = a_2[a_1, b_1, a_2] + [a_2, a_1, b_1]a_2 = [a_2a_1, b_1, a_2] - [a_2, a_1b_1, a_2] + [a_2, a_1, b_1a_2].$$

The real part of the right-hand side is zero; therefore, $n(a_2) = 0$, whence $n(a_1) = 0$, implying that $a_1 = a_2 = 0$ and $\mathfrak{Im}(a)\mathfrak{Im}(b) = 0$. Thus, we have a contradiction. ☐

Show that the additional conditions in item (1) of Lemma 8.11 are essential. We will need the following notation and some related assertions.

Notation 8.16. Consider $a, b \in \mathcal{A}_n$ such that $ba = 0$, $a \neq 0$, $b \neq 0$, $\mathfrak{Re}(a) = 0$, and $n(a) = \gamma_n n(b) \neq 0$.

(1) Set $c = (a, b), d = (0, b) \in \mathcal{A}_{n+1} = \mathcal{A}_n\{\gamma_n\}$.

(2) If we also have $\gamma_n > 0$ and $ab = 0$, then, for any $r \in \mathbb{R}$, $|r| < 1$, we denote $c(r) = \big(a, r(\sqrt{\gamma_n})^{-1}a + \sqrt{1-r^2}b\big) \in \mathcal{A}_{n+1}$.

Proposition 8.17. *Let $a, b \in \mathcal{A}_n$, $ba = 0$, $a \neq 0$, $\mathfrak{Re}(a) = 0$. Then the conditions $ab = 0$ and $\mathfrak{Re}(b) = 0$ are equivalent.*

Proof. Since $a \neq 0$, this assertion follows from the equality string

$$ab = \bar{\bar{b}\bar{a}} = \overline{-(2\mathfrak{Re}(b) - b)a} = \overline{-2\mathfrak{Re}(b)a + ba} = -2\mathfrak{Re}(b)\bar{a} = 2\mathfrak{Re}(b)a.$$ ☐

Proposition 8.18. *Let $a, b, c, c(r)$ be introduced in Notation 8.16. Then*

(1) $\mathfrak{Re}(c) = 0$, $n(c) = 0$;

(2) $\mathfrak{Re}(c(r)) = 0$, $n(c(r)) = 0$.

Proof. (1) From Lemma 3.20 it follows that

$$\mathfrak{Re}(c) = \mathfrak{Re}\big((a, b)\big) = \mathfrak{Re}(a) = 0,$$
$$n(c) = n\big((a, b)\big) = n(a) - \gamma_n n(b) = 0.$$

(2) Obviously, $\mathfrak{Re}\big(r(\sqrt{\gamma_n})^{-1}a+\sqrt{1-r^2}b\big)=0$, $\mathfrak{Re}(c(r))=0$. Now we verify that $n(c(r))=0$. Indeed,

$$
\begin{aligned}
n(c(r)) = n(a) - \gamma_n n\left(\frac{r}{\sqrt{\gamma_n}}a + \sqrt{1-r^2}b\right) &= \gamma_n\left(n(b) + \left(\frac{r}{\sqrt{\gamma_n}}a + \sqrt{1-r^2}b\right)^2\right)\\
&= \gamma_n\left(n(b) + \frac{r^2}{\gamma_n}a^2 + \frac{r\sqrt{1-r^2}}{\sqrt{\gamma_n}}(ab+ba) + (1-r^2)b^2\right)\\
&= \gamma_n\big(n(b) - r^2 n(b) - (1-r^2)n(b)\big) = 0.
\end{aligned}
$$

□

Proposition 8.19. *Let $a, b, c, c(r), d$ be introduced in Notation* 8.16. *Then*
(1) $C_{\mathcal{A}_{n+1}}(c) = \mathbb{R} \oplus \mathbb{R}d \oplus O_{\mathcal{A}_{n+1}}(c)$;
(2) $C_{\mathcal{A}_{n+1}}(c(r)) = \mathbb{R} \oplus \mathbb{R}d \oplus O_{\mathcal{A}_{n+1}}(c(r))$.

Proof. (1) It is readily seen that

$$cd = (a,b)(0,b) = (a0 + \gamma_n\bar{b}b, ba + b\bar{0}) = \gamma_n n(b) \in \mathbb{R}\setminus\{0\}.$$

Furthermore, $\mathfrak{Re}(c) = \mathfrak{Re}(d) = 0$, whence $dc = \bar{d}\bar{c} = \overline{cd} = cd$. Thus, $d \in C_{\mathcal{A}_{n+1}}(c)$. However, $d \notin \mathbb{R}\oplus O_{\mathcal{A}_{n+1}}(c)$ because $d \notin O_{\mathcal{A}_{n+1}}(c)$ and for any $r \in \mathbb{R}\setminus\{0\}$ we have $\mathfrak{Im}(c(d-r)) = -rc \neq 0$. From Lemma 8.10 we infer that $C_{\mathcal{A}_{n+1}}(c) = \mathbb{R} \oplus \mathbb{R}d \oplus O_{\mathcal{A}_{n+1}}(c)$.

(2) Show that $c(r)d \in \mathbb{R}\setminus\{0\}$. Indeed,

$$
\begin{aligned}
c(r)d &= \left(a, \frac{r}{\sqrt{\gamma_n}}a + \sqrt{1-r^2}b\right)(0,b)\\
&= \left(a0 + \gamma_n\bar{b}\left(\frac{r}{\sqrt{\gamma_n}}a + \sqrt{1-r^2}b\right), ba + \left(\frac{r}{\sqrt{\gamma_n}}a + \sqrt{1-r^2}b\right)\bar{0}\right)\\
&= \big(-r\sqrt{\gamma_n}ba + \gamma_n\sqrt{1-r^2}\bar{b}b, ba\big) = \gamma_n\sqrt{1-r^2}n(b) \in \mathbb{R}\setminus\{0\}.
\end{aligned}
$$

The fact that $C_{\mathcal{A}_{n+1}}(c(r)) = \mathbb{R} \oplus \mathbb{R}d \oplus O_{\mathcal{A}_{n+1}}(c(r))$ is proved similarly. □

Example 8.20. Both conditions of Notation 8.16 are satisfied, for instance, whenever

$$a = (e_1, e_4), \quad b = (e_2, e_7) \in \mathbb{S} = \mathcal{A}_4\{-1,-1,-1,-1\}, \quad \text{and} \quad \gamma_4 = 1.$$

9. Conclusion

The commutativity graphs of real Cayley–Dickson algebras are of special interest; however, a number of difficulties arises even for $n = 4$. It is conjectured that the commutativity graph of the algebra of sedenions possesses the following property:

Conjecture 9.1. The elements of $\mathbb{S}$ whose imaginary parts are zero divisors form a connected component in $\Gamma_C(\mathbb{S})$, and its diameter equals 3.

In the case of real Cayley–Dickson algebras of the main sequence, Lemma 8.11 completely describes the relationship between the centralizer and orthogonalizer of an arbitrary element. Moreover, it is obvious that $\Gamma_O(\mathcal{A})$ always is a subgraph of $\Gamma_C(\mathcal{A})$. In this connection, the following question arises:

Question 9.2. What is the relationship between $\Gamma_O(\mathcal{A}_n)$ and $\Gamma_C(\mathcal{A}_n)$ for a real Cayley–Dickson algebra $\mathcal{A}_n$ of the main sequence? For an arbitrary real Cayley–Dickson algebra?

Remark 9.3. The authors have recently discovered that the orthogonality graphs of formally real Jordan algebras (FRJAs for short) are studied in [17]. In particular, their clique numbers are computed, and it is proved that two FRJAs are isomorphic if and only if their orthogonality graphs are isomorphic.

Note that every real Cayley–Dickson algebra $(\mathcal{A}_n, +, \cdot)$ can be transformed into a Jordan algebra $(\mathcal{A}_n, +, \circ)$, where $a \circ b = \frac{1}{2}(ab + ba)$. Then the anticommutativity graph of $(\mathcal{A}_n, +, \cdot)$ is isomorphic to the orthogonality graph of $(\mathcal{A}_n, +, \circ)$. It can readily be seen that $(\mathcal{A}_n, +, \circ)$ is formally real if and only if $(\mathcal{A}_n, +, \cdot)$ either belongs to the main sequence or is isomorphic to the split-complex numbers. Thus, the classes of algebras discussed in [17] and in this paper do not coincide; however, they have a nontrivial intersection. It can be verified that the results obtained for the algebras from the above-mentioned intersection agree.

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