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AN ALGORITHM FOR DECOMPOSING REPRESENTATIONS OF FINITE GROUPS USING INVARIANT PROJECTIONS

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We describe an algorithm for decomposing permutation representations of finite groups over fields of characteristic zero into irreducible components. The algorithm is based on the fact that the components of the invariant inner product in invariant subspaces are operators of projecting to these subspaces. This allows us to reduce the problem to solving systems of quadratic equations. The current implementation of the suggested algorithm allows us to split representations with dimensions up to hundreds of thousands. Computational examples are given. Bibliography: 8 *titles.*

1. Introduction

The decomposition of linear representations of groups into irreducible representations is one of the main problems in the theory of groups and its applications in physics. In general, the problem of splitting a module over an associative algebra into irreducible components is very nontrivial. A quite complete overview of algorithms for solving this problem can be found in [1]. Currently, it is considered that the most efficient algorithm is a probabilistic algorithm of Las Vegas type called ***MeatAxe*** [2]. One of the main elements of the algorithm is the calculation of the characteristic polynomial of a randomly generated matrix of the module followed by the factorization of this polynomial. Processing the irreducible factors of the characteristic polynomial in the case of success allows one either to construct a decomposition of the module into submodules, or to prove that it is irreducible (a detailed description of the ***MeatAxe*** algorithm is given in Sec. 7.4 of the book [1]). The ***MeatAxe*** algorithm played an important role in solving the classification problem for finite simple groups, where it was applied to group representations in linear spaces over small finite fields (typically, over GF(2)). However, ***MeatAxe*** is not effective for representations over fields of zero characteristic, due to the rapid growth of numerical coefficients when calculating characteristic polynomials for large matrices, and because in the zero characteristic case a randomly generated matrix with high probability has an irreducible characteristic polynomial, which is useless for the work of ***MeatAxe***.

The quantum formalism is based on Hilbert spaces over fields of zero characteristic. Traditionally, one uses nonconstructive fields $\mathbb{C}$ or $\mathbb{R}$. Our goal was to develop an algorithm suitable for studying quantum mechanical models based on unitary representations of finite groups over constructive fields of zero characteristic [3, 4]. The computer implementation of our algorithm, let us call it ***IrreducibleProjectors***, splits representations of dimensions up to hundreds of thousands, which is as good as dimensions achievable for ***MeatAxe*** in the computationally easier context of finite fields. On the other hand, the ***IrreducibleProjectors*** algorithm, unlike ***MeatAxe***, is of little use in problems over finite fields, because it uses the notion of scalar product. In spaces over finite fields, introducing a scalar product, which would allow one to complete the computations, requires a number of arithmetic restrictions

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on the field characteristic. As a matter of fact, the ***MeatAxe*** and ***IrreducibleProjectors*** algorithms have different areas of application.

The ***IrreducibleProjectors*** algorithm requires the knowledge of the centralizer ring of the group representation in question. In the general case, the calculation of the centralizer ring reduces to a simple linear algebra problem, namely, solving a system of matrix equations of the form $AX = XA$. Here we consider only permutation representations, since (a) any linear representation of a finite group is a subrepresentation of a permutation representation, and (b) permutation representations underlie the aforementioned constructive quantum mechanical models. In the case of permutation representations, the calculation of the centralizer ring is especially simple: it reduces to constructing the orbits of the group on the Cartesian square of the set on which the group acts by permutations.

2. Basic notions and notation

Let G (or, in more detail, $\mathsf{G}(\Omega)$) be a permutation group acting *transitively* on the set $\Omega \cong \{1, \ldots, \mathsf{N}\}$. The action of an element $g \in \mathsf{G}$ on $i \in \Omega$ will be denoted by i^g. *A permutation representation* P is a representation of G by matrices of the form $\mathrm{P}(g)_{ij} = \delta_{i^g j}$. Since $\mathrm{P}(g)$ is a $(0,1)$-matrix, a permutation representation can be realized in a vector space over any field $\mathcal{F}$. We will assume that the representation space is the N-dimensional Hilbert space $\mathcal{H}_{\mathsf{N}}$. For $\mathcal{F}$ we can take a suitable subfield of the mth cyclotomic field, where m is the *exponent* of the group G. Such a field $\mathcal{F}$, being an Abelian extension of the field of rational numbers $\mathbb{Q}$, is a constructive dense subfield of $\mathbb{R}$ or $\mathbb{C}$. From the point of view of physics, $\mathcal{F}$ is indistinguishable from $\mathbb{R}$ or $\mathbb{C}$ and can be freely used in the formalism of quantum mechanics.

An orbit of G on the Cartesian square $\Omega \times \Omega$ is called an *orbital* [5]. The number of orbitals R is called the *rank* of the permutation group $\mathsf{G}(\Omega)$. If the set of orbitals contains some orbital Δ, then it necessarily contains the transposed orbital Δ^{T}. The set of orbitals of a transitive group contains the unique *diagonal* orbital $\Delta_1 = \{(i,i) \mid i \in \Omega\}$, which we will always consider to be the first element in the list of orbitals $\{\Delta_1, \ldots, \Delta_{\mathrm{R}}\}$. For transitive groups, there is a natural one-to-one correspondence between the orbitals and the orbits of stabilizers of arbitrary points i, i.e., subgroups $\mathsf{G}_i \leq \mathsf{G}$ such that $g \in \mathsf{G}_i \Rightarrow i^g = i$. The correspondence has the form

$$\Delta \leftrightarrow \Sigma_i = \{j \in \Omega \mid (i,j) \in \Delta\}\,.$$

Orbits of stabilizers will be called *suborbits*. Note that the sizes of orbitals and suborbits are related as follows: $|\Delta| = \mathsf{N}\,|\Sigma_i|$.

The invariance condition for a bilinear form A in the Hilbert space $\mathcal{H}_{\mathsf{N}}$ is expressed by the equations $A = \mathrm{P}(g)\,A\mathrm{P}\big(g^{-1}\big)\,,\ g \in \mathsf{G}$. In terms of components, these equations have the form $(A)_{ij} = (A)_{i^g j^g}$, which means that the basis of all invariant bilinear forms is in a one-to-one correspondence with the set of orbitals. Namely, each orbital $\Delta_r \in \{\Delta_1, \ldots, \Delta_{\mathrm{R}}\}$ corresponds to an $\mathsf{N} \times \mathsf{N}$ *base matrix* $\mathcal{A}_r$ with components

$$(\mathcal{A}_r)_{ij} = \begin{cases} 1 & \text{if} \quad (i,j) \in \Delta_r\,, \\ 0 & \text{if} \quad (i,j) \notin \Delta_r\,. \end{cases}$$

The matrices $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{\mathrm{R}}$ also form a basis of the *centralizer ring* (*centralizer algebra*) of the permutation representation P. The multiplication table for this basis is

$$\mathcal{A}_p\mathcal{A}_q = \sum_{r=1}^{\mathrm{R}} C_{pq}^{r}\mathcal{A}_r, \tag{1}$$

where C_{pq}^{r} are nonnegative integers such that $0 \leq C_{pq}^{r} < \mathsf{N}$. The centralizer ring is *commutative* if and only if the representation P is *multiplicity-free.*

To implement the algorithm and organize the output, we must introduce some ordering of the basis elements of the centralizer ring:

$$\mathcal{A}_1 \prec \mathcal{A}_2 \prec \ldots \prec \mathcal{A}_{\mathrm{R}}. \tag{2}$$

We use the following conventions:

(1) $\mathcal{A}_r \prec \mathcal{A}_s$ if $|\Delta_r| < |\Delta_s|$ (or, equivalently, $|(\Sigma_i)_r| < |(\Sigma_i)_s|$; comparing the lengths of the suborbits),

(2) $\mathcal{A}_r \prec \mathcal{A}_s$ if $\mathcal{A}_r = \mathcal{A}_r^{\mathrm{T}} \wedge \mathcal{A}_s \neq \mathcal{A}_s^{\mathrm{T}}$ (among the matrices with the same lengths of suborbits, symmetric matrices precede asymmetric ones),

(3) $\mathcal{A}_r \prec \mathcal{A}_s$ if $I_{\mathcal{A}_r} < I_{\mathcal{A}_s}$, $I_X = \min\left(i \mid (X)_{i1} = 1\right)$ (comparing the positions of the first nonzero element in the first columns of the matrices),

(4) if $\mathcal{A}_r \neq \mathcal{A}_r^{\mathrm{T}}$, then $\mathcal{A}_{r+1} = \mathcal{A}_r^{\mathrm{T}}$ (paired matrices are always adjacent, and inside the pair the rule (3) works automatically).

The application of the rules (1)–(4) in the specified order uniquely determines the sequence (2). According to these rules, the matrix of the diagonal orbital is the first element of the list (2): $\mathcal{A}_1 = \mathbb{1}_{\mathsf{N}}$.

3. Description of the algorithm

Let T be a unitary transformation matrix (we can always ensure that it is unitary) splitting a permutation representation P in the Hilbert space $\mathcal{H}_{\mathsf{N}}$ into M irreducible components:

$$T^{-1}\mathrm{P}(g)\,T = 1 \oplus \mathsf{U}_{d_2}(g) \oplus \cdots \oplus \mathsf{U}_{d_m}(g) \oplus \cdots \oplus \mathsf{U}_{d_M}(g)\,,$$

where U_{d_m} is an irreducible subrepresentation of dimension d_m and $\oplus$ denotes the direct sum of matrices, i.e., $A \oplus B = \mathrm{diag}(A, B)$.

The standard *scalar product* in a Hilbert space is represented in any orthonormal basis by the identity matrix $\mathbb{1}_{\mathsf{N}}$. In the splitting basis, we have the following expansion for the scalar product:

$$\mathbb{1}_{\mathsf{N}} = \mathbb{1}_{d_1=1} \oplus \cdots \oplus \mathbb{1}_{d_m} \oplus \cdots \oplus \mathbb{1}_{d_M}\,. \tag{3}$$

Here, $\mathbb{1}_{d_1=1} = \big(1\big)$ is the scalar product in the one-dimensional trivial subrepresentation, which is always present in any permutation representation. The preimage of the decomposition (3) in the original permutation basis has the form

$$\mathbb{1}_{\mathsf{N}} = \mathcal{B}_1 + \cdots + \mathcal{B}_m + \cdots + \mathcal{B}_M, \tag{4}$$

where $\mathcal{B}_m$ is defined by the relation

$$T^{-1}\mathcal{B}_m T = \mathbb{0}_{1+d_2+\cdots+d_{m-1}} \oplus \mathbb{1}_{d_m} \oplus \mathbb{0}_{d_{m+1}+\cdots+d_M} \equiv \mathcal{D}_m. \tag{5}$$

This relation shows that the matrices $\mathcal{B}_m$ are *idempotent*,

$$\mathcal{B}_m^2 = \mathcal{B}_m, \tag{6}$$

and *mutually orthogonal*,

$$\mathcal{B}_m\mathcal{B}_{m'} = \mathbb{0}_{\mathsf{N}} \ \text{ if } m \neq m'. \tag{7}$$

Relations (6) and (7), together with the *completeness* condition (4), mean that $\mathcal{B}_1, \ldots, \mathcal{B}_M$ is a *complete system of mutually orthogonal projections* in the Hilbert space $\mathcal{H}_{\mathsf{N}}$.

The set of *irreducible invariant* projections $\mathcal{B}_1, \ldots, \mathcal{B}_M$ contains complete information about the decomposition of the representation P into irreducible components. For example, the transformation matrix T can be calculated by solving the system of linear equations

$$\mathcal{B}_1 T - T\mathcal{D}_1 = \cdots = \mathcal{B}_M T - T\mathcal{D}_M = \mathbb{0}_{\mathsf{N}}\,.$$

Any invariant projection is a solution to the equation

$$X^2 - X = \mathbb{0}_{\mathsf{N}}, \tag{8}$$

where $X = x_1\mathcal{A}_1 + \cdots + x_{\mathrm{R}}\mathcal{A}_{\mathrm{R}}$ is the invariant bilinear form written in the basis (2). Using the multiplication table (1) and expanding (8) into components in the basis (2), we get a system of R quadratic equations for R unknowns $x_1, \ldots, x_{\mathrm{R}}$:

$$E(x_1, \ldots, x_{\mathrm{R}}) = 0 \sim \{E_1(x_1, \ldots, x_{\mathrm{R}}) = 0, \ldots, E_{\mathrm{R}}(x_1, \ldots, x_{\mathrm{R}}) = 0\}\,. \tag{9}$$

We will call the left-hand sides of these equations *idempotency polynomials*. In the basis (2), an irreducible invariant projection $\mathcal{B}_m$ has the form

$$\mathcal{B}_m = b_{m,1}\mathcal{A}_1 + b_{m,2}\mathcal{A}_2 + \cdots + b_{m,\mathrm{R}}\mathcal{A}_{\mathrm{R}}, \tag{10}$$

where the vector $B_m = [b_{m,1}, \ldots, b_{m,\mathrm{R}}]$ is a solution of the system of equations (9). Since the trace of a matrix is an invariant of a similarity transformation, relation (5) implies the equality

$$\operatorname{tr}\mathcal{B}_m = d_m. \tag{11}$$

Combining this equality with the fact that in the decomposition (10) only $\mathcal{A}_1$ has nonzero diagonal elements and $\operatorname{tr}\mathcal{A}_1 = \mathsf{N}$, we can fix the first coefficient in (10):

$$b_{m,1} = d_m/\mathsf{N}.$$

Thus, the possible values of x_1 in (9) are rational numbers d/N for some dimensions $d \in [1, \ldots, \mathsf{N}-1]$. Any positive integer d for which the polynomial system (9) has a solution is either an irreducible dimension d_m or a sum of such dimensions. The orthogonality condition (7) allows us to exclude from consideration dimensions that are not irreducible. In general, the orthogonality condition can be written as

$$BX = 0 \tag{12}$$

where $B = b_1\mathcal{A}_1 + \cdots + b_{\mathrm{R}}\mathcal{A}_{\mathrm{R}}$. Equation (12) is a system of linear equations in variables $x_1, \ldots, x_{\mathrm{R}}$ with parameters $b_1, \ldots, b_{\mathrm{R}}$. Using the multiplication table (1), the left-hand side of (12) can be represented as a system of R bilinear forms

$$O(b_1, \ldots, b_{\mathrm{R}}; x_1, \ldots, x_{\mathrm{R}}) = \begin{Bmatrix} O_1(b_1, \ldots, b_{\mathrm{R}}; x_1, \ldots, x_{\mathrm{R}})\,, \\ \vdots \\ O_{\mathrm{R}}(b_1, \ldots, b_{\mathrm{R}}; x_1, \ldots, x_{\mathrm{R}}) \end{Bmatrix}, \tag{13}$$

which we will call *orthogonality polynomials*.

The main part of the algorithm is organized as a cycle starting with $d = 1$ and ending when the sum of irreducible dimensions reaches the value N. The current dimension d is processed as follows:

1. The algorithm solves the system of equations
$$E(d/\mathsf{N}, x_2, \ldots, x_{\mathrm{R}}) = 0. \tag{14}$$
Moreover, without significant additional calculations,it finds the Hilbert dimension h of the corresponding polynomial ideal. The solution can always be realized algorithmically, since all roots of the system belong to Abelian extensions of the ring of rational numbers. Modern computer algebra systems, in particular, ***Maple***, do the job quite well.
2. If the system (14) is incompatible, then the current value d is not an irreducible dimension, and we pass to the next value of d.
3. If $h = 0$ and the system (14) has k solutions, then we get k (distinct, provided that $k > 1$) d-dimensional irreducible subrepresentations.

4. The fact that the dimension h of the polynomial ideal is positive means that there is a d-dimensional irreducible component of nontrivial multiplicity k. The corresponding component of the centralizer algebra has the structure of a Kronecker product $A \otimes \mathbb{1}_d$ where A is an arbitrary $k \times k$ matrix. From the idempotency condition
$$(A \otimes \mathbb{1}_d)^2 = A \otimes \mathbb{1}_d$$
distinguishing a projector, a restriction on the matrix A follows:
$$A^2 - A = 0. \tag{15}$$
The complete family of solutions to this equation[1] is a variety of dimension $h = \lfloor k^2/2 \rfloor$. Hence, the multiplicity can be calculated from the Hilbert dimension: $k = \left\lceil \sqrt{2h} \right\rceil$.

   Then, using a certain procedure, from the family of equivalent d-dimensional projections we choose k arbitrary, but mutually orthogonal, representatives.
5. Each of the k irreducible projections obtained at step 3 or 4 is processed as follows. The projection $\mathcal{B}_m$ is added to the list of irreducible projections. The corresponding invariant subspace is excluded from further consideration by adding the orthogonality polynomials $\mathcal{B}_m X$ to the set of polynomials (9):
$$E(x_1, x_2, \ldots, x_{\mathrm{R}}) \leftarrow E(x_1, x_2, \ldots, x_{\mathrm{R}}) \cup \{\mathcal{B}_m X\}\,.$$
6. After processing all k irreducible projections of the current dimension d as described at step 5, we pass to the next possible dimension.

The ***IrreducibleProjectors*** algorithm is implemented in the form of two procedures, called ***PreparePolynomialData*** and ***SplitRepresentation***.

The ***PreparePolynomialData*** procedure is implemented as a program in the language ***C***. The input to ***PreparePolynomialData*** is a set of permutations that generate the group $\mathsf{G}(\Omega)$. The program calculates a basis of the centralizer ring (2) and the multiplication table (1) and constructs the idempotency polynomials (9) and orthogonality polynomials (13). Then it builds the procedure code ***SplitRepresentation***. This code takes into account the specific features of concrete tasks. In particular, if the centralizer ring is noncommutative, then the decomposition contains multiple subrepresentations, for which additional functions are generated.

The program ***SplitRepresentation*** is a code in the language ***Maple*** generated by the program ***PreparePolynomialData***. This program performs the cycle described above. Systems of polynomial equations are processed using functions from the ***Groebner*** package implemented in the system ***Maple***.

The algorithms, their implementation, and related technical details are described in more detail in [6].

4. Computations

We give examples of computations using the programs ***PreparePolynomialData*** and ***SplitRepresentation***. The input data (permutations generating the representations under study) are taken from the "Sporadic groups" section of the Atlas of representations of finite groups [7]. This section contains subsections corresponding to the traditional classification of sporadic simple groups: "Mathieu groups," "Leech lattice groups," "Monster sections," and "Pariahs." Besides representations of simple groups, the Atlas contains representations of their extensions. Namely, if a group G has a nontrivial

[1] It is well known that any solution of the matrix equation (15) can be represented as $A = Q^{-1}\left(\mathbb{1}_r \oplus \mathbb{0}_{k-r}\right) Q$, where Q is an arbitrary invertible $k \times k$ matrix and r is an arbitrary positive integer such that $0 \le r \le k$.

(1) second homology group $H_2(\mathsf{G}, \mathbb{Z})$, called the *Schur multiplicator* and denoted by $\mathrm{M}(\mathsf{G})$, then there are nontrivial central extensions of G by subgroups of $\mathrm{M}(\mathsf{G})$;
(2) group of outer automorphisms $\mathrm{Out}(\mathsf{G})$, then there are nontrivial extensions in which G is a normal subgroup.

An arbitrary extension of a group B by a group A will be denoted by $A.B$; a decomposable extension, i.e., a semidirect product, will be denoted by $A \rtimes B$. According to the abbreviation used in the Atlas, cyclic groups C_n involved in extensions will be denoted by their orders n.

In the expressions for decompositions of representations below, irreducible components will be denoted by their dimensions in bold (possibly with additional indices to distinguish between nonequivalent subrepresentations of the same dimension). Permutation representations are denoted by their dimensions underlined. Irreducible projections will be denoted by $\mathcal{B}_{\mathbf{m}}$ where $\mathbf{m}$ is the corresponding irreducible subrepresentation.

The calculations were performed on a personal computer with a 3.30GHz Intel Core i3 2120 processor and 16 GB RAM.

4.1. A detailed example. Let us consider a clear example in detail, to illustrate the output data produced by the programs ***PreparePolynomialData*** and ***SplitRepresentation***.

The Held group He, belonging to the "Monster sections" subsection of the Atlas, has the following basic properties:

$$\mathrm{Ord(He)} = 4030387200 = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17, \quad \mathrm{M(He)} \cong 1, \quad \mathrm{Out(He)} \cong \mathsf{C}_2.$$

The program ***PreparePolynomialData***, applied to the 8330-dimensional representation of this group, besides generating the program code ***SplitRepresentation*** and the input data for it, outputs the following text:

```
___Action of He on 8330 points
Rank of He_on_8330: 7
Dimension: 8330
Suborbit lengths: 1, 105, 720, 840, 840', 1344, 4480.
Centralizer ring is commutative
=> permutation representation is multiplicity free
___Total time: 2.93 sec
___Technical information
Orbital matrices space: 57.9 MB
Orbital path space    : 35.6 MB
Total orbital space   : 93.5 MB
Maximum number of polynomial terms: 217
```

This text contains information on the rank of the representation, the lengths of the suborbits (the length of the suborbit of the second orbital in a pair of mutually transposed orbitals is primed), the presence or absence of multiple subrepresentations, as well as the time and memory spent to solve the problem.

The program ***SplitRepresentation*** produces the following decomposition:

$$\underline{\mathbf{8330}} \cong \mathbf{1} \oplus \mathbf{51} \oplus \overline{\mathbf{51}} \oplus \mathbf{680} \oplus \mathbf{1275} \oplus \mathbf{1920} \oplus \mathbf{4352}$$

$$\mathcal{B}_{\mathbf{1}} = \frac{1}{8330}\left(\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4+\mathcal{A}_5+\mathcal{A}_6+\mathcal{A}_7\right)$$

$$\mathcal{B}_{\mathbf{51}} = \frac{3}{490}\left(\mathcal{A}_1+\frac{\mathcal{A}_2}{3}-\frac{\mathcal{A}_3}{6}-\frac{1-\mathbf{i}\sqrt{7}}{12}\mathcal{A}_4-\frac{1+\mathbf{i}\sqrt{7}}{12}\mathcal{A}_5+\frac{\mathcal{A}_6}{6}\right)$$

$$\mathcal{B}_{\mathbf{680}} = \frac{4}{49}\left(\mathcal{A}_1+\frac{\mathcal{A}_2}{5}+\frac{\mathcal{A}_3}{120}+\frac{\mathcal{A}_4}{20}+\frac{\mathcal{A}_5}{20}-\frac{\mathcal{A}_7}{40}\right)$$

$$\mathcal{B}_{\mathbf{1275}} = \frac{15}{98}\left(\mathcal{A}_1+\frac{\mathcal{A}_2}{15}+\frac{\mathcal{A}_3}{15}-\frac{\mathcal{A}_4}{30}-\frac{\mathcal{A}_5}{30}\right)$$

$$\mathcal{B}_{\mathbf{1920}} = \frac{192}{833}\left(\mathcal{A}_1-\frac{2\mathcal{A}_2}{15}+\frac{\mathcal{A}_3}{120}+\frac{\mathcal{A}_4}{120}+\frac{\mathcal{A}_5}{120}+\frac{5\mathcal{A}_6}{192}-\frac{3\mathcal{A}_7}{320}\right)$$

$$\mathcal{B}_{\mathbf{4352}} = \frac{128}{245}\left(\mathcal{A}_1-\frac{\mathcal{A}_3}{48}-\frac{\mathcal{A}_6}{64}+\frac{\mathcal{A}_7}{128}\right)$$

```
Time:  1.4 sec
```

Here, **51** and $\overline{\mathbf{51}}$ are different complex conjugate representations of dimension 51.

4.2. Comparison with the implementation of *MeatAxe* in *Magma*. The implementation of the ***MeatAxe*** algorithm in the computer algebra system ***Magma*** is considered to be one of the best ones. The ***Magma*** database contains a 3906-dimensional permutation representation of the exceptional Lie type group $G_2(5)$. The decomposition of this representation into irreducible components over the field GF(2) is presented in [8] to illustrate the possibilities of ***MeatAxe***.

Applying our programs to this representation, we get the following data:

Rank: 4. Suborbit lengths: $1, 30, 750, 3125$.

$$\underline{\mathbf{3906}} \cong \mathbf{1}\oplus\mathbf{930}\oplus\mathbf{1085}\oplus\mathbf{1890}$$

$$\mathcal{B}_{\mathbf{1}} = \frac{1}{3906}\sum_{k=1}^{4}\mathcal{A}_k$$

$$\mathcal{B}_{\mathbf{930}} = \frac{5}{21}\left(\mathcal{A}_1+\frac{3}{10}\mathcal{A}_2+\frac{1}{50}\mathcal{A}_3-\frac{1}{125}\mathcal{A}_4\right)$$

$$\mathcal{B}_{\mathbf{1085}} = \frac{5}{18}\left(\mathcal{A}_1-\frac{1}{5}\mathcal{A}_2+\frac{1}{25}\mathcal{A}_3-\frac{1}{125}\mathcal{A}_4\right)$$

$$\mathcal{B}_{\mathbf{1890}} = \frac{15}{31}\left(\mathcal{A}_1-\frac{1}{30}\mathcal{A}_2-\frac{1}{30}\mathcal{A}_3+\frac{1}{125}\mathcal{A}_4\right)$$

Time **C**: 0.5 sec. Time **Maple**: 0.8 sec.

We see that in the case of zero characteristic, the representation can be split over the field $\mathbb{Q}$.

One cannot split this representation over $\mathbb{Q}$ with ***Magma***, due to memory exhaustion. However, one can reproduce the same set of dimensions of irreducible components as in the case of zero characteristic if one splits the representation over a finite field whose characteristic does not divide the order of the group. In this case, $|G_2(5)| = 5859000000 = 2^6\cdot 3^3\cdot 5^6\cdot 7\cdot 31$. Therefore, the smallest field "modeling" $\mathbb{Q}$ in the above sense is GF(11). We present the corresponding calculation session using ***Magma*** (the calculation time is displayed in seconds).

```
> load "g25";
Loading "/opt/magma.21-1/libs/pergps/g25"
The Lie group G( 2, 5 ) represented as a permutation
group of degree 3906.
```

```
Order: 5 859 000 000 = 2^6 * 3^3 * 5^6 * 7 * 31.
Group: G
> time Constituents(PermutationModule(G,GF(11)));
[
    GModule of dimension 1 over GF(11),
    GModule of dimension 930 over GF(11),
    GModule of dimension 1085 over GF(11),
    GModule of dimension 1890 over GF(11)
]
Time: 282.060
```

4.3. Some calculations for sporadic groups. For completeness, we have tried to select examples from both families of sporadic groups ("Pariah" and "Happy Family") and from all generations of "Happy Family." Not all sections of the Atlas have sufficiently representative data sets, so the examples below, along with the results solving difficult problems, contain almost trivial ones.

The results of the calculations presented in this section contain information on ranks, lengths of suborbits, structures of decompositions into irreducible components, and computation times. For brevity, we omit explicit expressions for irreducible projections, which can be quite cumbersome. An expression of the form ℓ^m in a list of lengths of suborbits means that there are m suborbits of length ℓ. Nonequivalent irreducible components of the same dimension are distinguished either by the complex conjugation symbol (overline), or by Greek subscripts, or else by the subscripts $\pm$, which mean that there are two components having the structure $A \pm B$. To distinguish one-dimensional representations from trivial ones, primes are used. Multiple subrepresentations are combined using underbraces. The execution times of the programs ***PreparePolynomialData*** and ***SplitRepresentation*** are indicated separately.

4.3.1. *Mathieu groups.* The five Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, and M_{24} are the first sporadic groups to be discovered. Each group M_n is isomorphic to a multiply transitive permutation group of n objects. Among all the Mathieu groups, only M_{12} and M_{22} have nontrivial Schur multiplicators and groups of outer automorphisms. As concerns the structure of decompositions, the most interesting case is that of extensions of the group M_{22}.

The main properties of the group M_{22} are as follows:

$$\mathrm{Ord}(M_{22}) = 443520 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11, \quad \mathrm{M}(M_{22}) \cong \mathsf{C}_{12}, \quad \mathrm{Out}(M_{22}) \cong \mathsf{C}_2.$$

(1) The 990-dimensional representation of the group $3.M_{22}$.

Rank: 13. Suborbit lengths: $1^3, 7^3, 42^3, 168^3, 336$.

$$\underline{\mathbf{990}} \cong \mathbf{1} \oplus \mathbf{21}_\alpha \oplus \mathbf{21}_\beta \oplus \overline{\mathbf{21}_\beta} \oplus \mathbf{55} \oplus \mathbf{99}_\alpha \oplus \mathbf{99}_\beta \oplus \overline{\mathbf{99}_\beta} \oplus \mathbf{105}_+ \oplus \overline{\mathbf{105}_+} \oplus \mathbf{105}_- \oplus \overline{\mathbf{105}_-} \oplus \mathbf{154}$$

Time **C**: 1 sec. Time **Maple**: 28 sec.

(2) The 2016-dimensional representation of the group $3.M_{22}$.

Rank: 16. Suborbit lengths: $1^3, 55^3, 66^3, 165^4, 330^3$.

$$\underline{\mathbf{2016}} \cong \mathbf{1} \oplus \mathbf{21}_\alpha \oplus \mathbf{21}_\beta \oplus \overline{\mathbf{21}_\beta} \oplus \mathbf{55} \oplus \mathbf{105}_+ \oplus \overline{\mathbf{105}_+} \oplus \mathbf{105}_- \oplus \overline{\mathbf{105}_-} \oplus \mathbf{154} \oplus \mathbf{210}_\alpha \oplus \mathbf{210}_\beta \oplus \overline{\mathbf{210}_\beta} \oplus \mathbf{231}_\alpha \oplus \mathbf{231}_\beta \oplus \overline{\mathbf{231}_\beta}$$

Time **C**: 2 sec. Time **Maple**: 1 h 15 min 52 sec.

(3) The 1980-dimensional representation of the group $6.M_{22}$.

Rank: 17. Suborbit lengths: $1^6, 14^3, 84^3, 336^5$.

$$\underline{\mathbf{1980}} \cong \mathbf{1} \oplus \mathbf{21}_\alpha \oplus \mathbf{21}_\beta \oplus \overline{\mathbf{21}_\beta} \oplus \mathbf{55} \oplus \mathbf{99}_\alpha \oplus \mathbf{99}_\beta \oplus \overline{\mathbf{99}_\beta} \oplus \mathbf{105}_+ \oplus \overline{\mathbf{105}_+} \oplus \mathbf{105}_- \oplus \overline{\mathbf{105}_-} \oplus \mathbf{120} \oplus \mathbf{154} \oplus \mathbf{210} \oplus \mathbf{330} \oplus \overline{\mathbf{330}}$$

Time **C**: 1 sec. Time **Maple**: 6 h 34 min 14 sec.

4.3.2. *Leech lattice groups.*

- The *Higman–Sims group* HS. Basic properties:

$$\mathrm{Ord(HS)} = 44352000 = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11, \quad \mathrm{M(HS)} \cong \mathsf{C}_2, \quad \mathrm{Out(HS)} \cong \mathsf{C}_2.$$

(1) The 5600-dimensional representation of the group HS.

Rank: 9. Suborbit lengths: $1, 55, 132, 165, 495, 660, 792, 1320, 1980$.

$$\underline{\mathbf{5600}} \cong \mathbf{1} \oplus \mathbf{22} \oplus \mathbf{77} \oplus \mathbf{154} \oplus \mathbf{175} \oplus \mathbf{770} \oplus \mathbf{825} \oplus \mathbf{1056} \oplus \mathbf{2520}$$

Time **C**: 2 sec. Time **Maple**: 2 sec.

(2) The 11200-dimensional representation of the group 2.HS.

Rank: 16. Suborbit lengths: $1^2, 110, 132^2, 165^2, 660^2, 792^2, 990, 1320^2, 1980^2$.

$$\underline{\mathbf{11200}} \cong \mathbf{1} \oplus \mathbf{22} \oplus \mathbf{56} \oplus \mathbf{77} \oplus \mathbf{154} \oplus \mathbf{175} \oplus \mathbf{176} \oplus \overline{\mathbf{176}} \oplus \mathbf{616} \oplus \overline{\mathbf{616}} \oplus \mathbf{770} \oplus \mathbf{825} \oplus \mathbf{1056} \oplus \mathbf{1980} \oplus \overline{\mathbf{1980}} \oplus \mathbf{2520}$$

Time **C**: 7 sec. Time **Maple**: 1 h 25 min 47 sec.

(3) The 1100-dimensional representation of the group $\mathrm{HS} \rtimes 2$.

Rank: 5. Suborbit lengths: $1, 28, 105, 336, 630$.

$$\underline{\mathbf{1100}} \cong \mathbf{1} \oplus \mathbf{77} \oplus \mathbf{154} \oplus \mathbf{175} \oplus \mathbf{693}$$

Time **C**: < 1 sec. Time **Maple**: < 1 sec.

(4) The 1408-dimensional representation of the group 2.HS.2.

Rank: 11. Suborbit lengths: $1^4, 50^4, 350^2, 504$.

$$\underline{\mathbf{1408}} \cong \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{22}_+ \oplus \mathbf{22}_- \oplus \mathbf{175}_+ \oplus \mathbf{175}_- \oplus \mathbf{308} \oplus \underbrace{\mathbf{352} \oplus \mathbf{352}}$$

Time **C**: < 1 sec. Time **Maple**: 3 sec.

- The *Janko group* J_2. Basic properties:

$$\mathrm{Ord}(J_2) = 604800 = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7, \quad \mathrm{M}(J_2) \cong \mathsf{C}_2, \quad \mathrm{Out}(J_2) \cong \mathsf{C}_2.$$

(1) The 1800-dimensional representation of the group J_2.

Rank: 18. Suborbit lengths: $1, 14^2, 21, 28, 42^3, 84^3, 168^6, 336$.

$$\underline{\mathbf{1800}} \cong \mathbf{1} \oplus \mathbf{36} \oplus \underbrace{\mathbf{63} \oplus \mathbf{63}} \oplus \underbrace{\mathbf{126} \oplus \mathbf{126}} \oplus \mathbf{160} \oplus \mathbf{175} \oplus \mathbf{288} \oplus \underbrace{\mathbf{336} \oplus \mathbf{336}}$$

Time **C**: 2 sec. Time **Maple**: 13 min 29 sec.

- The *Conway group* Co_1. Basic properties:

$$\mathrm{Ord}(\mathrm{Co}_1) = 4157776806543360000 = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23,$$
$$\mathrm{M}(\mathrm{Co}_1) \cong \mathsf{C}_2, \ \mathrm{Out}(\mathrm{Co}_1) \cong 1.$$

(1) The 98280-dimensional representation of the group Co_1.
Rank: 4. Suborbit lengths: $1, 4600, 46575, 47104$.

$$\underline{\mathbf{98280}} \cong \mathbf{1} \oplus \mathbf{299} \oplus \mathbf{17250} \oplus \mathbf{80730}$$

Time **C**: 43 min 12 sec. Time **Maple**: 6 sec.

Remark. The program ***PreparePolynomialData*** uses more than 8.8 Gb of memory for this task.

• The *Conway group* Co_2. Basic properties:

$$\mathrm{Ord}(\mathrm{Co}_2) = 42305421312000 = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \quad \mathrm{M}(\mathrm{Co}_2) \cong 1, \ \mathrm{Out}(\mathrm{Co}_2) \cong 1.$$

(1) The 4600-dimensional representation of the group Co_2.

Rank: 5. Suborbit lengths: $1^2, 891^2, 2816$.

$$\underline{\mathbf{4600}} \cong \mathbf{1} \oplus \mathbf{23} \oplus \mathbf{275} \oplus \mathbf{2024} \oplus \mathbf{2277}$$

Time **C**: < 1 sec. Time **Maple**: < 1 sec.

• The *Conway group* Co_3. Basic properties:

$$\mathrm{Ord}(\mathrm{Co}_3) = 495766656000 = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, \quad \mathrm{M}(\mathrm{Co}_3) \cong 1, \quad \mathrm{Out}(\mathrm{Co}_3) \cong 1.$$

(1) The 48600-dimensional representation of the group Co_3.

Rank: 8. Suborbit lengths: $1, 253, 506, 1771, 7590, 8855, 14168, 15456$.

$$\underline{\mathbf{48600}} \cong \mathbf{1} \oplus \mathbf{23} \oplus \mathbf{253} \oplus \mathbf{275} \oplus \mathbf{2024} \oplus \mathbf{5544} \oplus \mathbf{8855} \oplus \mathbf{31625}$$

Time **C**: 2 min 17 sec. Time **Maple**: 2 sec.

• The *McLaughlin group* McL. Basic properties:

$$\mathrm{Ord}(\mathrm{McL}) = 898128000 = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11, \quad \mathrm{M}(\mathrm{McL}) \cong \mathsf{C}_3, \ \mathrm{Out}(\mathrm{McL}) \cong \mathsf{C}_2.$$

(1) The 22275-dimentional representation (a) of the group McL.

Rank: 13. Suborbit lengths: $1, 112, 140, 210, 420, 672, 1680^2, 2240, 3360^3, 5040$.

$$\underline{\mathbf{22275}} \cong \mathbf{1} \oplus \mathbf{22} \oplus \underbrace{\mathbf{252} \oplus \mathbf{252}} \oplus \underbrace{\mathbf{1750} \oplus \mathbf{1750}} \oplus \mathbf{3520} \oplus \mathbf{5103} \oplus \mathbf{9625}$$

Time **C**: 23 sec. Time **Maple**: 11 sec.

(2) The 66825-dimensional representation of the group 3.McL.

Rank: 14. Suborbit lengths: $1^3, 630, 2240^3, 5040^3, 8064^3, 20160$.

$$\underline{\mathbf{66825}} \cong \mathbf{1} \oplus \mathbf{252} \oplus \mathbf{252} \oplus \mathbf{1750} \oplus \mathbf{2772} \oplus \overline{\mathbf{2772}} \oplus \mathbf{5103}_\beta \oplus \overline{\mathbf{5103}_\beta}$$
$$\oplus \mathbf{5103}_\alpha \oplus \mathbf{5544} \oplus \mathbf{6336} \oplus \overline{\mathbf{6336}} \oplus \mathbf{8064} \oplus \overline{\mathbf{8064}} \oplus \mathbf{9625}$$

Time **C**: 8 min 45 sec. Time **Maple**: 12 min 59 sec.

(3) The 22275-dimensional representation (a) of the group $\mathrm{McL} \rtimes 2$.

Rank: 11. Suborbit lengths: $1, 112, 210, 420, 1120, 1260, 2520^2, 3360, 4032, 6720$.

$$\underline{\mathbf{22275}} \cong \mathbf{1} \oplus \mathbf{22} \oplus \underbrace{\mathbf{252} \oplus \mathbf{252}} \oplus \mathbf{1750}_\alpha \oplus \mathbf{1750}_\beta \oplus \mathbf{3520} \oplus \mathbf{5103} \oplus \mathbf{9625}$$

Time **C**: 23 sec. Time **Maple**: 5 sec.

• The *Suzuki group* Suz. Basic properties:

$$\mathrm{Ord}(\mathrm{Suz}) = 448345497600 = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13, \quad \mathrm{M}(\mathrm{Suz}) \cong \mathsf{C}_6, \ \mathrm{Out}(\mathrm{Suz}) \cong \mathsf{C}_2.$$

(1) The 32760-dimensional representation of the group Suz.

Rank: 6. Suborbit lengths: $1, 891, 1980, 2816, 6336, 20736$.

$$\underline{\mathbf{32760}} \cong \mathbf{1} \oplus \mathbf{143} \oplus \mathbf{364} \oplus \mathbf{5940} \oplus \mathbf{12012} \oplus \mathbf{14300}$$

Time **C**: 54 sec. Time **Maple**: 2 sec.

(2) The 65520-dimensional representation of the group 2.Suz.
Rank: 10. Suborbit lengths: $1^2, 891^2, 2816^2, 3960, 12672, 20736^2$.

$$\underline{\mathbf{65520}} \cong \mathbf{1} \oplus \mathbf{143} \oplus \mathbf{364}_\alpha \oplus \mathbf{364}_\beta \oplus \overline{\mathbf{364}_\beta} \oplus \mathbf{5940} \oplus \mathbf{12012} \oplus \mathbf{14300} \oplus \mathbf{16016} \oplus \overline{\mathbf{16016}}$$

Time **C**: 6 min 9 sec. Time **Maple**: 11 sec.

(3) The 98280-dimensional representation of the group 3.Suz.
Rank: 14. Suborbit lengths: $1^3, 891^3, 2816^3, 5940, 19008, 20736^3$.

$$\underline{\mathbf{98280}} \cong \mathbf{1} \oplus \mathbf{78} \oplus \overline{\mathbf{78}} \oplus \mathbf{143} \oplus \mathbf{364} \oplus \mathbf{1365} \oplus \overline{\mathbf{1365}} \oplus \mathbf{4290} \oplus \overline{\mathbf{4290}} \oplus \mathbf{5940} \oplus \mathbf{12012} \oplus \mathbf{14300} \oplus \mathbf{27027} \oplus \overline{\mathbf{27027}}$$

Time **C**: 57 min 58 sec. Time **Maple**: 6 min 42 sec.
Remark. The ***PreparePolynomialData*** program uses more than 17.6 GB of memory for this task, which exceeds the RAM of our computer and slows the computations due to the use of hard drive space.

(4) The 1782-dimensional representation of the group Suz $\rtimes$ 2.

Rank: 3. Suborbit lengths: $1, 416, 1365$.

$$\underline{\mathbf{1782}} \cong \mathbf{1} \oplus \mathbf{780} \oplus \mathbf{1001}$$

Time **C**: < 1 sec. Time **Maple**: < 1 sec.

(5) The 5346-dimensional representation of the group 3.Suz $\rtimes$ 2.

Rank: 5. Suborbit lengths: $1, 2, 416, 832, 4095$.

$$\underline{\mathbf{5346}} \cong \mathbf{1} \oplus \mathbf{132} \oplus \mathbf{780} \oplus \mathbf{1001} \oplus \mathbf{3432}$$

Time **C**: 1 sec. Time **Maple**: < 1 sec.

4.3.3. *Monster sections.* The main properties of the *Held group* He and the results of calculations for its 8330-dimensional representation are described in Sec. 4.1.

(1) The 29155-dimensional representation of the group He.

Rank: 12. Suborbit lengths: $1, 90, 120, 384, 960^2, 1440, 2160, 2880^2, 5760, 11520$.

$$\underline{\mathbf{29155}} \cong \mathbf{1} \oplus \mathbf{51} \oplus \overline{\mathbf{51}} \oplus \mathbf{680} \oplus \underbrace{\mathbf{1275} \oplus \mathbf{1275}} \oplus \mathbf{1920} \oplus \mathbf{4352} \oplus \mathbf{7650} \oplus \mathbf{11900}$$

Time **C**: 42 sec. Time **Maple**: 11 sec.

(2) The 8330-dimensional representation of the group He $\rtimes$ 2.

Rank: 6. Suborbit lengths: $1, 105, 720, 1344, 1680, 4480$.

$$\underline{\mathbf{8330}} \cong \mathbf{1} \oplus \mathbf{102} \oplus \mathbf{680} \oplus \mathbf{1275} \oplus \mathbf{1920} \oplus \mathbf{4352}$$

Time **C**: 3 sec. Time **Maple**: 1 sec.

- The *Fischer group* Fi_{22}. Basic properties:

$\mathrm{Ord}(\mathrm{Fi}_{22}) = 64561751654400 = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$, $\quad \mathrm{M}(\mathrm{Fi}_{22}) \cong \mathsf{C}_6$, $\mathrm{Out}(\mathrm{Fi}_{22}) \cong \mathsf{C}_2$.

(1) The 61776-dimensional representation of the group Fi_{22}.

Rank: 4. Suborbit lengths: $1, 1575, 22400, 37800$.

$$\underline{\mathbf{61776}} \cong \mathbf{1} \oplus \mathbf{3080} \oplus \mathbf{13650} \oplus \mathbf{45045}$$

Time **C**: 10 min 6 sec. Time **Maple**: 3 sec.

(2) The 28160-dimensional representation of the group $2.\mathrm{Fi}_{22}$.

Rank: 5. Suborbit lengths: $1^2, 3159^2, 21840$.

$$\underline{\mathbf{28160}} \cong \mathbf{1} \oplus \mathbf{352} \oplus \mathbf{429} \oplus \mathbf{13650} \oplus \mathbf{13728}$$

Time **C**: 39 sec. Time **Maple**: 2 sec.

(3) The 56320-dimensional representation of the group $2.\mathrm{Fi}_{22} \rtimes 2$.

Rank: 9. Suborbit lengths: $1^2, 728, 1080^2, 3159^2, 21840, 25272$.

$$\underline{\mathbf{56320}} \cong \mathbf{1} \oplus \mathbf{1'} \oplus \mathbf{352} \oplus \overline{\mathbf{352}} \oplus \mathbf{429}_+ \oplus \mathbf{429}_- \oplus \mathbf{13650}_+ \oplus \mathbf{13650}_- \oplus \mathbf{27456}$$

Time **C**: 3 min 20 sec. Time **Maple**: 5 sec.

- The *Fischer group* Fi_{23}. Basic properties:

$$\mathrm{Ord}(\mathrm{Fi}_{23}) = 4089470473293004800 = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23,$$
$$\mathrm{M}(\mathrm{Fi}_{23}) \cong 1, \quad \mathrm{Out}(Fi_{23}) \cong 1.$$

(1) The 31671-dimensional representation of the group Fi_{23}.

Rank: 3. Suborbit lengths: $1, 3510, 28160$.

$$\underline{\mathbf{31671}} \cong \mathbf{1} \oplus \mathbf{782} \oplus \mathbf{30888}$$

Time **C**: 52 sec. Time **Maple**: 1 sec.

4.3.4. *Pariah.*

- The *Janko group* J_1. Basic properties:

$$\mathrm{Ord}(J_1) = 175560 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19, \quad \mathrm{M}(J_1) \cong 1, \quad \mathrm{Out}(J_1) \cong 1.$$

(1) The 1045-dimensional representation of the group J_1.

Rank: 11. Suborbit lengths: $1, 8, 28, 56^3, 168^5$.

$$\underline{\mathbf{1045}} \cong \mathbf{1} \oplus \mathbf{56}_+ \oplus \mathbf{56}_- \oplus \mathbf{76} \oplus \mathbf{77}_+ \oplus \mathbf{77}_- \oplus \mathbf{120}_\alpha \oplus \mathbf{120}_\beta \oplus \mathbf{120}_\gamma \oplus \mathbf{133} \oplus \mathbf{209}$$

Time **C**: < 1 sec. Time **Maple**: 22 sec.

- The *Janko group* J_3. Basic properties:

$$\mathrm{Ord}(J_3) = 50232960 = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19, \quad \mathrm{M}(J_3) \cong \mathsf{C}_3, \quad \mathrm{Out}(J_3) \cong \mathsf{C}_2.$$

(1) The 14688-dimensional representations (a) and (b) of the group J_3.
Rank: 14. Suborbit lengths: $1, 285, 342, 380, 570^2, 855^2, 1140^2, 1710^3, 3420$.

$$\underline{\mathbf{14688}} \cong \mathbf{1} \oplus \mathbf{85} \oplus \overline{\mathbf{85}} \oplus \underbrace{\mathbf{1140} \oplus \mathbf{1140}} \oplus \mathbf{1215}_+ \oplus \mathbf{1215}_- \oplus \mathbf{1615} \oplus \mathbf{1920}_\alpha \oplus \mathbf{1920}_\beta \oplus \mathbf{1920}_\gamma \oplus \mathbf{2432}$$

Time **C**: 11 sec. Time **Maple**: 1 min 52 sec.

Remark. The Atlas [7] contains two nonequivalent 14688-dimensional representations of the group J_3, (a) and (b), which have the same decomposition structure. The difference is manifested in the explicit expressions for irreducible projections (and in the structure of orbitals), which we do not present here. The execution times also coincide up to one second.

(2) The 6156-dimensional representation of the group $J_3 \rtimes 2$.
Rank: 7. Suborbit lengths: $1, 85, 120, 510, 680, 2040, 2720$.

$$\underline{\mathbf{6156}} \cong \mathbf{1} \oplus \mathbf{324} \oplus \mathbf{646} \oplus \mathbf{1140} \oplus \mathbf{1215}_{+} \oplus \mathbf{1215}_{-} \oplus \mathbf{1615}$$

Time **C**: 1 sec. Time **Maple**: 1 sec.

• The *Rudvalis group* Ru. Basic properties:

$$\mathrm{Ord(Ru)} = 145926144000 = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29, \quad \mathrm{M(Ru)} \cong \mathsf{C}_2, \ \mathrm{Out(Ru)} \cong 1.$$

(1) The 4060-dimensional representation of the group Ru.

Rank: 3. Suborbit lengths: $1, 1755, 2304$.

$$\underline{\mathbf{4060}} \cong \mathbf{1} \oplus \mathbf{783} \oplus \mathbf{3276}$$

Time **C**: < 1 sec. Time **Maple**: < 1 sec.

(2) The 16240-dimensional representation of the group 2.Ru.

Rank: 9. Suborbit lengths: $1^4, 2304^4, 7020$.

$$\underline{\mathbf{16240}} \cong \mathbf{1} \oplus \mathbf{28} \oplus \overline{\mathbf{28}} \oplus \mathbf{406} \oplus \mathbf{783} \oplus \mathbf{3276} \oplus \mathbf{3654} \oplus \mathbf{4032} \oplus \overline{\mathbf{4032}}$$

Time **C**: 12 sec. Time **Maple**: 2 sec.

4.4. Concluding remarks. For the program ***PreparePolynomialData***, the main restricting parameter is the dimension of the representation. A computer with 16 GB of RAM we use can deal with dimensions at most 100000. For example, processing the 98280-dimensional representation of the group 3.Suz requires 17.6 GB of memory, which causes the use of hard drive space, resulting in a significant slowdown of computations. The Windows 10 operating system we use can address up to 512 GB, so we can expect that if there is enough memory, the program will cope with representations of dimensions of several hundred thousand.

The main bottleneck of the ***SplitRepresentation*** program is that it is based on polynomial algebra methods, which are by nature algorithmically hard. The number of polynomial variables is equal to the rank R of the permutation representation to be splitted. In practice, the ***SplitRepresentation*** program has no difficulties with splitting representations of rank at most 17, although there are some examples with ranks 18 and 19. However, representations of finite groups often have low ranks. For example, the condition $\mathrm{R} \leq 17$ is satisfied for 761 out of 886, or 86%, permutation representations from the Atlas [7].

Systems of polynomial equations arising in the splitting algorithm using invariant projections are very special. In particular, all roots of these systems belong to Abelian extensions. It would be desirable to develop, instead of universal Gröbner bases methods, some approach that uses the specific features of polynomial systems arising in the problem in question.

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