DOI 10.1007/s10958-019-04381-z
Journal of Mathematical Sciences, Vol. 240, No. 5, August, 2019

RELATIONS BETWEEN SECOND-ORDER FUCHSIAN EQUATIONS AND FIRST-ORDER FUCHSIAN SYSTEMS

M. V. Babich* **and S. Yu. Slavyanov†**

UDC 517.289, 517.923, 517.926

Each component of any solution of a Fuchsian differential system satisfies a Fuchsian differential equation. The set of Fuchsian systems is fibered into equivalence classes. Each class consists of systems with similar sets of matrix residues, the conjugation matrix being the same for all elements of the set. We investigate the corresponding classes of scalar equations. Bibliography: 15 *titles.*

It has been discovered that the Fuchsian second-order equation with four singularities generates the Painlevé VI equation [1–3]. On the other hand, the isomonodromic property for an apropriate Fuchsian first-order system also leads to the Painlevé VI equation [4–7]. This paper reveals links between the above-mentioned approaches. These links simplify comparing the theory of Painlevé transcendents [8, 9] and the theory of Heun functions [10, 11].

1. The scalar equation and the matrix system

Any second-order differential equation

$$\psi'' + P\psi' + Q\psi = 0 \tag{1}$$

can be written as a differential system:

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}.$$

Let (1) be a Fuchsian equation. It is well known that there is a polynomial transformation

$$(\psi, \psi')^T \to g(z)(\psi, \psi')^T =: \vec{\psi}$$

such that the first-order system

$$\vec{\psi}' = A(z)\vec{\psi}, \quad A = g \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix} g^{-1} + g'g^{-1},$$

is Fuchsian. This means that $A(z)dz$ has only simple poles in $\overline{C}$.

Let us start from a system

$$\vec{\psi}' = A(z)\vec{\psi}. \tag{2}$$

To obtain a scalar equation, we exclude the second component ψ_2 of the vector $\vec{\psi} = (\psi, \psi_2)^T$ using $\vec{\psi}'' = (A' + A^2)\vec{\psi}$. The coefficients P, Q of the scalar equation (1) are

$$P = -\log' A_{12} - \mathrm{tr} A, \quad Q = \det A - A_{12} \left(\frac{A_{11}}{A_{12}} \right)'. \tag{3}$$

We consider the problem of recovering the Fuchsian system from a Heun equation, which is a Fuchsian equation with four singular points.

The roots $\rho_{1,2} = \rho_{1,2}(z_j)$ of the quadratic equation

$$\rho(\rho - 1) + \rho \,\mathrm{Res}|_{z=z_j} P + \mathrm{Res}|_{z=z_j} ((z - z_j)Q) = 0$$

*St.Petersburg Department of Steklov Institute of Mathematics and St.Petersburg State University, St.Petersburg, Russia, e-mail: mbabich@pdmi.ras.ru.

†St.Petersburg State University, St.Petersburg, Russia, e-mail: slav@ss2034.spb.edu.

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 468, 2018, pp. 221–227. Original article submitted August 14, 2018.

 1072-3374/19/2405-0646

are called the *characteristic exponents*. They are related to the eigenvalues Θ'_k, Θ_k of the residues $A^{(k)}$ of the matrix-valued function $A(z) = \sum_k \frac{A^{(k)}}{z-z_k}$:

$$\{\rho_1, \rho_2\} = \{\Theta'_k, \Theta_k + s_k + 1\},$$

where s_k is the degree of $A_{12}(z)$ at z_k:

$$A_{12}(z) \sim (z - z_k)^{s_k}.$$

A transformation of the form $\psi \to \prod_{k=1}^{4} (z - z_k)^{\alpha_k}\psi$, $\sum_k \alpha_k = 0$, is called an *S-homotopic transformation*. It shifts both the characteristic exponents and the eigenvalues:

$$\{\Theta'_k, \Theta_k\} \to \{\Theta'_k + \alpha_k, \Theta_k + \alpha_k\}.$$

Using an appropriate S-homotopic transformation, one can set $\Theta_k = 0$, $k = 1, 2, 3$. The formulas become simpler, and the matrices-residues $A^{(k)}$ for the system corresponding to the Heun equation

$$\sigma(z)\frac{d^2}{dz^2}\psi + \tau(z)\frac{d}{dz}\psi + (\alpha\beta z - \lambda)\psi = 0, \tag{4}$$

$$\sigma(z) = \prod_{j=1}^{3}(z - z_j), \quad \tau(z) = \sum_{j=1}^{3}(1 - \Theta_j)\sigma_j(z), \quad \sigma_j = \sigma(z)/(z - z_j),$$

can be written as

$$A^{(1)} = \begin{pmatrix} 0 & 0 \\ h & \Theta_1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} \frac{\alpha}{1-t} & -t \\ \frac{\alpha}{t(1-t)}(\frac{\alpha}{1-t} - \Theta_2) & \Theta_2 - \frac{\alpha}{1-t} \end{pmatrix},$$

$$A^{(3)} = \begin{pmatrix} \frac{-t\alpha}{1-t} & t \\ \frac{-\alpha}{1-t}(\frac{t\alpha}{1-t} + \Theta_3) & \Theta_3 + \frac{t\alpha}{1-t} \end{pmatrix}, \quad A^{(\infty)} = -\sum_k A^{(k)}.$$

We say that the sets $A^{(k)}$, $k = 1, 2, 3, 4$, are *equivalent* if the corresponding matrices are related by the same similarity transformation g, i.e., $\{A^{(k)}\} \sim \{g^{-1}A^{(k)}g\}$ for every k. Thus, we obtain an algebraically open set of equivalence classes.

In every class there is a unique representative such that

$$A^{(1)} = \begin{pmatrix} \Theta'_1 & \star \\ 0 & \Theta_1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} \star & -1 \\ \star & \star \end{pmatrix}, \quad A^{(3)} = \begin{pmatrix} \Theta'_3 & 0 \\ \star & \Theta_3 \end{pmatrix}.$$

All matrix elements $\star$ are uniquely determined by the matrix $A^{(\infty)} = -\sum_k A^{(k)}$, and there are no restrictions on $A^{(\infty)}$.

Consider the scalar equation for the first component of this system-representative. It is obvious that all other equations from the corresponding equivalence class form a one-dimensional submanifold in the manifold of all Heun equations. Its elements are parameterized by the direction of the eigenvector of $A^{(3)}$ correspondung to the eigenvalue Θ_3:

$$g_p = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \quad \{A^{(k)}\} \to \{g_p^{-1}A^{(k)}g_p\}.$$

The explicit formulas (3) for P and Q imply that P, Q have singularities at the zeros of A_{12}, not only at the poles z_k. These are apparent singularities of the scalar equation.

The Heun equation has exactly four singular points at z_k and has no apparent singularities. This is possible only if all zeros of $A_{12}dz$ coincide with some of z_k's.

It is easy to see that if a zero of $A_{12}dz$ coincides with z_k, then the corresponding matrix element of $(A^{(k)})_{12}$ vanishes. This means that there exists a lower triangular residue in the set.

Assume that the zeros of $A_{12}dz$ are distinct. This means that two residues, say $A^{(3)}$, $A^{(\infty)}$, are lower triangular. Hence,

$$A^{(1)} = \begin{pmatrix} \Theta_1' & 1 \\ 0 & \Theta_1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} \Theta_2' + \Theta_\Sigma & -1 \\ \Theta_\Sigma(\Theta_\Sigma + \Theta_2'') & \Theta_2 - \Theta_\Sigma \end{pmatrix},$$

$$A^{(3)} = \begin{pmatrix} \Theta_3' & 0 \\ h & \Theta_3 \end{pmatrix}, \quad A^{(\infty)} = \begin{pmatrix} \Theta_4' & 0 \\ -(\Theta_\Sigma(\Theta_\Sigma + \Theta_3'') + h) & \Theta_4 \end{pmatrix},$$

where $\Theta_j'' := \Theta_j' - \Theta_j, \Theta_\Sigma := \sum_{j=1}^{4} \Theta_j$, and $h \in \mathbb{C}$ is an accessory parameter.

In the case where both zeros of $A_{12}dz$ coincide with z_3, we obtain

$$A^{(1)} = \begin{pmatrix} \Theta_1' - pt & t \\ -p(pt - \Theta_1' + \Theta_1) & \Theta_1 + pt \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} \Theta_2' & 1-t \\ 0 & \Theta_2 \end{pmatrix},$$

$$A^{(3)} = \begin{pmatrix} \Theta_3' & 0 \\ a_{21}^{(3)} & \Theta_3 \end{pmatrix}, \quad A^{(4)} = \begin{pmatrix} -\Sigma_{11} & -1 \\ -\Sigma_{11}\Sigma_{22} + \Theta_4'\Theta_4 & -\Sigma_{22} \end{pmatrix},$$

$$\Sigma_{11} := -pt + \Theta_1' + \Theta_2' + \Theta_3', \quad \Sigma_{22} := pt + \Theta_1 + \Theta_2 + \Theta_3,$$

$$a_{21}^{(3)} = p(pt - \Theta_1' + \Theta_1) - (pt - \sum_j \Theta_j' + \Theta_4')(pt + \sum_j \Theta_j - \Theta_4) - \Theta_4'\Theta_4.$$

Here the accessory parameter is p. It is obvious that in this case

$$A_{12}(z)dz = t\frac{dz}{z} + (1-t)\frac{dz}{z-1} = \frac{(z-t)dz}{z(z-1)}$$

has poles at $z_1 = 0$, $z_2 = 1$, $z_4 = \infty$ and a zero at $z_3 = t$.

If the matrices are traceless, the formulas become simpler. In the case of distinct roots, we have

$$\psi'' + \left(\frac{1}{z} + \frac{1}{z-1}\right)\psi' + \left(\frac{h}{z(z-1)(z-t)} + \widetilde{Q}\right)\psi = 0, \quad \Theta_j + \Theta_j' = 0,$$

where

$$\begin{aligned} \widetilde{Q} = &-\frac{\Theta_3(2(\Theta_2 - \Theta_\Sigma) - 1)}{(z-1)(z-t)} - \frac{\Theta_3(2\Theta_1 - 1)}{(z-t)z} - \\ &- \frac{2\Theta_1\Theta_2 - (2\Theta_\Sigma + 1)(\Theta_1 + \Theta_2) + \Theta_\Sigma(\Theta_\Sigma + 1)}{z(z-1)} - \\ &- \frac{\Theta_1^2}{z^2} - \frac{\Theta_2^2}{(z-1)^2} - \frac{\Theta_3(\Theta_3 + 1)}{(z-t)^2}. \end{aligned}$$

2. Isomonodromic deformations, the Schlesinger system, and the Painlevé VI equation

The elements of a family of differential systems $d\Psi = A(z;t)dz\ \Psi$ have fixed monodromy if and only if there exists a 1-form $B(z,t)dt$ such that the form $Adz + Bdt$ is flat. L. Schlesinger introduced the following ansatz:

$$A(z;t)dz + B(z,t)dt = \sum_k A^{(k)} \frac{dz - dz_k}{z - z_k} =: \omega,$$

$A^{(k)} = A^{(k)}(t), z_k = z_k(t)$. It is the general form of ω in the case of generic eigenvalues of $A^{(k)}$.

The flatness condition

$$d\omega = \omega \wedge \omega$$

is equivalent to the multitime dynamical system

$$dA^{(k)} + \left[A^{(k)}, \sum_i \frac{dz_k - dz_i}{z_k - z_i}\right] = 0$$

on the Poisson space of sets $\{A^{(k)}\} \in \mathrm{gl}^M(n, \mathbb{C})$. The Hamiltonian defining the dynamics with respect to the "time" z_k is $H_k := h/dz_k$, $dz_s = 0$, $s \neq k$, where

$$h = \sum_{i,j} \operatorname{tr} A^{(i)} A^{(j)} \frac{dz_i - dz_j}{z_i - z_j}.$$

The eigenvalues of $A^{(k)}$ are determined by the monodromy. Fix them. All equations with equivalent sets $\{A^{(k)}\} \sim \{g^{-1} A^{(k)} g\}$ have the same monodromy. Again, choose a representative of the equivalence class:

$$A^{(1)} = \begin{pmatrix} \Theta_1' - pq & p \\ -q(pq - 2\Theta_1') & pq - \Theta_1' \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} pq - \Theta_\Sigma' + \Theta_2' & \Theta_\Sigma' - pq \\ pq - \Theta_\Sigma' + 2\Theta_2' & \Theta_\Sigma' - \Theta_2' - pq \end{pmatrix},$$

$$A^{(3)} = \begin{pmatrix} \Theta_3' & p(q-1) - \Theta_\Sigma' \\ 0 & -\Theta_3' \end{pmatrix}, \; A^{(4)} = \begin{pmatrix} \Theta_4' & 0 \\ q(p(q-1) - 2\Theta_1') + \Theta_\Sigma' - 2\Theta_2' & -\Theta_4' \end{pmatrix}.$$

The eigenvalues of $A^{(k)}$ are denoted by $\pm\Theta_k'$, and $\sum_{k=1}^{4} \Theta_k' =: \Theta_\Sigma'$. Calculating the Hamiltonian corresponding to the dynamics with respect to $z_3 = t$ gives

$$H = \sum_{k \neq 3} \operatorname{tr}\; A^{(3)} A^{(k)} \frac{dz_3 - dz_k}{z_3 - z_k}\Bigg|_{dz_k = 0} / dz_3 = \frac{1}{t(t-1)} \operatorname{tr} A^{(3)} \left((t-1)A^{(1)} + tA^{(4)}\right)$$

$$= \frac{1}{t(t-1)} \left(q(q-1)(q-t) \left(p^2 - p \left(\frac{\theta_1}{q} + \frac{\theta_2}{q-1} + \frac{\theta_3}{q-t} \right) \right) + 2\Theta_1' \theta_4 q \right) + \mathrm{const}_t,$$

where $\theta_1 = \Theta_\Sigma' - 2\Theta_2'$, $\theta_2 = \Theta_\Sigma' - 2\Theta_3'$, $\theta_3 = \Theta_\Sigma' - 2\Theta_4'$, $\theta_4 := \Theta_\Sigma'$ are parameters of the isomonodromic deformation, related to the set $2\Theta_1', 2\Theta_2', 2\Theta_3', 2\Theta_4'$ by the so-called Okamoto transformation [12], and const_t does not depend on p, q. This is the well-known Hamiltonian of the Painlevé VI system, the corresponding Euler–Lagrange equation being the Painlevé VI equation. The coordinate functions p, q are canonical on the symplectic space

$$\mathcal{O}^{(1)} \times \mathcal{O}^{(2)} \times \mathcal{O}^{(3)} \times \mathcal{O}^{(4)} /\!\!/ \mathrm{GL}(2),$$

which is the symplectic quotient of the product of orbits with respect to the diagonal (co)adjoint action of GL(2).

Consider the Heun equation with normalization $\psi(z) \to \prod_k (z - z_k)^{\alpha_k} \psi(z)$ such that one of the two exponents at all finite points z_1, z_2, z_3 vanish. The equation takes the form

$$\prod_{j=1}^{3} (z - z_j) \left(D^2\psi - \left(\sum_{j=1}^{3} \frac{\Theta_j - 1}{z - z_j} \right) D\psi \right) + (\alpha\beta z - \lambda)\psi = 0. \tag{5}$$

Let us compare it with the Hamiltonian of the system $P^{(\mathrm{VI})}$:

$$dp \wedge dq - d\left(\prod_{j=1}^{3} (q - z_j) \left(p^2 - \left(\sum_{j=1}^{3} \frac{\theta_j}{q - z_j} \right) p \right) + 2\Theta_1' \theta_4 q \right) \wedge \frac{dt}{t(1-t)}.$$

S. Yu. Slavyanov observed that the substitution

$$\{D = \frac{d}{dz}, z\} \to \{p, q\}$$

transforms the polynomial form of the Heun equation with three vanishing characteristic exponents into the Hamiltonian of the isomonodromic deformation equation $P^{(\mathrm{VI})}$. He called this procedure *antiquantization*. For more details, see [13,14]. This paper can be viewed as an addendum to the textbook [15].

Translated by M. V. Babich.

REFERENCES

1. R. Fuchs, "Über lineare homogene Differentialgleitungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich Singularitaeten," *Math. Ann.*, **163**, 301–321 (1907).
2. S. Slavyanov, "Painlevé equations as classical analogues of Heun equations," *J. Phys. A*, **29**, 7329–7335 (1996).
3. S. Slavyanov, "Antiquantization and the corresponding symmetries," *Theoret. Math. Phys.*, **185**, 1522–1526 (2015).
4. D. V. Anosov and A. A. Bolibruch, *The Riemann–Hilbert Problem*, Vieweg, Braunschweig (1994).
5. A. Bolibruch, *Monodromy Inverse Problems of the Analytic Theory of Differential Equations* [in Russian], MCCME, Moscow (2009).
6. A. Its and V. Novokshenov, *The Isomonodromic Deformation Method in the Theory of Painlevé Equations*, Lect. Notes Math., **1191**, Springer, Berlin–New York (1986).
7. A. Kapaev, "Lax pairs for Painlevé equations," in: *Isomonodromic Deformations and Applications in Physics*, CRM Proc. Lect. Notes, **31**, Amer. Math. Soc., Providence, Rhode Island (2002), pp. 37–48.
8. V. Gromak, I. Laine, and S. Shimomura, *Painlevé Differential Equations in the Complex Domain*, De Gruyter, Berlin (2002).
9. A. Bruno and A. Bathin (eds.), *Painleve Equations and Related Topics*, De Gruyter, Berlin–Boston (2012).
10. S. Slavyanov and W. Lay, *Special Functions: A Unified Theory Based on Singularities*, Oxford Univ. Press, Oxford–New York (2002).
11. A. Ronveau (ed.), *Heun's Differential Equation*, Oxford Univ. Press, Oxford–New York–Tokyo (1995).
12. A. Kazakov and S. Slavyanov, "Symmetries of Heun equations and Okamoto transformations for Kovalevskaya–Painlevé equations," *Theoret. Math. Phys.*, **155**, 721–732 (2008).
13. S. Slavyanov *et al.*, "Generation and removal of apparent singularities in linear ordinary differential equations with polynomial coefficients," *Theoret. Math. Phys.*, **189**, 1726–1733 (2016).
14. S. Slavyanov, "Symmetries and apparent singularities for simplest Fuchsian equations," *Theoret. Math. Phys.*, **193**, 1754–1760 (2017).
15. K. Iwasaki, H. Kimura, S. Shimomura, and M. Iosida, *From Gauss to Painlevé. A Modern Theory of Special Functions*, Vieweg, Braunschweig (1991).