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SYSTEMS WITH PARAMETERS, OR EFFICIENTLY SOLVING SYSTEMS OF POLYNOMIAL EQUATIONS 33 YEARS LATER. II

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Consider a system of polynomial equations with parametric coefficients over an arbitrary ground field. We show that the variety of parameters can be represented as a union of strata. For values of the parameters from each stratum, the solutions of the system are given by algebraic formulas depending only on this stratum. Each stratum is a quasiprojective algebraic variety with degree bounded from above by a subexponential function in the size of the input data. The number of strata is also subexponential in the size of the input data. Thus, here we avoid double exponential upper bounds on the degrees and solve a long-standing problem. Bibliography: 12 titles.

Introduction to the second part

This paper continues [8] and is the second part in a three-part series. In all parts, the numbering of theorems (respectively, lemmas, sections, and so on) is the same. It is continued from [8] and will be further continued in the third part, which will be prepared for publication. For example, in this paper the reference to Lemma 6 means Lemma 6 of [8]. Similarly, Sec. 3 means Sec. 3 from [8]. The list of references in this paper (apart from the reference to the first part [8], which is added here) coincides with that from [8]. In the present paper, we prove Theorem 1. In the last third part, we will prove Theorem 2.

In this second part, we use the construction for solving systems of polynomial equations with finitely many solutions in the projective space described in Sec. 3. Actually, to obtain an algorithm in the general case, we need only the part of this construction until Remark 6 (in particular, we will not use the tree T_0 introduced in Sec. 3). Of course, many other ideas are needed to prove Theorem 1 in full generality.

To prove assertion (d) of Theorem 1, we need to use estimates on the lengths of coefficients of absolutely irreducible factors of parametric polynomials with integer coefficients. Unfortunately, we forgot to give such estimates in the statement of Theorem 1 in [6]. But they are straightforward. Namely, under the conditions of Theorem 1 in [6], the following assertion holds.

(c) Assume that $k = \mathbb{Q}$ and $f \in \mathbb{Z}[a_1, \ldots, a_\nu, X_1, \ldots, X_n]$, and let the lengths of integer coefficients satisfy $\mathrm{l}(f) \leq M$ for a real number $M > 0$. Then all the polynomials $\psi^{(\beta)}_{\alpha,1}, \ldots, \psi^{(\beta)}_{\alpha,m_{\alpha,\beta}}$, $\lambda_{\alpha,0}, \lambda_{\alpha,1}$, H_j, F_j, f_j have integer coefficients with lengths bounded from above by $(M + n + \nu \log d')d^{O(1)}$ with an absolute constant in $O(1)$.

The proof of this assertion immediately follows from the construction described in [6] (we leave the details to the reader; perhaps, minor modifications are required in [6] to obtain (c)).

For better understanding, note that in many cases, to obtain a bound on the lengths of integer coefficients $\mathrm{l}(\Psi)$ of a polynomial Ψ, we estimate the logarithm of the sum of the absolute values of the coefficients of this polynomial. This gives a fortiori a required bound on $\mathrm{l}(\Psi)$.

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For example, assume that we have an $N \times N$ matrix $(\psi_{i,j})_{1\le i,j\le N}$ with

$$\psi_{i,j} \in \mathbb{Z}[a_1, \dots, a_\nu, X_1, \dots, X_n],$$

$\mathrm{l}(\psi_{i,j}) < M$, and $\deg_{a_1,\dots,a_n} \psi_{i,j} < d'$, $\deg_{X_1,\dots,X_n} \psi_{i,j} < d$ for all i, j. We would like to estimate the lengths of integer coefficients of the determinant $\det((\psi_{i,j})_{1\le i,j\le N})$. Then we proceed as follows. The sum of the absolute values of the integer coefficients of each polynomial $\psi_{i,j}$ is bounded from above by $2^M d^n (d')^\nu$. Hence, the sum of the absolute values of the integer coefficients of the determinant is bounded from above by $N! 2^{MN} d^{nN} (d')^{\nu N}$. Its logarithm is bounded from above by $(M + n\log d + \nu \log d')N^{O(1)}$. Hence, the same upper bound holds for the lengths of integer coefficients of this determinant.

Also, to prove assertion (d) of Theorem 1, we need a remark on Lemma 2 from Sec. 1. Namely,

(†) Under the conditions of Lemma 2, assume that $\mathrm{char}(k) = 0$. Then one can choose matrices $B_i \in M_{r,n}(\mathbb{Z})$ and $C_j \in M_{m,r}(\mathbb{Z})$ such that all entries $b_{i,\alpha,\beta}$ of the matrices B_i have lengths bounded from above by $n^{O(1)}$, and all entries $c_{j,\alpha,\beta}$ of the matrices C_j have lengths bounded from above by $m^{O(1)}$, with an absolute constant in $O(1)$.

This immediately follows from the proof of this lemma and [9]. More precisely, the construction in [9] is explicit. By [9], the matrices D_j from the proof of Lemma 2 can be chosen with integer coefficients of lengths bounded from above by $m^{O(1)}$ with an absolute constant in $O(1)$. Therefore, all entries $c_{j,\alpha,\beta}$ of the matrices C_j have lengths bounded from above by $m^{O(1)}$. Similarly, all entries $b_{i,\alpha,\beta}$ of the matrices B_i have lengths bounded from above by $n^{O(1)}$.

Note also that we implicitly used these assertions (c) and (†) in [8] to prove assertion (d) for the weakened (and weakened modified) Theorem 1 with $c = 0$. We would like to emphasize again that in this second part we will not need these results for the particular case $c = 0$. In [8], we proved them only to demonstrate the strength of the developed techniques.

At the end of this short introduction, we would like to correct the misprints noticed in [8].

1. In the formulation of condition (x) from the introduction, one must everywhere replace max by $\max_i$.
2. In Sec. 4, in the formulation of property (α_{n-c}), one must replace $d_w - d_{i-1}$ by $d_{i-1} - d_w$.
3. In Sec. 4, in formula (27), one must replace

$$\widetilde{h}_{a^*,j} = \sum_{j\le w\le m-1} q_{j,w} f_{a^*,w}$$

by

$$\widetilde{h}_{a^*,j} = f_{a^*,j-1} + \sum_{j\le w\le m-1} q_{j,w} f_{a^*,w}.$$

These misprints can easily be corrected from the context.

Now we are ready to proceed to the next section of the paper.

5. Several lemmas

In this section, c is an integer such that $-1 \le c \le n-1$. Let q be an integer such that $0 \le q \le c$ (so q exists if $c \ge 0$). Consider polynomials $h_{a^*,1}, \dots, h_{a^*,n-q} \in k_{a^*}[X_0, \dots, X_n]$ satisfying condition (α_{n-q}), see Sec. 4, and the following property:

(γ'_{n-q}) $\mathcal{Z}(h_{a^*,1}, \dots, h_{a^*,n-q}) = V'''_{a^*,q} \cup \bigcup_{q\le s\le c} V_{a^*,s}$, where the varieties $V_{a^*,s}$ are defined in the introduction and $V'''_{a^*,q}$ is some projective algebraic variety such that $\dim V'''_{a^*,q} = q$ or $V'''_{a^*,q} = \varnothing$.

Hence, each irreducible component of the algebraic variety $V'''_{a^*,q}$ is of dimension q.

Let ε be a new variable. Let $\mathbb{A}^1(\overline{k})$ have coordinate ε and $\mathbb{P}^n(\overline{k}) \times \mathbb{A}^1(\overline{k})$ have coordinates $((X_0 : \ldots : X_n), \varepsilon)$. We identify $\mathbb{P}^n(\overline{k})$ with the subvariety $\mathcal{Z}(\varepsilon) \subset \mathbb{P}^n(\overline{k}) \times \mathbb{A}^1(\overline{k})$. Consider the algebraic variety

$$\mathcal{Z}\Big(h_{a^*,1} - \varepsilon X_0^{d_0}, \ldots, h_{a^*,n-q} - \varepsilon X_{n-q-1}^{d_{n-q-1}}\Big). \tag{28}$$

It is a closed subvariety of $\mathbb{P}^n(\overline{k}) \times \mathbb{A}^1(\overline{k})$. Denote by $\overline{V}_{a^*,q}$ the union of all components E irreducible over $\overline{k}$ of the algebraic variety (28) such that E is not contained in any hyperplane $\mathcal{Z}(\varepsilon - c)$, $c \in \overline{k}$. Put $V''_{a^*,q} = \overline{V}_{a^*,q} \cap \mathcal{Z}(\varepsilon) \subset \mathbb{P}^n(\overline{k})$. One can easily prove (cf. [2]) that every irreducible component of $V''_{a^*,q}$ is of dimension $n-q$ and $V''_{a^*,q} \supset V_{a^*,q} \cup V'''_{a^*,q}$. Denote by $V'_{a^*,q}$ the union of all components E of V''_{a^*} irreducible over $\overline{k}$ such that $E \subset \mathcal{Z}(f_{a^*,0}, \ldots, f_{a^*,m-1})$. Thus, $V''_{a^*,q} \supset V'_{a^*,q} \supset V_{a^*,q}$. Moreover, obviously, the closure of $V'_{a^*,q} \setminus \Big(\bigcup_{q+1 \le s \le c} V'_{a^*,s}\Big)$ with respect to the Zariski topology in $\mathbb{P}^n(\overline{k})$ coincides with $V_{a^*,q}$. Denote by $V''''_{a^*,q}$ the union of all components E irreducible over $\overline{k}$ of the algebraic variety $V'_{a^*,q}$ such that $E \not\subset V_{a^*,q}$. So, we have $V'_{a^*,q} = V_{a^*,q} \cup V''''_{a^*,q}$ and $V''_{a^*,q} = V_{a^*,q} \cup V'''_{a^*,q} \cup V''''_{a^*,q}$, and each irreducible component of the algebraic variety $V''_{a^*,q}$ is an irreducible component of only one of the varieties $V_{a^*,q}, V'''_{a^*,q}, V''''_{a^*,q}$.

In what follows, $s = q$. In this section, $Y_0, \ldots, Y_n \in k[X_0, \ldots, X_n]$ are arbitrary linear forms in $X_0, \ldots, X_n$ linearly independent over k (we do not assume now that necessarily $(Y_0, \ldots, Y_{s+1}) \in \mathcal{L}_s^{s+1} \times \mathcal{L}'_s$).

Let us extend the ground field k to $k(\varepsilon)$. Denote by $\widetilde{V}_{a^*,s}$ the algebraic variety (28) regarded as a subvariety in $\mathbb{P}^n(\overline{k(\varepsilon)})$.

Let $t_1, \ldots, t_s$ be elements algebraically independent over k. Let $k^\circ = k(t_1, \ldots, t_s)$, $k^\circ_{a^*} = k_{a^*}(t_1, \ldots, t_s)$, and $\overline{k}^\circ = \overline{k}(t_1, \ldots, t_s)$ be purely transcendental extensions of the fields k, k_{a^*}, and $\overline{k}$, respectively.

Let $Y_i = \sum_{0 \le j \le n} y_{i,j} X_j$, $0 \le i \le n$, where $y_{i,j} \in \overline{k}$. Put

$$\delta^{(0)} = \det((y_{i,j})_{0 \le i,j \le n}). \tag{29}$$

Then $\delta^{(0)} \neq 0$. For an arbitrary polynomial $g \in \overline{k(\varepsilon)}[X_0, \ldots, X_n]$, we define polynomials $g^{(0)} \in \overline{k(\varepsilon)}[Y_0, \ldots, Y_n]$ and $g^\circ \in \overline{k(\varepsilon)}[Y_0, Y_{s+1}, \ldots, Y_n]$. Namely,

$$g^{(0)}(Y_0, \ldots, Y_n) = (\delta^{(0)})^{\deg_{X_0,\ldots,X_n} g} g$$

and

$$g^\circ = g^{(0)}(Y_0, t_1 Y_0, \ldots, t_s Y_0, Y_{s+1}, \ldots, Y_n).$$

Hence, if g is a homogeneous polynomial in $X_0, \ldots, X_n$, then g° is a homogeneous polynomial in $Y_0, Y_{s+1}, \ldots, Y_n$ with coefficients in $\overline{k(\varepsilon)}(t_1, \ldots, t_s)$. If $g \in \overline{k}[\varepsilon, X_0, \ldots, X_n]$, then, obviously, $g^\circ \in \overline{k}^\circ[\varepsilon, X_0, \ldots, X_n]$.

Let the projective algebraic variety $\mathbb{P}^{n-s}(\overline{k^\circ(\varepsilon)})$ (and also $\mathbb{P}^{n-s}(\overline{k^\circ})$) have homogeneous coordinates $Y_0, Y_{s+1}, \ldots, Y_n$.

Let $\varphi_1, \ldots, \varphi_{m_1} \in k_{a^*}[\varepsilon, X_0, \ldots, X_n]$ be homogeneous polynomials in $X_0, \ldots, X_n$ such that $\overline{V}_{a^*,s} = \mathcal{Z}(\varphi_1, \ldots, \varphi_{m_1})$. Set

$$\mathrm{con}(\overline{V}_{a^*,s}) = \mathcal{Z}(\varphi_1, \ldots, \varphi_{m_1}) \subset \mathbb{A}^{n+1}(\overline{k}) \times \mathbb{A}^1(\overline{k})$$

where $\mathbb{A}^{n+1}(\overline{k})$ has coordinates $X_0, \ldots, X_n$ and $\mathbb{A}^1(\overline{k})$ has coordinate ε. Obviously, the irreducible components of the varieties $V_{a^*,s}$ and $\mathrm{con}(V_{a^*,s})$ are in a natural one-to-one correspondence.

Set $\overline{\varphi}_i = \varphi_i(0, X_0, \dots, X_n)$, $1 \le i \le m_1$. Then, obviously, $V''_{a^*,s} = \mathcal{Z}(\overline{\varphi}_1, \dots, \overline{\varphi}_{m_1})$. Put

$$\overline{C}_{a^*,s} = \mathcal{Z}(\varphi_1^\circ, \dots, \varphi_{m_1}^\circ) \subset \mathbb{P}^{n-s}(\overline{k^\circ}) \times \mathbb{A}^1(\overline{k^\circ}),$$
$$\widetilde{C}_{a^*,s} = \mathcal{Z}(\varphi_1^\circ, \dots, \varphi_{m_1}^\circ) \subset \mathbb{P}^{n-s}(\overline{k^\circ(\varepsilon)}),$$
$$C''_{a^*,s} = \overline{C}_{a^*,s} \cap \mathcal{Z}(\varepsilon) = \mathcal{Z}(\overline{\varphi}_1^\circ, \dots, \overline{\varphi}_{m_1}^\circ) \subset \mathbb{P}^{n-s}(\overline{k^\circ}).$$

Similarly to $\mathrm{con}(\overline{V}_{a^*,s})$, we define the varieties

$$\mathrm{con}(\overline{C}_{a^*,s}) = \mathcal{Z}(\varphi_1^\circ, \dots, \varphi_{m_1}^\circ) \subset \mathbb{A}^{n-s+1}(\overline{k^\circ}) \times \mathbb{A}^1(\overline{k^\circ}),$$
$$\mathrm{con}(\widetilde{C}_{a^*,s}) = \mathcal{Z}(\varphi_1^\circ, \dots, \varphi_{m_1}^\circ) \subset \mathbb{A}^{n-s+1}(\overline{k^\circ(\varepsilon)}),$$
$$\mathrm{con}(C''_{a^*,s}) = \mathcal{Z}(\overline{\varphi}_1^\circ, \dots, \overline{\varphi}_{m_1}^\circ) \subset \mathbb{A}^{n-s+1}(\overline{k^\circ}).$$

In all these formulas, the affine space $\mathbb{A}^{n-s+1}$ has coordinates $Y_0, Y_{s+1}, \dots, Y_n$.

Note that $\widetilde{V}_{a^*,s} = \mathcal{Z}(\varphi_1, \dots, \varphi_{m_1}) \subset \mathbb{P}^n(\overline{k(\varepsilon)})$ and $\widetilde{C}_{a^*,s} = \mathcal{Z}(h^\circ_{a^*,1}, \dots, h^\circ_{a^*,n-s}) \subset \mathbb{P}^{n-s}(\overline{k^\circ(\varepsilon)})$.

Lemma 9. *The following assertions hold true.*

(a) *The intersection $\widetilde{V}_{a^*,s} \cap \mathcal{Z}(Y_0, Y_1, \dots, Y_s)$ is empty in $\mathbb{P}^n(\overline{k(\varepsilon)})$.*

(b) *$\widetilde{C}_{a^*,s}$ is a finite subset in $\mathbb{P}^{n-s}(\overline{k^\circ(\varepsilon)})$, and $\widetilde{C}_{a^*,s} \cap \mathcal{Z}(Y_0) = \varnothing$.*

(c) *The intersection $V''_{a^*,s} \cap \mathcal{Z}(Y_0, Y_1, \dots, Y_s)$ is empty in $\mathbb{P}^n(\overline{k})$ if and only if $C''_{a^*,s}$ is a finite subset in $\mathbb{P}^{n-s}(\overline{k^\circ})$ and $C''_{a^*,s} \cap \mathcal{Z}(Y_0) = \varnothing$.*

Proof. This is straightforward. For example, assertion (a) essentially follows from the fact that $\mathcal{Z}(X_0, X_1, \dots, X_{n-s-1}, Y_0, \dots, Y_s) = \varnothing$ in $\mathbb{P}^n(\overline{k})$. Assertion (b) is equivalent to (a), and hence (b) is also true. The lemma is proved. □

Denote by $\mathcal{E}_1$ the set of all components of the algebraic variety $\overline{V}_{a^*,s}$ irreducible over $\overline{k}$.

Denote by $\mathcal{E}_5$ set of all components of the algebraic variety $V''_{a^*,s}$ irreducible over $\overline{k}$.

For any affine algebraic variety W defined over a field K, we will denote by $K[W]$ the ring of regular functions on W defined over K, and by $K(W)$ the total quotient ring of the ring $K[W]$.

The following assertions (I)–(IV) are straightforward. Their detailed proofs are left to the reader.

(I) Denote by $\mathcal{E}_2$ the set of all components of the algebraic variety $\widetilde{V}_{a^*,s}$ defined and irreducible over $\overline{k}(\varepsilon)$. Take S_2 to be the multiplicatively closed set $\overline{k}[\varepsilon] \setminus \{0\}$. Then, by the definition of the algebraic variety $\overline{V}_{a^*,s}$, the variety $\mathrm{con}(\widetilde{V}_{a^*,s})$ is defined over the field $\overline{k}(\varepsilon)$, and one can identify the ring of regular functions $\overline{k}(\varepsilon)[\mathrm{con}(\widetilde{V}_{a^*,s})]$ with $S_2^{-1}\overline{k}[\mathrm{con}(\overline{V}_{a^*,s})]$ (it is the localization of the ring $\overline{k}[\mathrm{con}(\overline{V}_{a^*,s})]$ with respect to the multiplicatively closed set S_2). Hence, the total quotient ring $\overline{k}(\varepsilon)(\mathrm{con}(\widetilde{V}_{a^*,s}))$ coincides with $\overline{k}(\mathrm{con}(\overline{V}_{a^*,s}))$. Therefore, there is a natural bijection $\iota_{1,2} : \mathcal{E}_1 \to \mathcal{E}_2$, and $\widetilde{V}_{a^*,s} = \bigcup_{W \in \mathcal{E}_2} W$.

(II) Denote by $\mathcal{E}_3$ the set of all components of the algebraic variety $\overline{C}_{a^*,s}$ defined and irreducible over $\overline{k}^\circ$. By Lemma 9(a), one can make the identification

$$t_i = Y_i/Y_0 \in \overline{k}(\mathrm{con}(\overline{V}_{a^*,s})), \quad 1 \le i \le s, \tag{30}$$

i.e., the functions Y_i/Y_0, $1 \le i \le s$, from the ring $\overline{k}(\mathrm{con}(\overline{V}_{a^*,s}))$ are algebraically independent over $\overline{k}$. Take S_3 to be the multiplicatively closed set $\overline{k}[t_1, \dots, t_s] \setminus \{0\}$. Then the variety $\overline{C}_{a^*,s}$ is defined over the field $\overline{k}^\circ$, and one can identify the ring of regular functions $\overline{k}^\circ[\mathrm{con}(\overline{C}_{a^*,s})]$ with $S_3^{-1}\overline{k}[\mathrm{con}(\overline{V}_{a^*,s})][t_1, \dots, t_s]$. Hence, the total quotient ring $\overline{k}^\circ(\mathrm{con}(\overline{C}_{a^*,s}))$ coincides with $\overline{k}(\mathrm{con}(\overline{V}_{a^*,s}))$. Therefore, there is a natural bijection $\iota_{1,3} : \mathcal{E}_1 \to \mathcal{E}_3$, and $\overline{C}_{a^*,s} = \bigcup_{W \in \mathcal{E}_3} W$.

(III) Denote by $\mathcal{E}_4$ the set of all components of the algebraic variety $\widetilde{C}_{a^*,s}$ defined and irreducible over $\overline{k}^\circ(\varepsilon)$. Take S_4 to be the multiplicatively closed set $\overline{k}[\varepsilon, t_1, \dots, t_s] \setminus \{0\}$. Then the variety $\widetilde{C}_{a^*,s}$ is defined over the field $\overline{k}^{\circ}(\varepsilon)$, and, taking into account (30), one can identify the ring of regular functions $\overline{k}^{\circ}(\varepsilon)[\mathrm{con}(\widetilde{C}_{a^*,s})]$ with $S_4^{-1}\overline{k}[\mathrm{con}(\overline{V}_{a^*,s})][t_1,\dots,t_s]$. Hence, the total quotient ring $\overline{k}^{\circ}(\varepsilon)(\mathrm{con}(\widetilde{C}_{a^*,s}))$ coincides with $\overline{k}(\mathrm{con}(\overline{V}_{a^*,s}))$. Therefore, there is a natural bijection $\iota_{1,4} : \mathcal{E}_1 \to \mathcal{E}_4$, and $\widetilde{C}_{a^*,s} = \bigcup_{W\in\mathcal{E}_4} W$.

(IV) Denote by $\mathcal{E}_6$ the set of all components of the algebraic variety $C''_{a^*,s}$ defined and irreducible over $\overline{k}^{\circ}$. Assume that $V''_{a^*,s} \cap \mathcal{Z}(Y_0,\dots,Y_s) = \varnothing$ in $\mathbb{P}^n(\overline{k}) \times \mathbb{A}^1(\overline{k})$. Now, by Lemma 9(c), one can make the identification

$$t_i = Y_i/Y_0 \in \overline{k}(\mathrm{con}(V''_{a^*,s})), \quad 1 \le i \le s. \tag{31}$$

Take S_6 to be the multiplicatively closed set $\overline{k}[t_1,\dots,t_s] \setminus \{0\}$. Then the variety $C''_{a^*,s}$ is defined over the field $\overline{k}^{\circ}$, and one can identify the ring of regular functions $\overline{k}^{\circ}[\mathrm{con}(C''_{a^*,s})]$ with $S_6^{-1}\overline{k}[\mathrm{con}(V''_{a^*,s})][t_1,\dots,t_s]$. Hence, the total quotient ring $\overline{k}^{\circ}(\mathrm{con}(C''_{a^*,s}))$ coincides with $\overline{k}(\mathrm{con}(\overline{V}_{a^*,s}))$. Therefore, there is a natural bijection $\iota_{5,6} : \mathcal{E}_5 \to \mathcal{E}_6$, and $C''_{a^*,s} = \bigcup_{W\in\mathcal{E}_6} W$.

Let $K \supset \overline{k}$ be an extension of the ground field linearly disjoint with $\overline{k}(\varepsilon)$ over $\overline{k}$. Put $K^{\circ} = K(t_1,\dots,t_s)$. Let $L \in K[X_0,\dots,X_n]$ be a linear form with coefficients from K. Hence $L^{\circ} \in K^{\circ}[Y_0,Y_{s+1},\dots,Y_n]$.

Let $W \in \mathcal{E}_4$. Then the component W is defined and irreducible over the field $K^{\circ}(\varepsilon)$ by (III). Therefore, for every $W \in \mathcal{E}_4$ there is a polynomial $\Phi_W \in K[\varepsilon,t_1,\dots,t_s,Y_0,Z]$ irreducible over K such that the polynomial $\Phi_W(\varepsilon,t_1,\dots,t_s,Y_0,L^{\circ})$ vanishes on W. It is uniquely defined up to a nonzero factor from K. The polynomial Φ_W is homogeneous in Y_0,Z, and, by Lemma 9(b), we have $\mathrm{lc}_Z\Phi_W \in K[t_1,\dots,t_s,\varepsilon]$ and $\deg_Z \Phi_W \ge 1$. Let $\eta \in W$. Then, obviously, $\Phi_W(\varepsilon,t_1,\dots,t_s,1,Z)$ is a minimal polynomial over the field $K^{\circ}(\varepsilon)$ of the element $(L^{\circ}/Y_0)(\eta)$. Conversely, if $\Psi_W \in K[\varepsilon,t_1,\dots,t_s,Z]$ is a minimal polynomial of the element $(L^{\circ}/Y_0)(\eta)$ over the field $K^{\circ}(\varepsilon)$, then its homogenization $Y_0^{\deg_Z \Psi_W}\Psi_W(\varepsilon,t_1,\dots,t_s,Z/Y_0)$ coincides with Φ_W up to a factor from $K^{\circ}(\varepsilon)$. Put $\Phi_W^{\vee} = \Phi_W(\varepsilon,Y_1/Y_0,\dots,Y_s/Y_0,Y_0,Z)$. We will write $\Phi_W = \Phi_{W,L}$, $\Phi_W^{\vee} = \Phi_{W,L}^{\vee}$, $\Psi_W = \Psi_{W,L}$ when the dependence on L is important.

Assume that $V''_{a^*,s} \cap \mathcal{Z}(Y_0,\dots,Y_s) = \varnothing$ in $\mathbb{P}^n(\overline{k})$. Let $W \in \mathcal{E}_6$. Then the component W is defined and irreducible over the field K° by (IV). Therefore, for every $W \in \mathcal{E}_6$ there is a polynomial $\Phi_W \in K[t_1,\dots,t_s,Y_0,Z]$ irreducible over K such that the polynomial $\Phi_W(t_1,\dots,t_s,Y_0,L^{\circ})$ vanishes on W. It is uniquely defined up to a nonzero factor from K. The polynomial Φ_W is homogeneous in Y_0,Z, and, by Lemma 9(c), we have $\mathrm{lc}_Z\Phi_W \in K[t_1,\dots,t_s]$ and $\deg_Z \Phi_W \ge 1$. Let $\eta \in W$. Then, obviously, $\Phi_W(t_1,\dots,t_s,1,Z)$ is a minimal polynomial over the field K° of the element $(L^{\circ}/Y_0)(\eta)$. Conversely, if $\Psi_W \in K[t_1,\dots,t_s,Z]$ is a minimal polynomial of the element $(L^{\circ}/Y_0)(\eta)$ over the field K°, then its homogenization $Y_0^{\deg_Z \Psi_W}\Psi_W(t_1,\dots,t_s,Z/Y_0)$ coincides with Φ_W up to a factor from K°. Put $\Phi_W^{\vee} = \Phi_W(Y_1/Y_0,\dots,Y_s/Y_0,Y_0,Z)$. We will write $\Phi_W = \Phi_{W,L}$, $\Phi_W^{\vee} = \Phi_{W,L}^{\vee}$, $\Psi_W = \Psi_{W,L}$ when the dependence on L is important.

Lemma 10. (a) *Let $W \in \mathcal{E}_4$, $\eta \in W$, and $\iota_{1,4}(W') = W$, see* (III)*. Then, in the above notation,* $\mathrm{lc}_Z\Phi_W \in K[\varepsilon]$. *Hence, the element $(L/Y_0)(\eta)$ is integral over the ring $K(\varepsilon)[t_1,\dots,t_s]$. Furthermore, we have $\Phi_W^{\vee} \in K[\varepsilon,Y_0,\dots,Y_s,Z]$. The polynomial $\Phi_W^{\vee}$ is irreducible in the ring $K[\varepsilon,Y_0,\dots,Y_s,Z]$, and $\Phi_W^{\vee}(\varepsilon,Y_0,\dots,Y_s,L)$ vanishes on the variety W'. Besides,*

$$\mathrm{lc}_Z\Phi_W^{\vee} \in K[\varepsilon].$$

(b) *Assume that $V''_{a^*,s} \cap \mathcal{Z}(Y_0,\dots,Y_s) = \varnothing$ in $\mathbb{P}^n(\overline{k})$. Let $W \in \mathcal{E}_6$, $\eta \in W$, and $\iota_{5,6}(W') = W$, see* (IV)*. Then, in the above notation,* $\mathrm{lc}_Z\Phi_W \in K$. *Hence, the element $(L/Y_0)(\eta)$ is integral over the ring $K[t_1,\dots,t_s]$. Furthermore, the polynomial $\Phi_W^{\vee} \in K[Y_0,\dots,Y_s,Z]$ is irreducible (in this ring), $\Phi_W^{\vee}(Y_0,\dots,Y_s,L)$ vanishes on the variety W', and $\mathrm{lc}_Z\Phi_W^{\vee} \in K$.*

Proof. (a) Indeed, one can represent $\Phi_W^\vee$ in the form $\Phi_W^\vee = Q/Y_0^e$ where $Q \in K[\varepsilon, Y_0, \dots, Y_s, Z]$ is an irreducible polynomial, $Q \neq \lambda Y_0$ for $\lambda \in K$, and e is a nonnegative integer. The polynomial $Q(\varepsilon, Y_0, \dots, Y_s, L)$ vanishes on W' by (III). If $e \geq 1$ or $\mathrm{lc}_Z \Phi_W \notin K[\varepsilon]$, then $\mathrm{lc}_Z Q \notin K[\varepsilon]$. This contradicts Lemma 9(a) and proves the required claim (a).

The proof of claim (b) is similar to that of claim (a) and is left to the reader. □

Corollary 1. *Assume that $K = \overline{k}(t)$ where t is a transcendental element over the field $\overline{k}^\circ(\varepsilon)$. Let $L = L_1 + tL_2$ where $L_1, L_2 \in \overline{k}[X_0, \dots, X_n]$ are linear forms.*

(a) *Under the conditions of Lemma* 10(a)*, one can choose $\Phi_{W,L} \in \overline{k}[t, \varepsilon, t_1, \dots, t_s, Y_0, Z]$ such that $\Phi_{W,L}$ is irreducible in the last ring and $\mathrm{lc}_Z \Phi_{W,L} \in k[\varepsilon]$. For such a choice of $\Phi_{W,L}$, we have $\Phi_{W,L}|_{t=0} = \lambda \Phi_{W,L_1}^e$ where $0 \neq \lambda \in \overline{k}[\varepsilon]$ and $e = e_{W,L_1,L_2} \geq 1$. Hence, $\Phi_{W,L}|_{t=0}/\mathrm{lc}_Z(\Phi_{W,L}) = (\Phi_{W,L_1}/\mathrm{lc}_Z(\Phi_{W,L_1}))^e$.*

(b) *Furthermore, under the conditions of Lemma* 10(b)*, one can choose a polynomial $\Phi_{W,L} \in \overline{k}[t, t_1, \dots, t_s, Y_0, Z]$ such that $\Phi_{W,L}$ is irreducible in the last ring and $\mathrm{lc}_Z \Phi_{W,L} \in \overline{k}$. For such a choice of $\Phi_{W,L}$, we have $\Phi_{W,L}|_{t=0} = \lambda \Phi_{W,L_1}^e$ where $0 \neq \lambda \in \overline{k}$ and $e = e_{W,L_1,L_2} \geq 1$. Hence,*

$$\Phi_{W,L}|_{t=0}/\mathrm{lc}_Z(\Phi_{W,L}) = (\Phi_{W,L_1}/\mathrm{lc}_Z(\Phi_{W,L_1}))^e.$$

Proof. (a) By Lemma 10(a), the elements $(L_i/Y_0)(\eta)$, $i = 1, 2$, are integral over the ring $\overline{k}(\varepsilon)[t_1, \dots, t_s]$. Hence, the element $((L_1 + tL_2)/Y_0)(\eta)$ is integral over the ring $\overline{k}(\varepsilon)[t, t_1, \dots, t_s]$. Therefore, one can choose $\Psi_{W,L} \in \overline{k}[\varepsilon, t, t_1, \dots, t_s, Z]$ such that $\mathrm{lc}_Z \Psi_{W,L} \in \overline{k}[\varepsilon]$. We take $\Phi_{W,L}$ to be the homogenization of $\Psi_{W,L}$, see above. Then, obviously, $\mathrm{lc}_Z \Phi_{W,L} \in \overline{k}[\varepsilon]$ and $\Psi_{W,L}$ is irreducible in the ring $\overline{k}[\varepsilon, t, t_1, \dots, t_s, Y_0, Z]$.

Each root of the polynomial $\Psi_{W,L}$ has the form $Z = ((L_1 + tL_2)/Y_0)(\eta^{(1)})$ where $\eta^{(1)} \in W$. Therefore, each root of the polynomial $\Psi_{W,L}|_{t=0}$ has the form $Z = (L_1/Y_0)(\eta^{(1)})$ where $\eta_1 \in W$. The polynomial Ψ_{W,L_1} is irreducible in the ring $\overline{k}[\varepsilon, t_1, \dots, t_s, Z]$. Hence, $\Psi_{W,L}|_{t=0} = \lambda \Psi_{W,L_1}^e$ where $0 \neq \lambda \in \overline{k}[\varepsilon]$ and $e \geq 1$. It remains to take the homogenization of the last equality. Claim (a) is proved.

The proof of claim (b) is similar to that of claim (a) and is left to the reader. □

Remark 9. In what follows, for $W \in \mathcal{E}_4$ and $L = L^{(1)} + tL^{(2)}$ (for arbitrary linear forms $L^{(1)}, L^{(2)} \in \overline{k}[X_0, \dots, X_n]$), using the notation $\Phi_{W,L}$, we will always assume that the polynomial $\Phi_{W,L} \in \overline{k}[\varepsilon, t, t_1, \dots, t_s, Y_0, Z]$ is irreducible in this ring.

Assume that $V''_{a^*,s} \cap \mathcal{Z}(Y_0, \dots, Y_s) = \varnothing$ in $\mathbb{P}^n(\overline{k})$. Then, analogously, for $W \in \mathcal{E}_6$ and $L = L^{(1)} + tL^{(2)}$, using the notation $\Phi_{W,L}$, we will always assume that the polynomial $\Phi_{W,L} \in \overline{k}[t, t_1, \dots, t_s, Y_0, Z]$ is irreducible in this ring.

Let $L_1 \in k[X_0, \dots, X_n]$ be a linear form. Let $W \in \mathcal{E}_4$ and $\iota_{1,4}(W') = W$, $\iota_{1,3}(W') = W''$. Let $\psi_1, \dots, \psi_{m_2} \in \overline{k}[\varepsilon, X_0, \dots, X_n]$ be polynomials such that $W' = \mathcal{Z}(\psi_1, \dots, \psi_{m_2})$. Put $\overline{\psi}_i = \psi_i(0, X_0, \dots, X_n)$, $1 \leq i \leq m_2$. Hence $W = \mathcal{Z}(\psi_1^\circ, \dots, \psi_{m_2}^\circ)$ and $W'' \cap \mathcal{Z}(\varepsilon) = \mathcal{Z}(\overline{\psi}_1^\circ, \dots, \overline{\psi}_{m_2}^\circ)$. Put

$$\Delta_1 = \Delta_{k_{a^*}^\circ(\varepsilon); Y_0, Y_{s+1}, \dots, Y_n; \psi_1^\circ, \dots, \psi_{m_2}^\circ; Y_0, L_1^\circ,} \in k_{a^*}^\circ(\varepsilon)[U_0, U_1],$$
$$\Delta_2 = \Delta_{k_{a^*}^\circ; Y_0, Y_{s+1}, \dots, Y_n; \overline{\psi}_1^\circ, \dots, \overline{\psi}_{m_2}^\circ; Y_0, L_1^\circ,} \in k_{a^*}^\circ[U_0, U_1],$$
$$\Delta_3 = \Delta_{k_{a^*}^\circ(\varepsilon); Y_0, Y_{s+1}, \dots, Y_n; h_{a^*,1}^\circ, \dots, h_{a^*,s}^\circ; Y_0, L_1^\circ} \in k_{a^*}^\circ(\varepsilon)[U_0, U_1],$$

see the notation in Remark 6 from Sec. 3.

Lemma 11. (a)

$$\Delta_3(Z, -Y_0) = \prod_{W \in \mathcal{E}_4} (\Phi_{W,L_1}/\mathrm{lc}_Z(\Phi_{W,L_1}))^{e'_{W,L_1}}$$

for some integers $e'_{W,L_1} \geq 1$.

(b) *Assume that* $V''_{a^*,s} \cap \mathcal{Z}(Y_0,\dots,Y_s) = \varnothing$ *in* $\mathbb{P}^n(\overline{k})$ (*see also Lemma* 9(c)). *Then*

$$\Delta_2(Z,-Y_0) = \prod_{\substack{W_1\in\mathcal{E}_6,\\ W_1\subset W''}} (\Phi_{W_1,L_1}/\mathrm{lc}_Z(\Phi_{W_1,L_1}))^{e_{W,W_1,L_1}}$$

for some integers $e_{W,W_1,L_1} \geq 1$.

(c) $\Delta_1(Z,-Y_0) = (\Phi_{W,L_1}/\mathrm{lc}_Z(\Phi_{W,L_1}))^{e''_{W,L_1}}$ *for an integer* $e''_{W,L_1} \geq 1$.

Proof. (a) Let $L_2 \in k[X_0,\dots,X_n]$ be a linear form such that $(L_2/Y_0)(\eta_1) \neq (L_2/Y_0)(\eta_2)$ for all pairwise distinct $\eta_1,\eta_2 \in \widetilde{C}_{a^*,s}$. Put $L = L_1 + tL_2$ and

$$\Delta_4 = \Delta_{k^\circ_{a^*}(\varepsilon,t);Y_0,Y_{s+1},\dots,Y_n;h^\circ_{a^*,1},\dots,h^\circ_{a^*,s};Y_0,L^\circ} \in k^\circ_{a^*}(\varepsilon,t)[U_0,U_1].$$

Then, by Lemma 4 and Remark 6, we have $\Delta_4 \in k^\circ_{a^*}(\varepsilon)[t,U_0,U_1]$ and $\Delta_4|_{t=0} = \Delta_3$. By Lemma 4 and since all polynomials $\Phi_{W,L}$, $W \in \mathcal{E}_4$, are pairwise distinct and irreducible in the ring $\overline{k}^\circ(\varepsilon)[t,Y_0,Z]$, we have

$$\Delta_4(Z,-Y_0) = \prod_{W\in\mathcal{E}_4} (\Phi_{W,L}/\mathrm{lc}_Z(\Phi_{W,L}))^{e'_{W,L}}$$

for some integers $e'_{W,L} \geq 1$. Now, applying Corollary 1(a), we establish claim (a).

The proofs of claims (b) and (c) are similar to that of claim (a) and are left to the reader. □

Lemma 12. *Let* $V''_{a^*,s}\cap\mathcal{Z}(Y_0,\dots,Y_s)=\varnothing$ *in* $\mathbb{P}^n(\overline{k})$. *Let* $W\in\mathcal{E}_4$ *and* $\iota_{1,4}(W')=W$, $\iota_{1,3}(W')=W''$, *and let* L_1, Δ_1, Δ_2 *be as above. Then* $\mathrm{lc}_Z(\Phi_{W,L_1})|_{\varepsilon=0} \neq 0$ (*recall that* $\mathrm{lc}_Z(\Phi_{W,L_1}) \in \overline{k}[\varepsilon]$), $\Delta_1|_{\varepsilon=0} = \Delta_2$, *and*

$$(\Phi_{W,L_1}/(\mathrm{lc}_Z(\Phi_{W,L_1})))|_{\varepsilon=0} = \prod_{\substack{W_1\in\mathcal{E}_6,\\ W_1\subset W''}} (\Phi_{W_1,L_1}/(\mathrm{lc}_Z(\Phi_{W_1,L_1}))^{e'_{W,W_1,L_1}} \tag{32}$$

for some integers $e'_{W,W_1,L_1} \geq 1$.

Proof. In Sec. 3, we have defined the matrix $\mathcal{A} = (\mathcal{A}',\mathcal{A}'')$. Consider the case $\nu = 0$. In the definition of the matrix $\mathcal{A}$ with $\nu = 0$, replace n, k, $(X_0,\dots,X_n)$, $(f_0,\dots,f_{m-1})$, $(Y_0,\dots,Y_n)$ by $n-s$, $\overline{k}^\circ(\varepsilon)$, $(Y_0,Y_{s+1},\dots,Y_n)$, $(\psi^\circ_1,\dots,\psi^\circ_{m_2})$, $(Y_0,L^\circ_1,0,\dots,0)$, respectively. We will denote the obtained matrix again by $\mathcal{A} = (\mathcal{A}',\mathcal{A}'')$. Now, the entries of the matrix $\mathcal{A}'$ belong to $k_{a^*}[\varepsilon,t_1,\dots,t_s, Y_0,Y_{s+1},\dots,Y_n]$, and the entries of the matrix $\mathcal{A}''$ are linear forms in U_0,U_1 with coefficients from the latter ring. Let γ be the number of rows of $\mathcal{A}$.

Let $\mathrm{rank}(\mathcal{A}'|_{\varepsilon=0}) = \gamma'$. Then, by Lemma 4(b), we have $\mathrm{rank}(\mathcal{A}|_{\varepsilon=0}) = \gamma$ and, by Lemma 4(c), the number of roots in $\mathbb{P}^{n-s}(\overline{k^\circ})$ of the system

$$\overline{\psi}^\circ_1 = \ldots = \overline{\psi}^\circ_{m_2} = 0 \tag{33}$$

counting multiplicities is equal to $\gamma-\gamma'$. Let $\mathcal{A}'_1$ be a submatrix of $\mathcal{A}'$ of size $\gamma\times\gamma'$ such that $\mathrm{rank}(\mathcal{A}'_1|_{\varepsilon=0}) = \gamma'$. Let $\mathcal{A}''_1$ be a submatrix of $\mathcal{A}''$ of size $\gamma\times(\gamma-\gamma')$ such that the $\gamma\times\gamma$ matrix $(\mathcal{A}'_1|_{\varepsilon=0},\mathcal{A}''_1|_{\varepsilon=0})$ is of rank γ. Set $\mathcal{A}_1 = (\mathcal{A}'_1,\mathcal{A}''_1)$. Put $\Delta_W = \det(\mathcal{A}_1) \in k_{a^*}[\varepsilon,t_1,\dots,t_s,U_0,U_1]$. This is a homogeneous polynomial in U_0,U_1.

Then, by Lemma 4(c), the polynomial $\Delta_W|_{\varepsilon=0}$ coincides with Δ_2 up to a nonzero factor from $k^\circ_{a^*}$. Hence, $\Delta_W \neq 0$ and Δ_1 divides Δ_W.

Note that $\mathrm{lc}_{U_0}(\Delta_W) \in \overline{k}[\varepsilon,t_1,\dots,t_s]$. We have $Y_0(\eta) \neq 0$ for every $\eta \in W$ (respectively, for every $\eta \in W''\cap\mathcal{Z}(\varepsilon)$). Hence, by Lemma 4(c), we have $\deg_{U_0}\Delta_W = \deg_{U_0,U_1}\Delta_W = \deg_{U_0,U_1}\Delta_W|_{\varepsilon=0} = \deg_{U_0,U_1}\Delta_2 = \deg_{U_0}\Delta_2 = \deg_{U_0}\Delta_W|_{\varepsilon=0}$. Therefore, $(\mathrm{lc}_{U_0}\Delta_W)|_{\varepsilon=0} \neq 0$, the polynomial $(\Delta_W/\mathrm{lc}_{U_0}\Delta_W)|_{\varepsilon=0}$ is defined, and $(\Delta_W/\mathrm{lc}_{U_0}\Delta_W)|_{\varepsilon=0} = \Delta_2$.

Let us show that $\deg_{U_0,U_1} \Delta_W = \deg_{U_0,U_1} \Delta_1$. By Lemma 4(c), it suffices to prove that the number δ of roots (in $\mathbb{P}^{n-s}(\overline{k^\circ(\varepsilon)})$, counting multiplicities) of the system

$$\psi_1^\circ = \ldots = \psi_{m_2}^\circ = 0 \tag{34}$$

is equal to the number $\overline{\delta}$ of roots (in $\mathbb{P}^{n-s}(\overline{k^\circ})$, counting multiplicities) of the system (33). Indeed, $\delta = \deg_{U_0,U_1} \Delta_1$ and $\overline{\delta} = \deg_{U_0,U_1} \Delta_2 = \deg_{U_0,U_1} \Delta_W$. Note that we have proved that $\delta \le \overline{\delta}$. It remains to prove that $\delta \ge \overline{\delta}$.

Recall that $\psi_i^\circ \in \overline{k}^\circ[\varepsilon, Y_0, Y_{s+1}, \ldots, Y_n]$. Put

$$\psi_i' = \psi_i^\circ(\varepsilon, 1, Y_{s+1}, \ldots, Y_n), \quad 1 \le i \le m_2.$$

We identify the ring of regular functions $\overline{k}^\circ[W \setminus \mathcal{Z}(Y_0)]$ with $\overline{k}^\circ(\varepsilon)[Y_{s+1}, \ldots, Y_n]/(\psi_1', \ldots, \psi_{m_2}')$. Now, by the well-known definition of the multiplicities,

$$\delta = \dim_{\overline{k}^\circ(\varepsilon)} \overline{k}^\circ[W \setminus \mathcal{Z}(Y_0)] = \dim_{\overline{k}^\circ(\varepsilon)} \overline{k}^\circ(\varepsilon)[Y_{s+1}, \ldots, Y_n]/(\psi_1', \ldots, \psi_{m_2}').$$

We identify the ring of regular functions $\overline{k}^\circ[W'' \setminus \mathcal{Z}(Y_0)]$ with $\bar{k}^\circ[\varepsilon, Y_{s+1}, \ldots, Y_n]/(\psi_1', \ldots, \psi_{m_2}')$. The element ε is not a zero-divisor in the ring $\bar{k}^\circ[W'' \setminus \mathcal{Z}(Y_0)]$ by the definition of the variety $V''_{a^*,s}$. We have

$$\overline{\delta} = \dim_{\overline{k}^\circ} \overline{k}^\circ[W'' \setminus \mathcal{Z}(Y_0)]/(\varepsilon) = \dim_{k^\circ} \overline{k}^\circ[\varepsilon, Y_{s+1}, \ldots, Y_n]/(\psi_1', \ldots, \psi_{m_2}', \varepsilon).$$

Let $z_1, \ldots, z_{\overline{\delta}} \in \overline{k}^\circ[W'' \setminus \mathcal{Z}(Y_0)]$ be functions such that their residues

$$z_i \bmod (\varepsilon) \in \overline{k}^\circ[W'' \setminus \mathcal{Z}(Y_0)]/(\varepsilon)$$

are linearly independent over $\overline{k}^\circ$. We claim that $z_1, \ldots, z_{\overline{\delta}}$ are linearly independent over $\overline{k}^\circ(\varepsilon)$ in $\overline{k}^\circ(\varepsilon)[Y_{s+1}, \ldots, Y_n]/(\psi_1', \ldots, \psi_{m_2}')$. Assume the contrary. Then there is a linear relation $c_1 z_1 + \ldots + c_{\overline{\delta}} z_{\overline{\delta}} = 0$ where $c_i \in \overline{k}^\circ[\varepsilon]$ and not all of these coifficients are zeros. Since ε is not a zero-divisor in $\overline{k}^\circ[W'' \setminus \mathcal{Z}(Y_0)]$, we may assume without loss of generality that ε does not divide at least one of c_i. Now, taking the residues $\bmod(\varepsilon)$, we see that the elements $z_i \bmod (\varepsilon)$ are linearly dependent over the field $\overline{k}^\circ$. This is a contradiction. Thus, $\delta \ge \overline{\delta}$.

Therefore, $\delta = \overline{\delta}$ and $\Delta_W/\mathrm{lc}_{U_0}(\Delta_W) = \Delta_1$. Hence, the polynomial $\Delta_1|_{\varepsilon=0}$ is defined, and $\Delta_1|_{\varepsilon=0} = \Delta_2$.

Put $\Delta_W(Z, -Y_0) = \Delta_W|_{U_0=Z, U_1=-Y_0}$, i.e., in this notation we regard Δ_W as an element of $\overline{k}^\circ(\varepsilon)[U_0, U_1]$. Obviously, $\mathrm{lc}_Z(\Delta_W(Z, -Y_0)) = \mathrm{lc}_{U_0}\Delta_W \in \overline{k}[\varepsilon, t_1, \ldots, t_s]$. By Lemma 11(c), we have

$$\Delta_W(Z, -Y_0)/\mathrm{lc}_Z(\Delta_W(Z, -Y_0)) = (\Phi_{W,L_1}/\mathrm{lc}_Z(\Phi_{W,L_1}))^{e''_{W,L_1}}.$$

Recall that the polynomial Φ_{W,L_1} is irreducible. Hence, $\Delta_W(Z, -Y_0) = \lambda_{W,L_1}\Phi_{W,L_1}^{e''_{W,L_1}}$ where $\lambda_{W,L_1} \in \overline{k}[\varepsilon, t_1, \ldots, t_s]$. This implies

$$0 \ne \mathrm{lc}_Z(\Delta_W(Z, -Y_0))|_{\varepsilon=0} = (\lambda_{W,L_1}|_{\varepsilon=0}) \cdot (\mathrm{lc}_Z(\Phi_{W,L_1})|_{\varepsilon=0})^{e''_{W,L_1}}.$$

Therefore, $\mathrm{lc}_Z(\Phi_{W,L_1})|_{\varepsilon=0} \ne 0$, the polynomial $(\Phi_{W,L_1}/\mathrm{lc}_Z(\Phi_{W,L_1}))|_{\varepsilon=0}$ is defined, and

$$\Delta_2(Z, -Y_0) = (\Delta_W/\mathrm{lc}_Z(\Delta_W))|_{\varepsilon=0} = ((\Phi_{W,L_1}/\mathrm{lc}_Z(\Phi_{W,L_1}))|_{\varepsilon=0})^{e''_{W,L_1}}.$$

Hence, by Lemma 11(b),

$$((\Phi_{W,L_1}/\mathrm{lc}_Z(\Phi_{W,L_1}))|_{\varepsilon=0})^{e''_{W,L_1}} = \prod_{\substack{W_1 \in \mathcal{E}_6,\\ W_1 \subset W''}} (\Phi_{W_1,L_1}/(\mathrm{lc}_Z(\Phi_{W_1,L_1}))^{e_{W,W_1,L_1}}. \tag{35}$$

Let us replace the ground field k by $k(t)$ where t is a transcendental element over k, and choose a linear form $L_2 \in k[X_0, \ldots, X_n]$ such that $(L_2/Y_0)(\eta_1) \ne (L_2/Y_0)(\eta_2)$ for all pairwise distinct

$\eta_1, \eta_2 \in W'' \cap \mathcal{Z}(\varepsilon)$, cf. the proof of Lemma 11. Put $L = L_1 + tL_2$. Then (35) holds true with L in place of L_1. All polynomials $\Phi_{W_1,L}$, $W_1 \in \mathcal{E}_6$, $W_1 \subset W''$, are pairwise distinct and irreducible in the ring $\overline{k}[t, \varepsilon, t_1, \dots, t_s, Y_0, Z]$, see the remark after the proof of Corollary 1. We have already seen that $\mathrm{lc}_Z(\Phi_{W,L}) \in \overline{k}[\varepsilon]$ and $\mathrm{lc}_Z(\Phi_{W_1,L}) \in \overline{k}$. Hence, $1 \le e''_{W,L}/e_{W,W_1,L} \in \mathbb{Z}$, and (32) (see the statement of Lemma 12) is fulfilled for L in place of L_1. Now, applying Corollary 1, we get (32) also for L_1. The lemma is proved. □

Corollary 2. *The degree of the projective algebraic variety* $\deg V''_{a^*,s}$ *satisfies the inequality*

$$\deg V''_{a^*,s} \le d_0 \cdot \ldots \cdot d_{n-s-1} = D'_{n-s}.$$

Proof. Applying Lemma 7 to the algebraic variety $V''_{a^*,s}$ (in place of V) with $D = \deg V''_{a^*,s}$, we obtain linear forms $Y_0, \dots, Y_{s+1}$ and a polynomial Φ_s. There are linear forms

$$Y_{s+2}, \dots, Y_n \in \overline{k}[X_0, \dots, X_n]$$

such that $Y_0, \dots, Y_n$ are linearly independent over $\overline{k}$. Put $L_1 = Y_{s+1}$. The polynomials Φ_s° and $\prod_{W \in \mathcal{E}_5} \Phi^\vee_{\iota_{5,6}(W),L_1}$ coincide up to a nonzero factor from $\overline{k}$, see (IV). Now, by Lemma 7(b), we have

$$\deg V''_{a^*,s} = \deg_Z \Phi_s^\circ = \sum_{W \in \mathcal{E}_5} \deg_Z \Phi^\vee_{\iota_{5,6}(W),L_1} = \sum_{W_1 \in \mathcal{E}_6} \deg_Z \Phi_{W_1,L_1}.$$

By Lemma 12 and the definition of the algebraic variety $V''_{a^*,s}$, for every $W_1 \in \mathcal{E}_6$ there is $W \in \mathcal{E}_4$ such that $W_1 \subset W''$ (in the notation of Lemma 12) and (32) holds true. Hence, we have

$$\sum_{W_1 \in \mathcal{E}_6} \deg_Z \Phi_{W_1,L_1} \le \sum_{W \in \mathcal{E}_4} \deg_Z \Phi_{W,L_1} \le \sum_{W \in \mathcal{E}_4} \deg W = \deg \widetilde{C}_{a^*,s}.$$

But $\widetilde{C}_{a^*,s} = \mathcal{Z}(h^\circ_{a^*,1}, \dots, h^\circ_{a^*,n-s})$. Therefore, by Bézout's theorem, $\deg \widetilde{C}_{a^*,s} \le d_0 \cdot \ldots \cdot d_{n-s-1} = D'_{n-s}$. The corollary is proved. □

Let the field $K \supset \overline{k}$ be as above and $L \in K[X_0, \dots, X_n]$ be an arbitrary linear form. Recall that $K^\circ = K(t_1, \dots, t_s)$. Put

$$\Delta_L = \Delta_{K^\circ(\varepsilon);Y_0,Y_{s+1},\dots,Y_n;h^\circ_{a^*,1},\dots,h^\circ_{a^*,s};Y_0,L^\circ} \in K(\varepsilon)[t_1, \dots, t_s, U_0, U_1], \tag{36}$$

see the notation in Remark 6. So, if $L = L_1 \in \overline{k}[X_0, \dots, X_n]$, then $\Delta_L = \Delta_3$, see above. Put $\Delta_L^{(3)} = \Delta_L(Y_1/Y_0, \dots, Y_s/Y_0, Z, -Y_0)$.

Denote by $K[\varepsilon]_{(\varepsilon)}$ the local ring of the prime ideal $(\varepsilon) \subset K[\varepsilon]$, i.e., $z \in K[\varepsilon]_{(\varepsilon)}$ if and only if $z \in K(\varepsilon)$ and one can represent z in the form $z = z_1/z_2$ where $z_1, z_2 \in K[\varepsilon]$ and ε does not divide z_2.

Lemma 13. *Let $L_{s+1}, \dots, L_n \in K[X_0, \dots, X_n]$ be linear forms such that the linear forms $Y_0, \dots, Y_s, L_{s+1}, \dots, L_n$ are linearly independent over K. Then the following assertions are equivalent.*

(a) $V''_{a^*,s} \cap \mathcal{Z}(Y_0, \dots, Y_s) = \varnothing$ *in* $\mathbb{P}^n(\overline{k})$.

(b) *For $s+1 \le i \le n$, we have $\Delta^{(3)}_{L_i} \in K[\varepsilon]_{(\varepsilon)}[Y_0, \dots, Y_s, Z]$ and* $(\mathrm{lc}_Z \Delta^{(3)}_{L_i})|_{\varepsilon=0} \ne 0$.

Assume that condition (b) *is fulfilled. Then, obviously, $\mathrm{lc}_Z \Delta^{(3)}_{L_i} = \mathrm{lc}_{U_0} \Delta_{L_i}$ for $s+1 \le i \le n$. Assume additionally that $K = \overline{k}(t)$ (see above) and $L_i \in \overline{k}[t][X_0, \dots, X_n]$. Then we have $\Delta_{L_i} \in K[\varepsilon]_{(\varepsilon)}[t, t_1, \dots, t_s, U_0, U_1]$, $\Delta^{(3)}_{L_i} \in K[\varepsilon]_{(\varepsilon)}[t, Y_0, \dots, Y_s, Z]$, and $\mathrm{lc}_Z \Delta^{(3)}_{L_i} = \mathrm{lc}_{U_0} \Delta_L \in K[\varepsilon]_{(\varepsilon)} \setminus \varepsilon K[\varepsilon]_{(\varepsilon)}$ for all i.*

Proof. Assertion (a) implies (b) by Lemma 11(a) and Lemma 12 (with the ground field K in place of k and L_i in place of L_1).

Conversely, under the conditions of (b), the polynomial $\Delta^{(3)}_{L_i}(Y_1, \dots, Y_s, L_i)$ vanishes on $V_{a^*,s}$ for every i. Put $\psi = \prod_{s+1 \le i \le n} \mathrm{lc}_Z \Delta^{(3)}_{L_i} \in K[\varepsilon]$. Hence $\psi(0) \neq 0$, and the morphism

$$V''_{a^*,s}(\overline{K}) \times (\mathbb{A}^1(\overline{K}) \setminus \mathcal{Z}(\psi)) \to \mathbb{P}^s(\overline{K}) \times \big(\mathbb{A}^1(\overline{K}) \setminus \mathcal{Z}(\psi)\big),\ \big((X_0 : \dots : X_n), \varepsilon\big) \mapsto \big((Y_0 : \dots : Y_s), \varepsilon\big),$$

is finite dominant. Therefore, the restriction of this morphism $V''_{a^*,s}(\overline{K}) \times \{0\} \to \mathbb{P}^s(\overline{K}) \times \{0\}$ is also finite dominant. This implies (a).

Let us prove the last assertion of the lemma. For every $\eta \in \widetilde{C}_{a^*,s}$, the elements $(X_i/Y_0)(\eta)$, $0 \le i \le n$, are integral over the ring $\overline{k}(\varepsilon)[t_1, \dots, t_s]$. Hence, the elements $(L_i/Y_0)(\eta)$, $s+1 \le i \le n$, are integral over $\overline{k}(\varepsilon)[t, t_1, \dots, t_s]$, cf. the proof of Corollary 1. Therefore, $\mathrm{lc}_Z(\Phi_{W,L_i}) \in \overline{k}(\varepsilon)$ for $s+1 \le i \le n$ and $W \in \mathcal{E}_4$. Now, the required assertion follows from (b) and Lemma 11(a) (with the ground field K in place of k and L_i in place of L_1). The lemma is proved. □

Remark 10. Another independent proof of Lemma 13 can be deduced from Lemma 3.9 in [7, pp. 186–189]. In [7], the proof of the latter lemma is, in a sense, more direct. It does not use a result similar to Lemma 12.

6. The main recursion

Lemma 14. *It suffices to prove Theorem* 1 *in the case* $c \le n-1$.

Proof. Indeed, let c, c' be the integers from the statement of Theorem 1 and $c = n$. Put $c_1 = n-1$, $c'_1 = \min\{c', n-1\}$. Assume that Theorem 1 is proved for (c_1, c'_1, A_1) in place of (c, c', A). Let $\alpha^{(n)} \notin A_1$. Let $f_{i,i_0,\dots,i_n}$, $0 \le i \le m-1$, $i_1, \dots, i_n \ge 0$, $i_0 + \ldots + i_n = d_i$, be the family of all coefficients from $k[a_1, \dots, a_\nu]$ of the polynomials $f_0, \dots, f_{m-1}$. Set $\mathcal{W}_{\alpha^{(n)}} = \mathcal{Z}(f_{i,i_0,\dots,i_n}, \forall\, i, i_0, \dots, i_n)$. Put $A = A_1 \cup \{\alpha^{(n)}\}$. Then (4) is a stratification of the set $\mathcal{U}_n$.

Since Theorem 1 is proved for (c_1, c'_1), all the objects from (iv)–(xiii) for the stratification (4) with the initial values of (c, c') are also obtained, and Theorem 1 with (c, c') is proved (actually, if $c' = n$ and $\alpha \in A_1$, then even more objects corresponding to α are constructed). The lemma is proved. □

In what follows, we will assume that $-1 \le c \le n-1$. Let $a^* \in \mathcal{U}_c$. Let s be an integer, $0 \le s \le c$. The variety $V_{a^*,s}$ is defined in the introduction. We will also use the notation $V''_{a^*,s}$, $V'_{a^*,s}$, $V'''_{a^*,s}$, $V''''_{a^*,s}$, and so on introduced in Sec. 5.

Let $W_{a^*,s} = V_{a^*,s}$ or $W_{a^*,s} = V'_{a^*,s}$. Put $\mathcal{W}_{\alpha_0} = \{a^*\}$. Let $(Y_0, \dots, Y_{s+1}) \in \mathcal{L}_s^{s+1} \times \mathcal{L}'_s$, see the introduction. Let us replace in conditions (iv)–(xii) from the introduction the field k by k_{a^*}, the index α by α_0, the varieties $V_{a^*,s}$, $V_{a^*,s,r}$ by $W_{a^*,s}$, $W_{a^*,s,r}$. Denote by (iv)′–(xii)′ the resulting new conditions.

Consider also the following condition.

(xiv)′ The degrees in $a_1, \dots, a_\nu$ of all nonzero polynomials $\Phi_{\alpha_0,s,r}$, H_j, Δ_j, Φ_j, $\lambda_{\alpha_0,s,r,0}$, $\lambda_{\alpha_0,s,r,1}$, $G_{\alpha_0,s,r}$, $G_{\alpha_0,s,r,i}$, G_j, $G_{j,i}$, $\Psi_{\alpha_0,s,r,i_1,i_2}$, Ψ_{j,i_1,i_2}, $j \in J_{\alpha_0,s,r}$, are equal to 0 (i.e., these elements do not depend on $a_1, \dots, a_\nu$).

To avoid a confusion, we need new notation for the objects introduced in (iv)′–(xii)′.

If $W_{a^*,s} = V_{a^*,s}$, then $\Phi_{\alpha_0,s,r}$, $\Delta_{\alpha_0,s,r}$, $\lambda_{\alpha_0,s,r,0}, \lambda_{\alpha_0,s,r,1}$, $G_{\alpha_0,s,r}$, $G_{\alpha_0,s,r,i}$, $J_{\alpha_0,s,r}$ will be denoted by

$$\Phi_{a^*,s,r},\ \Delta_{a^*,s,r},\ \lambda_{a^*,s,r,0}, \lambda_{a^*,s,r,1},\ G_{a^*,s,r},\ G_{a^*,s,r,i},\ J_{a^*,s,r}, \tag{37}$$

respectively.

If $W_{a^*,s} = V'_{a^*,s}$, then $\Phi_{\alpha_0,s,r}$, $\Delta_{\alpha_0,s,r}$, $\lambda_{\alpha_0,s,r,0}, \lambda_{\alpha_0,s,r,1}$, $G_{\alpha_0,s,r}$, $G_{\alpha_0,s,r,i}$, $J_{\alpha_0,s,r}$ will be denoted by

$$\Phi^{(1)}_{a^*,s,r},\ \Delta^{(1)}_{a^*,s,r},\ \lambda^{(1)}_{a^*,s,r,0}, \lambda^{(1)}_{a^*,s,r,1},\ G^{(1)}_{a^*,s,r},\ G^{(1)}_{a^*,s,r,i},\ J^{(1)}_{a^*,s,r}, \tag{38}$$

respectively.

We will assume without loss of generality that all sets of indices $J_{a^*,s,r}$, $J^{(1)}_{a^*,s,r}$ are pairwise disjoint, i.e., $\sum\limits_{s,r}(\#J_{a^*,s,r} + \#J^{(1)}_{a^*,s,r}) = \#(\bigcup\limits_{s,r}(J_{a^*,s,r} \cup J^{(1)}_{a^*,s,r}))$.

For arbitrary $W_{a^*,s}$, the other objects introduced in (iv)$'$-(xii)$'$, namely,

$$H_j,\ \Delta_j,\ \Phi_j, \Xi_{j,a^*},\ W_{j,a^*,\xi},\ G_j,\ G_{j,i},\ \Psi_{\alpha_0,s,r,i_1,i_2},\ \Psi_{j,i_1,i_2},$$

will be denoted by

$$H_{a^*,j},\ \Delta_{a^*,j},\ \Phi_{a^*,j}, \Xi_{a^*,j},\ W_{a^*,j,\xi},\ G_{a^*,j},\ G_{a^*,j,i},\ \Psi_{a^*,s,r,i_1,i_2},\ \Psi_{a^*,j,i_1,i_2}, \tag{39}$$

respectively.

Using the construction of Sec. 4, we get polynomials $h_{a^*,1}, \dots, h_{a^*,n-c}$ satisfying conditions (α_{n-c}) and (β_{n-c}).

In what follows, we assume that $0 \le c \le n-1$. We will use a decreasing recursion on q, where $0 \le q \le c$. The base of the recursion is the case $q = c$. The last step of this recursion is the case $q = c'_{a^*} \le c$ with $\dim V'''_{a^*,q} \le c' - 1$. So, if $V'''_{a^*,q} \ne \varnothing$, then $q = c'_{a^*} = c' - 1 \ge 0$. Put $c_{a^*} = \dim V_{a^*}$. Then, obviously, $c' - 1 \le c'_{a^*} \le c_{a^*} \le c$.

Assume that at the previous steps with numbers $c, \dots, q+1$ of this recursion we have constructed polynomials $h_{a^*,1}, \dots, h_{a^*,n-q} \in k_{a^*}[X_0, \dots, X_n]$ and $q_{a^*,j,w}$ for $1 \le j \le n-q$, $j \le w \le m-1$ satisfying conditions (α_{n-q}), see Sec. 4, and (γ'_{n-q}), see Sec. 5.

For the base $q = c$, polynomials $h_{a^*,1}, \dots, h_{a^*,n-c}$ are already constructed, see above.

Let $0 \le q \le c-1$. Then we assume additionally that at the previous step with number $q+1$, the following objects are obtained. Put $s = q+1$. An element $(Y_0, \dots, Y_{s+1}) \in \mathcal{L}_s^{s+1} \times \mathcal{L}'_s$ is constructed. We will write $(Y_0, \dots, Y_{s+1}) = (Y_{s,0}, \dots, Y_{s,s+1})$ if the dependence on s of these linear forms is important. The linear forms $Y_0, \dots, Y_{s+1}$ satisfy the following properties.

For the case $W_{a^*,s} = V_{a^*,s}$, conditions (iv)$'$–(xii)$'$ hold true and all the objects (37), (39) are obtained.

For the case $W_{a^*,s} = V'_{a^*,s}$, conditions (iv)$'$, (v)$'$, (xi)$'$ hold true and all the polynomials $\Phi^{(1)}_{a^*,s,r}$, $\Psi^{(1)}_{a^*,s,r,i_1,i_2}$ are obtained (actually, one can satisfy (iv)$'$–(xii)$'$ and obtain all the objects (38), (39) also in this case, but for the variety $V'_{a^*,s}$ it suffices to have only all $\Phi^{(1)}_{a^*,s,r}$, $\Psi^{(1)}_{a^*,s,r,i_1,i_2}$ to perform the recursive step).

Assume that $0 \le q < c$ but $\dim V'''_{a^*,q+1} > c' - 1$, or $q = c$. Now we are going to describe the qth recursive step of our construction.

In what follows, $s = q$ through the whole section. First, we will find the variety $V''_{a^*,s}$ and some objects related to it. We will enumerate the elements $(Y_0, \dots, Y_{s+1}) \in \mathcal{L}_s^{s+1} \times \mathcal{L}'_s$. Recall that the linear forms $Y_0, \dots, Y_n$ are linearly independent over k, see Sec. 4. Put $L_{s+1} = Y_{s+1}$ and $L_i = Y_{s+1} + tY_i$, $s+2 \le i \le n$, and let

$$\widetilde{\Delta}_{a^*,L_i} = \widetilde{\Delta}_{k^\circ_{a^*}(t,\varepsilon);Y_0,Y_{s+1},\dots,Y_n;h^\circ_{a^*,1},\dots,h^\circ_{a^*,s};Y_0,L_i} \in k^\circ_{a^*}(t,\varepsilon)[U_0,U_1]$$

for $s+1 \le i \le n$, see Remark 6 in Sec. 3. Note that, in fact, $\widetilde{\Delta}_{a^*,L_i} \in k_{a^*}[\varepsilon, t, t_1, \dots, t_s, U_0, U_1]$. We find all polynomials $\widetilde{\Delta}_{a^*,L_i}$ using the construction from Sec. 3. By Lemma 9, we have $\widetilde{\Delta}_{a^*,L_i} \ne 0$ for $s+1 \le i \le n$. Further, by Lemma 2,

$$\deg_{U_0,U_1} \widetilde{\Delta}_{a^*,L_i} = \deg_{U_0,U_1} \widetilde{\Delta}_{a^*,L_{s+1}}, \quad s+2 \le i \le n. \tag{40}$$

Let us compute, applying [6], the polynomials

$$\delta_{a^*,L_i} = \mathrm{GCD}_{\varepsilon,t,t_1,\dots,t_s,U_0,U_1}(\mathrm{lc}_{U_0}\widetilde{\Delta}_{a^*,L_i}, \widetilde{\Delta}_{a^*,L_i}) \in k_{a^*}[\varepsilon, t, t_1, \dots, t_s]$$

for $s+1 \le i \le n$. After that, using Lemma 2 from [6] and Noether normalization (see [6] for more details), we compute the polynomial $\Delta^{(4)}_{a^*,L_i} \in k_{a^*}[\varepsilon, t, t_1, \dots, t_s, U_0, U_1]$ coinciding with $\widetilde{\Delta}_{a^*,L_i}/\delta_{a^*,L_i}$ up to a nonzero factor from k_{a^*} for $s+1 \le i \le n$. Note that $\Delta^{(4)}_{a^*,L_i}/\mathrm{lc}_{U_0}\Delta^{(4)}_{a^*,L_i} = \Delta_{L_i}$ in the notation of Sec. 5, see (36). We apply Lemma 11(a) with $(k(t), k_{a^*}(t), L_i, \widetilde{\Delta}_{a^*,L_i})$ in place of $(k, k_{a^*}, L_1, \Delta_3)$. Then, by assertion (a) of this modified lemma and Remark 9, we have $\mathrm{lc}_{U_0}\Delta^{(4)}_{a^*,L_i} \in k_{a^*}[\varepsilon]$. Now, by Lemma 2, we have $\Delta^{(4)}_{a^*,L_i}|_{t=0} = \lambda_i \Delta^{(4)}_{a^*,L_{s+1}}$ where $\lambda_i \in k_{a^*}[\varepsilon]$ for $s+2 \le i \le n$.

Put $\Delta^{(3)}_{a^*,L_i} = \Delta^{(4)}_{a^*,L_i}(\varepsilon, t, Y_1/Y_0, \dots, Y_s/Y_0, Z, -Y_0)$, $s+1 \le i \le n$. If for at least one i with $s+1 \le i \le n$ we have $\Delta^{(3)}_{a^*,L_i} \notin k_{a^*}[\varepsilon, t, Y_0, \dots, Y_s, Z]$ or $\Delta^{(3)}_{a^*,L_i} \in k_{a^*}[\varepsilon, t, Y_0, \dots, Y_s, Z]$ but $\mathrm{lc}_Z\Delta^{(3)}_{a^*,L_i} \notin k_{a^*}[\varepsilon] \setminus (\varepsilon)$, then we proceed to the next element $(Y_0, \dots, Y_{s+1}) \in \mathcal{L}_s^{s+1} \times \mathcal{L}'_s$. By Corollary 2, Lemma 7 applied to the variety $V''_{a^*,s}$, and Lemma 13, there is $(Y_0, \dots, Y_{s+1}) \in \mathcal{L}_s^{s+1} \times \mathcal{L}'_s$ such that condition (41) stated below is fulfilled.

In what follows, we assume that

$$\Delta^{(3)}_{a^*,L_i} \in k_{a^*}[\varepsilon, t, Y_0, \dots, Y_s, Z] \ \&\ \mathrm{lc}_Z\Delta^{(3)}_{a^*,L_i} \in k_{a^*}[\varepsilon] \setminus (\varepsilon), \quad s+1 \le i \le n. \qquad (41)$$

Then, by Lemma 13, we have $V''_{a^*,s} \cap \mathcal{Z}(Y_0, \dots, Y_s) = \varnothing$, and the polynomials $\Delta^{(4)}_{a^*,L_i}|_{\varepsilon=0} \in k_{a^*}[t, t_1, \dots, t_s, U_0, U_1]$ are defined. In this case, put $\Delta^{(5)}_{a^*,L_i} = \Delta^{(4)}_{a^*,L_i}|_{\{\varepsilon=0,\,U_0=Z,\,U_1=-Y_0\}} = \Delta^{(4)}(0, t, t_1, \dots, t_s, Z, -Y_0) \in k_{a^*}[t, t_1, \dots, t_s, Y_0, Z]$. Note that the condition $\mathrm{lc}_Z\Delta^{(3)}_{a^*,L_i} \in k_{a^*}[\varepsilon] \setminus (\varepsilon)$ implies $\mathrm{lc}_Z\Delta^{(3)}_{a^*,L_i} = \mathrm{lc}_{U_0}\Delta^{(4)}_{a^*,L_i}$ and $\deg_Z \Delta^{(3)}_{a^*,L_i} = \deg_{U_0,U_1}\Delta^{(4)}_{a^*,L_i}$. Therefore, $\mathrm{lc}_Z\Delta^{(5)}_{a^*,L_i} \in k_{a^*}$ and $\deg_Z \Delta^{(5)}_{a^*,L_i} = \deg_{U_0,U_1}\Delta^{(4)}_{a^*,L_i}$.

Lemma 15. *Assume that* (41) *holds true. Then there is a family of integers* $e_\eta \ge 1$, $\eta \in C''_{a^*,s}$, *such that*

$$\Delta^{(5)}_{a^*,L_i} = \mathrm{lc}_Z(\Delta^{(5)}_{a^*,L_i}) \cdot \prod_{\eta \in C''_{a^*,s}} \Big(Z - ((Y_{s+1} + tY_i)/Y_0)(\eta)Y_0\Big)^{e_\eta} \quad \text{for } s+2 \le i \le n \qquad (42)$$

and

$$\Delta^{(5)}_{a^*,L_{s+1}} = \mathrm{lc}_Z(\Delta^{(5)}_{a^*,L_{s+1}}) \cdot \prod_{\eta \in C''_{a^*,s}} \Big(Z - (Y_{s+1}/Y_0)(\eta)Y_0\Big)^{e_\eta}.$$

Hence, for $s+2 \le i \le n$ *the polynomial* $\Delta^{(5)}_{a^*,L_i}|_{t=0}$ *coincides with* $\Delta^{(5)}_{a^*,L_{s+1}}$ *up to a nonzero factor from* k_{a^*}.

Proof. Let $u_{s+2}, \dots, u_n$ be algebraically independent elements over the field k. Put $K = \overline{k}(u_{s+2}, \dots, u_n)$ and $L = Y_{s+1} + \sum_{s+2 \le i \le n} u_i Y_i$,

$$\widetilde{\Delta}_{a^*} = \widetilde{\Delta}_{K^\circ(\varepsilon); Y_0, Y_{s+1}, \dots, Y_n; h^\circ_{a^*,1}, \dots, h^\circ_{a^*,s}; Y_0, L} \in K^\circ_{a^*}(t, \varepsilon)[U_0, U_1].$$

Actually, $\widetilde{\Delta}_{a^*} \in k_{a^*}[\varepsilon, t, u_{s+2}, \dots, u_n, t_1, \dots, t_s, U_0, U_1]$. By Lemma 9(b) and Lemma 4 (with $\widetilde{C}_{a^*,s}$ in place of V_{a^*}), we have $\mathrm{lc}_{U_0}\widetilde{\Delta}_{a^*} \in k_{a^*}[\varepsilon, t, u_{s+2}, \dots, u_n, t_1, \dots, t_s]$ and $\deg_{U_0}\widetilde{\Delta}_{a^*} = \deg_{U_0,U_1}\widetilde{\Delta}_{a^*}$. For every $W \in \mathcal{E}_4$, we choose $\Phi_{W,L} \in \overline{k}[\varepsilon, t, u_{s+2}, \dots, u_n, t_1, \dots, t_s, Y_0, Z]$ to be an irreducible polynomial in this ring, see Sec. 5. For every $W \in \mathcal{E}_4$, we have $\mathrm{lc}_Z\Phi_{W,L} \in \overline{k}[\varepsilon]$, cf.

the proof of Corollary 1(a) (we leave the details to the reader; actually, the required assertion follows immediately from Lemma 9(b)).

Let

$$\delta_{a^*} = \mathrm{GCD}_{\varepsilon,t,u_{s+2},\dots,u_n,t_1,\dots,t_s,U_0,U_1}(\mathrm{lc}_{U_0}\widetilde{\Delta}_{a^*}, \widetilde{\Delta}_{a^*}).$$

So, $\delta_{a^*} \in k_{a^*}[\varepsilon, t, u_{s+2}, \dots, u_n, t_1, \dots, t_s]$. Here one can regard $\widetilde{\Delta}_{a^*}$ as a polynomial in U_0, U_1 with coefficients from the last ring, and then δ_{a^*} is the greatest common divisor of all these coefficients. Put $\Delta^{(4)}_{a^*} = \widetilde{\Delta}_{a^*}/\delta_{a^*}$. We apply Lemma 11(a) with $(K, K, L, \widetilde{\Delta}_{a^*})$ in place of $(k, k_{a^*}, L_1, \Delta_3)$. Then assertion (a) of this modified lemma implies that

$$(\Delta^{(4)}_{a^*}/\mathrm{lc}_{U_0}\Delta^{(4)}_{a^*})|_{\{U_0=Z,\,U_1=-Y_0\}} = \prod_{W\in\mathcal{E}_4} (\Phi_{W,L}/\mathrm{lc}_Z\Phi_{W,L})^{e'_{W,L}} \qquad (43)$$

for some integers $e'_{W,L} \geq 1$. Hence, by the Gauss lemma, $\mathrm{lc}_{U_0}\Delta^{(4)}_{a^*}$ coincides with

$$\prod_{W\in\mathcal{E}_4} (\mathrm{lc}_Z\Phi_{W,L})^{e'_{W,L}}$$

up to a nonzero factor from $\overline{k}$. Thus, $\mathrm{lc}_{U_0}\Delta^{(4)}_{a^*,L} \in k_{a^*}[\varepsilon]$. Recall that (41) holds true. Now, by Lemma 13(a) and Lemma 9(b), we have $\mathrm{lc}_Z\Phi_{W,L} \in \overline{k}[\varepsilon]\setminus(\varepsilon)$. Therefore, $\mathrm{lc}_{U_0}\Delta^{(4)}_{a^*} \in k_{a^*}[\varepsilon]\setminus(\varepsilon)$.

By Lemma 4(b) with $\widetilde{C}_{a^*,s}$ in place of V_{a^*}, we have

$$\Delta^{(4)}_{a^*} = \mathrm{lc}_{U_0}(\Delta^{(4)}_{a^*}) \cdot \prod_{\eta\in\widetilde{C}_{a^*,s}} (U_0 + (L/Y_0)(\eta)U_1)^{e_\eta} \qquad (44)$$

and

$$\Delta^{(4)}_{a^*,L_i} = \mathrm{lc}_{U_0}(\Delta^{(4)}_{a^*,L_i}) \cdot \prod_{\eta\in\widetilde{C}_{a^*,s}} (U_0 + (L_i/Y_0)(\eta)U_1)^{e_\eta}\,, \quad s+1 \leq i \leq n, \qquad (45)$$

where e_η is the multiplicity of the root $\eta \in \widetilde{C}_{a^*,s}$ of the system $h^\circ_{a^*,1} = \ldots = h^\circ_{a^*,s} = 0$.

Put $v_{s+1,j} = 0$ for $s+2 \leq j \leq n$. For $s+2 \leq i \leq n$, $s+2 \leq j \leq n$, put $v_{i,j} = 0$ if $i \neq j$ and $v_{i,j} = t$ if $i = j$.

Let i be an integer, $s+1 \leq i \leq n$. Denote $\overline{\Delta}^{(4)}_{a^*,L_i} = \Delta^{(4)}_{a^*}|_{\{u_j=v_{i,j}\,\forall j\}}$, i.e., we substitute $u_j = v_{i,j}$ for all $j = s+2, \dots, n$ into the polynomial $\Delta^{(4)}_{a^*}$ and denote by $\overline{\Delta}^{(4)}_{a^*,L_i}$ the obtained polynomial. This substitution transforms the linear form L into L_i. Hence, by (44) and (45), the polynomials $\overline{\Delta}^{(4)}_{a^*,L_i}$ and $\Delta^{(4)}_{a^*,L_i}$ coincide up to a nonzero factor from the local ring $k_{a^*}[\varepsilon]_{(\varepsilon)}$ for $s+1 \leq i \leq n$. Hence, $\overline{\Delta}^{(4)}_{a^*,L_i}|_{\{\varepsilon=0,\,U_0=Z,\,U_1=-Y_0\}}$ coincides with $\Delta^{(5)}_{a^*,L_i}$ up to a nonzero factor from k_{a^*}.

Furthermore, put $\Delta^{(5)}_{a^*} = \Delta^{(4)}_{a^*}|_{\{\varepsilon=0,\,U_0=Z,\,U_1=-Y_0\}}$. Hence $\mathrm{lc}_Z\Delta^{(5)}_{a^*} = \mathrm{lc}_{U_0}\Delta^{(4)}_{a^*}|_{\{\varepsilon=0\}} \in k_{a^*}$. For every $W \in \mathcal{E}_6$, choose $\Phi_{W,L} \in \overline{k}[t, u_{s+2}, \dots, u_n, t_1, \dots, t_s, Y_0, Z]$. For every $W \in \mathcal{E}_6$, we have $\mathrm{lc}_Z\Phi_{W,L} \in \overline{k}$, cf. the proof of Corollary 1(b) (we leave the details to the reader; actually, the required assertion follows immediately from Lemma 9(c)). By (43) and Lemma 12 (with the ground field K in place of k), see (32), we have $\Delta^{(5)}_{a^*}/(\mathrm{lc}_{U_0}\Delta^{(5)}_{a^*}) = \prod_{W\in\mathcal{E}_6} (\Phi_{W,L}/(\mathrm{lc}_{U_0}\Phi_{W,L}))^{e_{W,L}}$ for some integers $e_{W,L} \geq 1$.

But, obviously, $\Phi_{W,L}/(\mathrm{lc}_{U_0}\Phi_{W,L}) = \prod_{\eta\in W} (Z - (L/Y_0)(\eta))$ for every $W \in \mathcal{E}_6$. Therefore,

$$\Delta^{(5)}_{a^*} = \mathrm{lc}_Z(\Delta^{(5)}_{a^*}) \cdot \prod_{\eta\in C''_{a^*,s}} (Z - (L/Y_0)(\eta)Y_0)^{e_\eta} \qquad (46)$$

for some integers $e_\eta \geq 1$.

Put $\overline{\Delta}^{(5)}_{a^*,L_i} = \Delta^{(5)}_{a^*}|_{\{u_j=v_{i,j}\,\forall j\}}$ for $s+1 \le i \le n$. Then, by (46),

$$\overline{\Delta}^{(5)}_{a^*,L_i} = \mathrm{lc}_Z(\Delta^{(5)}_{a^*}) \cdot \prod_{\eta \in C''_{a^*,s}} (Z - (L_i/Y_0)(\eta)Y_0)^{e_\eta} \tag{47}$$

for $s+1 \le i \le n$. On the other hand, obviously,

$$\overline{\Delta}^{(5)}_{a^*,L_i} = \Delta^{(4)}_{a^*}|_{\{u_j=v_{i,j}\,\forall j;\, \varepsilon=0,\, U_0=Z,\, U_1=-Y_0\}} = \overline{\Delta}^{(4)}_{a^*,L_i}|_{\{\varepsilon=0,\, U_0=Z,\, U_1=-Y_0\}},$$

and, as we have seen, the last polynomial coincides with $\Delta^{(5)}_{a^*,L_i}$ up to a nonzero factor from k_{a^*} for $s+1 \le i \le n$. The lemma is proved. □

Put $\Delta^{(6)}_{a^*,L_i} = \Delta^{(5)}_{a^*,L_i}|_{Y_0=1}$ for $s+1 \le i \le n$. Now, using the construction of [6, Sec. 2], compute the polynomials

$$\Delta_{a^*,L_i,j} = \mathrm{SQF}_{j,t,t_1,\dots,t_s,Z}(\Delta^{(6)}_{a^*,L_i}) \in k_{a^*}[t,t_1,\dots,t_s,Z], \quad 1 \le j \le \deg_Z \Delta^{(6)}_{a^*,L_i},$$

giving the square-free decomposition of the polynomial $\Delta^{(6)}_{a^*,L_i}$ in the sense of (48), see below. Actually, $\Delta_{a^*,L_i,j} \in k_{a^*}[t,t_1,\dots,t_s,Z]$. The polynomials $\Delta_{a^*,L_i,j}$ are separable (i.e., do not have multiple factors in $\overline{k}[t,t_1,\dots,t_s,Z]$).

Recall that the integer $\rho = \rho_s$ is defined in the introduction, see (iv) with $s = q$. If the characteristic exponent p is equal to 1, then $B_{0,i} = \{1,\dots,\deg_Z \Delta^{(6)}_{a^*,L_i}\}$, $B_{1,i} = \varnothing$. If $p > 1$, then $B_{r,i} = \{jp^r : 1 \le j \le (\deg_Z \Delta^{(6)}_{a^*,L_i})/p^r\}$ for every integer $r \ge 0$, see [6, Sec. 2]. By definition, put $r(j) = r$ if and only if $j \in B_r \setminus B_{r+1}$.

In this notation, the polynomial

$$\prod_{0 \le r \le \rho} \prod_{j \in B_{r,i} \setminus B_{r+1,i}} \Delta^{j/p^r}_{a^*,L_i,j}(t^{p^r}, t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r}) = \lambda'_{a^*,i} \Delta^{(6)}_{a^*,L_i}, \tag{48}$$

where $0 \ne \lambda'_{a^*,i} \in k_{a^*}$, and the polynomials $\Delta_{a^*,L_i,j}(t^{p^{r(j)}}, t_1^{p^{r(j)}}, \dots, t_s^{p^{r(j)}}, Z^{p^{r(j)}})$, where $1 \le j \le \deg_{U_0} \Delta^{(6)}_{a^*,L_i}$, are pairwise relatively prime in the ring $k_{a^*}[t,t_1,\dots,t_s,Z]$, see [6, Sec. 2]. Hence, for every j we have $0 \le \deg_Z \Delta_{a^*,L_i,j} \le (\deg_Z \Delta^{(6)}_{a^*,L_i})/j$.

Denote by $\mathrm{Res}_Z(\Delta_{a^*,L_i,j}, \partial\Delta_{a^*,L_i,j}/\partial Z)$ the discriminant with respect to Z of the polynomial $\Delta_{a^*,L_i,j}$. If for at least one pair (i,j), $s+1 \le i \le n$, $1 \le j \le \deg_Z \Delta^{(5)}_{a^*,L_i}$, the polynomial $\Delta_{a^*,L_i,j}$ is not separable with respect to Z (i.e., $\mathrm{Res}_Z(\Delta_{a^*,L_i,j}, \partial\Delta_{a^*,L_i,j}/\partial Z) = 0$), then we proceed to the next element $(Y_0,\dots,Y_{s+1}) \in \mathcal{L}_s^{s+1} \times \mathcal{L}'_s$. By Corollary 2 and Lemma 7 applied to the variety $V''_{a^*,s}$, there is $(Y_0,\dots,Y_{s+1}) \in \mathcal{L}_s^{s+1} \times \mathcal{L}'_s$ such that condition (49) stated below is fulfilled.

In what follows, we assume that

$$\mathrm{Res}_Z(\Delta_{a^*,L_i,j}, \partial\Delta_{a^*,L_i,j}/\partial Z) \ne 0, \quad s+1 \le i \le n,\ 1 \le j \le \deg_Z \Delta^{(6)}_{a^*,L_i}. \tag{49}$$

Put

$$g_{a^*,L_i,r} = \prod_{j \in B_{r,i} \setminus B_{r+1,i}} \Delta_{a^*,L_i,j} \in k_{a^*}[t,t_1,\dots,t_s,Z], \quad 0 \le r \le \rho.$$

Therefore, each polynomial $g_{a^*,L_i,r}$ is separable with respect to Z. Again by (48), we have $\deg_Z g_{a^*,L_i,r} \le \deg_Z(\Delta^{(6)}_{a^*,L_i})/p^r$. Note that

$$\sum_{0 \le r \le \rho} \deg_Z g_{a^*,L_i,r} = \#\{(Y_{s+1}/Y_0 + tY_i/Y_0)(\eta) : \eta \in C''_{a^*,s}\} \tag{50}$$

(we leave the details to the reader). So, if $i = s+1$, then one can omit "$+tY_i/Y_0$" in (50).

The following lemma is similar to Lemma 6.

Lemma 16. *Let* (41) *and* (49) *hold true. In the notation of* (42), *let* $e_\eta = p^{r_\eta} e'_\eta$ *where* r_η, e'_η *are integers,* $0 \le r_\eta \le \rho$, $e'_\eta \ge 1$, $\mathrm{GCD}\,(e'_\eta, p) = 1$ *for every* $\eta \in C''_{a^*,s}$. *Then the following conditions are equivalent.*

(a) $\#\{(Y_{s+1}/Y_0)(\eta) \,:\, \eta \in C''_{a^*,s}\} = \#C''_{a^*,s}$,

(b) $\sum_{0\le r\le\rho} \deg_Z g_{a^*,L_i,r} = \sum_{0\le r\le\rho} \deg_Z g_{a^*,L_{s+1},r}$ *for all* i,

(c) $\deg_Z g_{a^*,L_i,r} = \deg_Z g_{a^*,L_{s+1},r}$ *for all* i, r,

(d) *for* $0 \le r \le \rho$ *the polynomial* $g_{a^*,L_{s+1},r}(t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r})$ *coincides with*

$$\prod_{\eta\in C''_{a^*,s},\, r_\eta = r} (Z - (Y_{s+1}/Y_0)(\eta))^{p^r}$$

up to a nonzero factor from $\overline{k}$, *and for* $s+2 \le i \le n$ *and* $0 \le r \le \rho$ *the polynomial* $g_{a^*,L_i,r}(t^{p^r}, t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r})$ *coincides with*

$$\prod_{\eta\in C''_{a^*,s},\, r_\eta = r} (Z - (Y_{s+1}/Y_0)(\eta) - t(Y_i/Y_0)(\eta))^{p^r}$$

up to a nonzero factor from $\overline{k}$.

Proof. The proof is similar to the proof of Lemma 6 and is left to the reader. □

If assertion (c) of Lemma 16 is not fulfilled, then we proceed to the next element

$$(Y_0, \dots, Y_{s+1}) \in \mathcal{L}_s^{s+1} \times \mathcal{L}'_s.$$

By Corollary 2 and Lemma 7 applied to the variety $V''_{a^*,s}$, there is $(Y_0, \dots, Y_{s+1}) \in \mathcal{L}_s^{s+1} \times \mathcal{L}'_s$ such that assertion (c) of Lemma 16 is satisfied.

7. The end of the description of the main recursion

In what follows, we assume that assertion (c) of Lemma 16 holds true. Now, by Lemma 16(d), for $s+1 \le i \le n$, $0 \le r \le \rho$ we have (in the notation introduced in Sec. 5)

$$g_{a^*,L_i,r}(t^{p^r}, t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r})/\mathrm{lc}_Z(g_{a^*,L_i,r}) = \prod_{W\in\mathcal{E}_{6,r}} \Psi_{W,L_i}^{p^r},$$

where $\mathcal{E}_{6,r} \subset \mathcal{E}_6$. The subset $\mathcal{E}_{6,r}$ does not depend on L_i. It depends only on r. We have $W \in \mathcal{E}_{6,r}$ if and only if $r_\eta = r$ for every $\eta \in W$.

Furthermore, $\bigcup_{0\le r\le\rho} \mathcal{E}_{6,r} = \mathcal{E}_6$ and $\mathcal{E}_{6,r_1} \cap \mathcal{E}_{6,r_2} = \varnothing$ for $0 \le r_1 \ne r_2 \le \rho$. Let $\iota_{5,6}(\mathcal{E}_{5,r}) = \mathcal{E}_{6,r}$, see Sec. 5. Put

$$V''_{a^*,s,r} = \bigcup_{W\in\mathcal{E}_{5,r}} W, \qquad C''_{a^*,s,r} = \bigcup_{W\in\mathcal{E}_{6,r}} W, \qquad 0 \le r \le \rho.$$

Denote by $V'_{a^*,s,r}$ (respectively, $V_{a^*,s,r}$, $V'''_{a^*,s,r}$, $V''''_{a^*,s,r}$) the union of all irreducible components $W \in \mathcal{E}_{5,r}$ such that $W \subset V'_{a^*,s}$ (respectively, $W \subset V_{a^*,s}$, $W \subset V'''_{a^*,s}$, $W \subset V''''_{a^*,s}$).

Denote by $C'_{a^*,s}$ (respectively, $C_{a^*,s}$, $C'''_{a^*,s}$, $C''''_{a^*,s}$) the union of all irreducible components $\iota_{5,6}(W)$ where $W \in \mathcal{E}_5$ and $W \subset V'_{a^*,s}$ (respectively, $W \subset V'_{a^*,s}$, $W \subset V'''_{a^*,s}$, $W \subset V''''_{a^*,s}$).

Denote by $C'_{a^*,s,r}$ (respectively, $C_{a^*,s,r}$, $C'''_{a^*,s,r}$, $C''''_{a^*,s,r}$) the union of all irreducible components $\iota_{5,6}(W)$ where $W \in \mathcal{E}_{5,r}$ and $W \subset V'_{a^*,s}$ (respectively, $W \subset V_{a^*,s}$, $W \subset V'''_{a^*,s}$, $W \subset V''''_{a^*,s}$).

Thus, $C''_{a^*,s,r}$ (respectively, $C'_{a^*,s,r}$, $C_{a^*,s,r}$, $C'''_{a^*,s,r}$, $C''''_{a^*,s,r}$) is the subset of all η from the set $C''_{a^*,s}$ (respectively, $C'_{a^*,s}$, $C_{a^*,s}$, $C'''_{a^*,s}$, $C''''_{a^*,s}$) such that $r_\eta = r$.

Note that if $n-s \geq m$, then, by property (α_{n-s}) (see Secs. 5 and 4), we have $V'''_{a^*,s} = \varnothing$ and, therefore, also $V'''_{a^*,s,r} = \varnothing$ for $0 \leq r \leq \rho$.

Put $g_{a^*,r} = g_{a^*,L_{s+1},r} \in k_{a^*}[t_1, \dots, t_s, Z]$. Now, by Lemma 16(d), for every r and every i the polynomial $g_{a^*,L_i,r}|_{t=0} = g_{a^*,L_i,r}(0, t_1, \dots, t_s, Z)$ coincides with $g_{a^*,r}$ up to a nonzero factor from k_{a^*}. Let $\mu_{a^*,r} = \mathrm{lc}_Z g_{a^*,r}$ (respectively, $\mu_{a^*,L_i,r} = \mathrm{lc}_Z g_{a^*,L_i,r}$, $0 \leq i \leq n$). Replacing $g_{a^*,r}$ by $g_{a^*,r} \prod\limits_{s+2 \leq j \leq n} \mu_{a^*,L_j,r}$ and each $g_{a^*,L_i,r}$ by $g_{a^*,L_i,r}\mu_{a^*,r} \prod\limits_{s+2 \leq j \neq i \leq n} \mu_{a^*,L_j,r}$, we will assume without loss of generality that $g_{a^*,L_i,r}(0, Z) = g_{a^*,r}$ for $s+2 \leq i \leq n$.

If $\deg_Z g_{a^*,r} = 0$, then put $\Phi_{a^*,s,r} = 1$, $\lambda_{a^*,s,r,0} = \lambda_{a^*,s,r,1} = 1$, $J_{a^*,s,r} = \varnothing$, $\Phi^{(1)}_{a^*,s,r} = 1$ and $\Psi_{a^*,s,r,i_1,i_2} = 1$, $\Psi^{(1)}_{a^*,s,r,i_1,i_2} = 1$ for all i_1, i_2, see the beginning of the section.

In what follows, we assume that $\deg_Z g_{a^*,r} > 0$. Consider the separable $\overline{k}$-algebra

$$\Lambda_r = \overline{k}[t_1, \dots, t_s, Z]/(g^{1/p^r}_{a^*,r}(t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r})).$$

Put $\theta_{a^*,r} = Z \bmod g^{1/p^r}_{a^*,r}(t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r}) \in \Lambda_r$ and

$$\theta'_{a^*,r,i} = -\left(\frac{\partial g_{a^*,L_i,r}}{\partial t}\right) \Big/ \left(\frac{\partial g_{a^*,L_i,r}}{\partial Z}\right)\Bigg|_{\substack{t_1 \to t_1^{p^r},\, \dots,\, t_s \to t_s^{p^r}, \\ t \to 0,\, Z \to \theta^{p^r}_{a^*,r}}}, \quad s+2 \leq i \leq n$$

(this means that we substitute $t_i^{p^r}$ for t_i, $1 \leq i \leq s$, 0 for t, and $\theta^{p^r}_{a^*,r}$ for Z).

Denote by $\overline{k}(t_1, \dots, t_s)[\theta_{a^*,r}]$ the localization of Λ_r with respect to the multiplicatively closed set $\overline{k}[t_1, \dots, t_s] \setminus \{0\}$. Denote by $\overline{k}(V''_{a^*,s,r} \setminus \mathcal{Z}(Y_0))$ the total quotient ring of the ring of regular functions $\overline{k}[V''_{a^*,s,r} \setminus \mathcal{Z}(Y_0)]$ of the algebraic variety $V''_{a^*,s,r} \setminus \mathcal{Z}(Y_0)$. Then (cf. Sec. 3) there is a natural isomorphism of $\overline{k}$-algebras

$$\overline{k}(V''_{a^*,s,r} \setminus \mathcal{Z}(Y_0)) \to \overline{k}(t_1, \dots, t_s)[\theta_{a^*,r}]$$

such that $Y_i/Y_0 \mapsto t_i$ for $1 \leq i \leq s$, $Y_{s+1}/Y_0 \mapsto \theta_{a^*,r}$, and $(Y_i/Y_0)^{p^r} \mapsto \theta'_{a^*,r,i}$ for $s+2 \leq i \leq n$.

Consider the separable k-algebra $\Lambda^{(1)}_r = \overline{k}[t_1, \dots, t_s, Z]/(g_{a^*,r})$. Put $\theta^{(1)}_{a^*,r} = Z \bmod g_{a^*,r} \in \Lambda^{(1)}_r$.

For $s+2 \leq i \leq n$, we have $g_{a^*,L_i,r} = g_{a^*,r} + \sum\limits_{j \geq 0} g_{a^*,L_i,r,j} t^j \in \overline{k}[t, t_1, \dots, t_s, Z]$ where $g_{a^*,L_i,r,j} \in \overline{k}[t_1, \dots, t_s, Z]$. Set $g'_{a^*,r} = \frac{\partial}{\partial Z}(g_{a^*,r})$. We have

$$-\left(\frac{\partial g_{a^*,L_i,r}}{\partial t}\right) \Big/ \left(\frac{\partial g_{a^*,L_i,r}}{\partial Z}\right)\Bigg|_{t=0, Z=\theta^{(1)}_{a^*,r}} = -\left(g_{a^*,L_i,r,1}/g'_{a^*,r}\right)|_{Z=\theta^{(1)}_{a^*,r}} \in \Lambda^{(1)}_r.$$

Let $\delta_{a^*,r}$ be the discriminant of the polynomial $g_{a^*,r}$ with respect to Z. Then one can write $-\left(g_{a^*,L_i,r,1}/g'_{a^*,r}\right)|_{Z=\theta^{(1)}_{a^*,r}} = (\delta_{a^*,L_i,r}|_{Z=\theta^{(1)}_{a^*,r}})/\delta_{a^*,r}$, where $\delta_{a^*,L_i,r} \in k_{a^*}[t_1, \dots, t_s, Z]$, $\deg_Z \delta_{a^*,L_i,r} < \deg_Z g_{a^*,r}$, and the coefficients from k_{a^*} of $\delta_{a^*,L_i,r}$ are polynomials in the coefficients of all $f_{a^*,j}$, $0 \leq j \leq m-1$, cf. the construction of $\delta_{a^*,i,r}$ in Sec. 3. Therefore,

$$\theta'_{a^*,r,i} = \frac{\delta_{a^*,L_i,r}(t_1^{p^r}, \dots, t_s^{p^r}, \theta^{p^r}_{a^*,r})}{\delta_{a^*,r}(t_1^{p^r}, \dots, t_s^{p^r})}$$

for all r, i.

Also set $\delta_{a^*,L_{s+1},r} = Z\delta_{a^*,r}$ for $0 \leq r \leq \rho$.

In what follows, we assume that $Y_i = X_i$ for $s+2 \leq i \leq n$, cf. Sec. 5. Recall that in Sec. 5, for any polynomial $F \in \overline{k(\varepsilon)}[X_0, \dots, X_n]$ a polynomial $F^\circ \in \overline{k(\varepsilon)}[t_1, \dots, t_s, Y_0, Y_{s+1} \dots, Y_n]$ is defined. Below, we will use this definition with t in place of ε, i.e., with the field $\overline{k(t)}$ in place of $\overline{k(\varepsilon)}$. Now, for any polynomial $F \in \overline{k}[t, X_0, \dots, X_n]$ there is a unique polynomial $G \in \overline{k}[t, t_1, \dots, t_s, Y_0, Y_{s+1} \dots, Y_n]$ such that

$$(F^{p^r})^\circ = G(t^{p^r}, t_1^{p^r}, \dots, t_s^{p^r}, Y_0^{p^r}, Y_{s+1}^{p^r}, \dots, Y_n^{p^r}).$$

By definition, put

$$F^\nabla = G(t, t_1, \dots, t_s, \delta_{a^*,r}, \delta_{a^*,L_{s+1},r}, \dots, \delta_{a^*,L_n,r}) \in \overline{k}[t, t_1, \dots, t_s, Z]. \tag{51}$$

So, by our construction, the polynomial $g_{a^*,r}$ divides $h^\nabla_{a^*,i}$ for $1 \le i \le n-s$ (we leave the details to the reader).

Assume that $s \ge n-m+1$ and hence $n-s \le m-1$. Then the polynomial $\widetilde{h}_{a^*,n-s+1}$ is defined by (27) with $j = n-s+1$, see also a correction to this formula at the end of the introduction to the second part. Let us extend the ground field k to $k(t)$. So, now the polynomial $(\widetilde{h}_{a^*,n-s+1})^\nabla$ is defined according to (51). Put

$$g^{(4)}_{a^*,r} = \mathrm{GCD}_{\,t,t_1,\dots,t_s,Z}\left(g_{a^*,r}, (\widetilde{h}_{a^*,n-s+1})^\nabla\right).$$

Here $\mathrm{GCD}_{\,t,t_1,\dots,t_s,Z}$ is an algorithm corresponding to a computation forest, see [6, Sec. 2]. Hence $g^{(4)}_{a^*,r} \in k_{a^*}[t, t_1, \dots, t_s, Z]$, $\deg_{t_1,\dots,t_s,Z} g^{(4)}_{a^*,r} = D^{O(1)}_{n-s+1}$, and $\deg_t g^{(4)}_{a^*,r}$ is bounded from above by $D^{O(1)}_{n-s+1}$. Put $g^{(5)}_{a^*,r} = \mathrm{LC}_t(g^{(4)}_{a^*,r})$, i.e., $g^{(5)}_{a^*,r}$ is the leading coefficient of the polynomial $g^{(4)}_{a^*,r}$ with respect to t, see [5] for a precise definition of the computation forest $\mathrm{LC}_{\dots}$. Then, obviously, $g^{(5)}_{a^*,r}$ is the greatest common divisor of the polynomials $g_{a^*,r}$ and $f^\nabla_{a^*,i}$, $0 \le i \le m-1$, in the ring $k_{a^*}[t_1, \dots, t_s, Z]$.

If $s < n-m+1$, put $g^{(5)}_{a^*,r} = g_{a^*,r}$. In this case, $n-s-1 \ge m-1$ and, therefore, by property (α_{n-s}) (see Secs. 5 and 4), the polynomial $g_{a^*,r}$ divides $f^\nabla_{a^*,i}$ for $0 \le i \le m-1$.

Let us return to the case of arbitrary s. Now, by the definitions of $V'_{a^*,s}$ and $C'_{a^*,s}$ and Lemma 16(d), the polynomial $g^{(5)}_{a^*,r}(t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r})$ coincides with $\prod_{\eta \in C'_{a^*,s,r}} (Z - (Y_{s+1}/Y_0)(\eta))^{p^r}$ up to a nonzero factor from k_{a^*}. Put $e_5 = \deg_Z g^{(5)}$. Set

$$\Phi^{(1)}_{a^*,s,r} = Y_0^{e_5} g^{(5)}_{a^*,r}(Y_1/Y_0, \dots, Y_s/Y_0, Y_{s+1}/Y_0).$$

Then $\Phi^{(1)}_{a^*,s,r} \in k_{a^*}[Y_0, \dots, Y_{s+1}]$ is a homogeneous polynomial in $Y_0, \dots, Y_{s+1}$. Furthermore, all the assertions of (iv)$'$ and (v)$'$ for $W_{a^*,s} = V'_{a^*,s}$ (see the beginning of the section) hold true.

Let $s \le n-2$. Let $Y^{(i_1)} \in \mathcal{L}'_s$, $0 \le i_1 \le \varkappa_{2,s}$, see the introduction. Let i_2 be an integer, $s+2 \le i_2 \le n$. Consider the resultant

$$\varphi^{(5)}_{a^*,r,i_1,i_2} = \mathrm{Res}_{Z_1}\left(\delta_{a^*,r} Z - ((Y^{(i_1)} + tX_{i_2})^\nabla|_{Z=Z_1}),\ g^{(5)}_{a^*,r}(t_1, \dots, t_s, Z_1)\right).$$

Then $\varphi^{(5)}_{a^*,r,i_1,i_2} \in k_{a^*}[t, t_1, \dots, t_s, Z]$, and $\varphi^{(5)}_{a^*,r,i_1,i_2}(t^{p^r}, t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r})$ coincides with

$$\delta^{e_5}_{a^*,r}(t_1^{p^r}, \dots, t_s^{p^r}) \prod_{\eta \in C'_{a^*,s,r}} \left(Z - ((Y^{(i_1)}/Y_0) + t(X_{i_2}/Y_0))(\eta)\right)^{p^r}$$

up to a nonzero factor from k_{a^*}.

Using Lemma 2 from [6] and Noether normalization, we compute the polynomial

$$\psi^{(5)}_{a^*,r,i_1,i_2} \in k_{a^*}[t, t_1, \dots, t_s, Z]$$

coinciding with $\varphi^{(5)}_{a^*,r,i_1,i_2}/\delta^{e_5}_{a^*,r}$ up to a nonzero factor from k_{a^*}. Now $\mathrm{lc}_Z \psi^{(5)}_{a^*,r,i_1,i_2} \in k_{a^*}$, and $\psi^{(5)}_{a^*,r,i_1,i_2}(t^{p^r}, t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r})$ coincides with $\prod_{\eta\in C'_{a^*,s,r}} \left(Z - ((Y^{(i_1)}/Y_0) + t(X_{i_2}/Y_0))(\eta)\right)^{p^r}$ up to a nonzero factor from k_{a^*}. Set

$$\Psi^{(1)}_{a^*,s,r,i_1,i_2} = Y_0^{e_5}\psi^{(5)}_{a^*,r,i_1,i_2}(t, Y_1/Y_0, \dots, Y_s/Y_0, Z/Y_0).$$

Then $\Psi^{(1)}_{a^*,s,r,i_1,i_2} \in k_{a^*}[t, Y_0, \dots, Y_s, Z]$ is a homogeneous polynomial in $Y_0, \dots, Y_s, Z$. Furthermore, by Lemma 8, all the assertions of (xi)$'$ for $W_{a^*,s} = V'_{a^*,s}$ (see the beginning of the section) hold true.

Using Lemma 2 from [6] and Noether normalization, we compute the polynomial

$$g^{(6)}_{a^*,r} \in k_{a^*}[t_1, \dots, t_s, Z]$$

coinciding with $g_{a^*,r}/g^{(5)}_{a^*,r}$ up to a nonzero factor from k_{a^*}. By the definitions of the algebraic varieties $V'''_{a^*,s}$ and $C'''_{a^*,s}$ and Lemma 16(d), the polynomials $g^{(6)}_{a^*,r}(t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r})$ and $\prod_{\eta\in C'''_{a^*,s,r}} (Z - (Y_{s+1}/Y_0)(\eta))^{p^r}$ coincide up to a nonzero factor from k_{a^*}.

Let $s = c$. Then put $g^{(8)}_{a^*,s,r} = 1$ for $0 \le r \le \rho_s$.

Let $s < c$. We are going to define and construct $g^{(8)}_{a^*,s,r}$ in this case. Let $s+1 \le s_1 \le c$. Recall that, by the recursive assumption, at step s_1 linear forms $Y_{s_1,0}, \dots, Y_{s_1,s_1+1}$ and polynomials $\Phi^{(1)}_{a^*,s_1,r_1} \in k_{a^*}[Y_{0,s_1}, \dots, Y_{s_1,s_1}, Z]$, $0 \le r_1 \le \rho_{s_1}$, are obtained. Furthermore, if $s_1 \le n-2$, then also polynomials $\Psi^{(1)}_{a^*,s_1,r_1,i_1,i_2} \in k_{a^*}[t, Y_{0,s_1}, \dots, Y_{s_1,s_1}, Z]$, $0 \le i_1 \le \varkappa_{2,s_1}$, $s_1+2 \le i_2 \le n$, $1 \le r_1 \le \rho_{s_1} = \log_p D_{n-s_1}$, are obtained. If $c < n-1$, put $\varphi_{a^*,n-1} = 1$. If $c = n-1$, put

$$\varphi_{a^*,n-1} = \prod_{0\le r_1\le \rho_{n-1}} \Phi^{(1)}_{a^*,n-1,r_1}(Y^{p^{r_1}}_{n-1,0}, \dots, Y^{p^{r_1}}_{n-1,n}).$$

If $s_1 \le n-2$, put

$$\Phi^{(1)}_{a^*,s_1,r_1,i_1,i_2} = \Psi^{(1)}_{a^*,s_1,r_1,i_1,i_2}(t^{p^{r_1}}, Y^{p^{r_1}}_{s_1,0}, \dots, Y^{p^{r_1}}_{s_1,s_1}, (Y^{(i_1)} + tX_{i_2})^{p^{r_1}})$$

(recall that here $Y^{(i_1)} \in \mathcal{L}_{s_1,\varkappa_{2,s_1}}$). By (x)$'$ (see the recursive assumption at the beginning of the section), if $s_1 \le n-2$ then

$$V'_{a^*,s_1,r_1} = \mathcal{Z}(\Phi^{(1)}_{a^*,s_1,r_1,i_1,i_2}, \forall i_1, i_2) \cap \mathbb{P}^n(\overline{k}),$$

and if $c = n-1$ then $V'_{a^*,n-1} = \mathcal{Z}(\varphi_{a^*,n-1})$.

Let u_1, u_2 be transcendental elements over the field $k(t)$. Let us extend the ground field k to $k(t)$. So, now (see (51)), a polynomial $\varphi^{\triangledown}_{a^*,n-1} \in k_{a^*}[t_1, \dots, t_s, Z]$ is defined if $c = n-1$, and also all polynomials $(\Phi^{(1)}_{a^*,s_1,r_1,i_1,i_2})^{\triangledown} \in k_{a^*}[t, t_1, \dots, t_s, Z]$ are defined if $s_1 \le n-2$. Put $c_1 = \min\{c, n-2\}$ and

$$\varphi_{a^*,s,r} = \varphi^{\triangledown}_{a^*,n-1} \cdot \prod_{s+1\le s_1\le c_1} \prod_{0\le r_1\le\rho_{s_1}} \left(\sum_{\substack{0\le i_1\le \varkappa_{2,s_1},\\ s_1+2\le i_2\le n}} u_1^{i_1} u_2^{i_2} (\Phi^{(1)}_{a^*,s_1,r_1,i_1,i_2})^{\triangledown} \right).$$

Set

$$g^{(7)}_{a^*,r} = \mathrm{GCD}_{t,u_1,u_2,t_1,\dots,t_s,Z}\left(g^{(5)}, \varphi_{a^*,s,r}\right).$$

Hence $g^{(7)}_{a^*,r} \in k_{a^*}[t, u_1, u_2, t_1, \dots, t_s, Z]$, and the degrees $\deg_{t_1,\dots,t_s,Z} g^{(7)}_{a^*,r}$ and $\deg_{t,u_1,u_2} g^{(7)}_{a^*,r}$ are bounded from above by $D^{O(1)}_{n-s+1}$. Put

$$g^{(8)}_{a^*,r} = \mathrm{LC}_t(\mathrm{LC}_{u_1}(\mathrm{LC}_{u_2}(g^{(7)}_{a^*,r}))).$$

So, for arbitrary $s \le c$, by the definitions of the algebraic varieties $V''''_{a^*,s}$ and $C''''_{a^*,s}$ and Lemma 16(d), the polynomial $g^{(8)}_{a^*,r}(t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r})$ coincides with

$$\prod_{\eta \in C''''_{a^*,s,r}} (Z - (Y_{s+1}/Y_0)(\eta))^{p^r}$$

up to a nonzero factor from k_{a^*} (we leave the details to the reader).

Using Lemma 2 from [6] and Noether normalization, we compute the polynomial

$$g^{(9)}_{a^*,r} \in k_{a^*}[t_1, \dots, t_s, Z_1]$$

coinciding with $g^{(6)}_{a^*,r}/g^{(8)}_{a^*,r}$ up to a nonzero factor from k_{a^*}. By the definitions of the algebraic varieties $V_{a^*,s}$ and $C_{a^*,s}$ and Lemma 16(d), the polynomials $g^{(9)}_{a^*,r}(t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r})$ and $\prod_{\eta \in C_{a^*,s,r}} (Z - (Y_{s+1}/Y_0)(\eta))^{p^r}$ coincide up to a nonzero factor from k_{a^*}. Put $e_9 = \deg_Z g^{(9)}_{a^*,r}$. Set

$$\Phi_{a^*,s,r} = Y_0^{e_9} g^{(9)}_{a^*,r}(Y_1/Y_0, \dots, Y_s/Y_0, Y_{s+1}/Y_0).$$

Then $\Phi_{a^*,s,r} \in k_{a^*}[Y_0, \dots, Y_{s+1}]$ is a homogeneous polynomial in $Y_0, \dots, Y_{s+1}$. Furthermore, all the assertions of (iv)′ and (v)′ for $W_{a^*,s} = V_{a^*,s}$ (see the beginning of the section) hold true.

Now assume that $s \le n-2$. In this case, let $Y^{(i_1)} \in \mathcal{L}'_s$, $0 \le i_1 \le \varkappa_{2,s}$, see the introduction. Let i_2 be an integer, $s+2 \le i_2 \le n$. Consider the resultant

$$\varphi^{(9)}_{a^*,r,i_1,i_2} = \mathrm{Res}_{Z_1}\left(\delta_{a^*,r} Z - ((Y^{(i_1)} + tX_{i_2})^{\nabla}|_{Z=Z_1}),\ g^{(9)}_{a^*,r}(t_1, \dots, t_s, Z_1)\right).$$

Then $\varphi^{(9)}_{a^*,r,i_1,i_2} \in k_{a^*}[t, t_1, \dots, t_s, Z]$, and $\varphi^{(9)}_{a^*,r,i_1,i_2}(t^{p^r}, t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r})$ coincides with

$$\delta^{e_9}_{a^*,r}(t_1^{p^r}, \dots, t_s^{p^r}) \prod_{\eta \in C_{a^*,s,r}} \left(Z - ((Y^{(i_1)}/Y_0) + t(X_{i_2}/Y_0))(\eta)\right)^{p^r}$$

up to a nonzero factor from k_{a^*}.

Using Lemma 2 from [6] and Noether normalization, we compute the polynomial

$$\psi^{(9)}_{a^*,r,i_1,i_2} \in k_{a^*}[t, t_1, \dots, t_s, Z]$$

coinciding with $\varphi^{(9)}_{a^*,r,i_1,i_2}/\delta^{e_9}_{a^*,r}$ up to a nonzero factor from k_{a^*}. Now $\mathrm{lc}_Z \psi^{(9)}_{a^*,r,i_1,i_2} \in k_{a^*}$, and $\psi^{(9)}_{a^*,r,i_1,i_2}(t^{p^r}, t_1^{p^r}, \dots, t_s^{p^r},\ Z^{p^r})$ coincides with $\prod_{\eta \in C_{a^*,s,r}} \left(Z - ((Y^{(i_1)}/Y_0) + t(X_{i_2}/Y_0))(\eta)\right)^{p^r}$ up to a nonzero factor from k_{a^*}. Set

$$\Psi_{a^*,s,r,i_1,i_2} = Y_0^{e_9} \psi^{(9)}_{a^*,r,i_1,i_2}(t, Y_1/Y_0, \dots, Y_s/Y_0, Z/Y_0).$$

Then $\Psi_{a^*,s,r,i_1,i_2} \in k_{a^*}[t, Y_0, \dots, Y_s, Z]$ is a homogeneous polynomial in $Y_0, \dots,\ Y_s, Z$. Furthermore, by Lemma 8, all the assertions of (xi)′ for $W_{a^*,s} = V_{a^*,s}$ (see the beginning of the section) hold true.

Now we return to the case of arbitrary s, c with $0 \le s \le c \le n-1$. We apply the modified version of Theorem 1 from [6] (see Remark 2 from the introduction) and construct the decomposition of the polynomial $\Phi_{a^*,s,r}$ into absolutely irreducible factors in the ring

$\overline{k}[Y_0, \ldots, Y_{s+1}]$. This decomposition can be obtained by a multivalued computation forest, see the remark at the end of Sec. 2.

So, we get a finite (or empty) family of polynomials $H_{a^*,j} \in k_{a^*}[Z]$, $j \in J_{a^*,s,r}$, their discriminants $0 \neq \Delta_{a^*,j} \in \overline{k}$, the set of roots $\Xi_{a^*,j}$ of each polynomial $H_{a^*,j}$, nonzero elements $\lambda_{a^*,s,r,0}$, $\lambda_{a^*,s,r,1} \in k_{a^*}$, the polynomials $\Phi_{a^*,j} \in k_{a^*}[Z, Y_1, \ldots, Y_{s+1}]$, $j \in J_{a^*,s,r}$, the irreducible (over $\overline{k}$) components $W_{a^*,j,\xi}$, $\xi \in \Xi_{a^*,j}$, $j \in J_{a^*,s,r}$, of the algebraic variety $V_{a^*,s,r}$. These objects satisfy properties (vi)′–(viii)′ for $W_{a^*,s} = V_{a^*,s}$ (see the beginning of the section), with only one exception. Namely, applying the construction from [6], we get $\varphi_{a^*,j,0} = \mathrm{lc}_{Y_{s+1}} \Phi_{a^*,j} \in k_{a^*}[Z]$ with $\deg_Z \varphi_{a^*,j,0} < \deg_Z H_{a^*,j}$. But in (vi)′ we need $\varphi_{a^*,j,0} \in k_{a^*}$.

Still, we can satisfy the last condition replacing each polynomial $\Phi_{a^*,j}$ by a new polynomial. Namely, we proceed as follows. There are polynomials $A_j, B_j \in k_{a^*}[Z]$ such that $\deg_Z A_j < \deg_Z H_{a^*,j}$, $\deg_Z B_j < \deg_Z \varphi_{a^*,j,0}$, and $A_j \varphi_{a^*,j,0} + B_j H_{a^*,j} = \mathrm{Res}_Z(\varphi_{a^*,j,0}, H_{a^*,j})$ where $0 \neq \mathrm{Res}_Z(\varphi_{a^*,j,0}, H_{a^*,j}) \in k_{a^*}$. Let $\deg_{Y_{s+1}} \Phi_{a^*,j} = e_1$. Put

$$\widetilde{\Phi}_{a^*,j} = A_j \Phi_{a^*,j} + (\mathrm{Res}_Z(\varphi_{a^*,j,0}, H_{a^*,j}) - A_j \varphi_{a^*,j,0}) Y_{s+1}^{e_1}.$$

Let $\deg_Z \widetilde{\Phi}_{a^*,j} = e_2$, $\deg_Z H_{a^*,j} = e_3$, and $e_{2,3} = \max\{e_2 - e_3 + 1, 0\}$. Furthermore, using Lemma 2 from [6], we can write $(\mathrm{lc}_Z H_{a^*,j})^{e_{2,3}} \widetilde{\Phi}_{a^*,j} = Q_{a^*,j} H_j + R_{a^*,j}$ where $Q_{a^*,j}, R_{a^*,j} \in k_{a^*}[Y_0, \ldots, Y_{s+1}, Z]$, $\deg_Z R_{a^*,j} < \deg_Z H_{a^*,j}$, and

$$\mathrm{lc}_{Y_{s+1}} R_{a^*,j} = (\mathrm{lc}_Z H_{a^*,j})^{e_{2,3}} \mathrm{Res}_Z(\varphi_{a^*,j,0}, H_{a^*,j}) \in k_{a^*}.$$

Finally, we replace each $\Phi_{a^*,j}$ by $R_{a^*,j}$. This involves also replacing $\lambda_{a^*,s,r,0}$, $\lambda_{a^*,s,r,1}$. Actually, one can take the new elements $\lambda_{a^*,s,r,0} = \mathrm{lc}_Z \Phi_{a^*,s,r}$, $\lambda_{a^*,s,r,1} = \prod_{j \in J_{s,r}} \mathrm{lc}_Z R_{a^*,j}$. Now, properties (vi)′–(viii)′ are fulfilled for $W_{a^*,s} = V_{a^*,s}$.

Recall that $\delta^{(0)} = \det((y_{i,j})_{0 \leq i,j \leq n})$, see (29). Put $G_{a^*,s,r} = (\delta^{(0)})^{p^r} \delta_{a^*,r}$. We are going to define $G_{a^*,s,r,i}$ for $0 \leq i \leq n$. Let us write $\delta^{(0)} X_i = \sum_{0 \leq j \leq n} x_{i,j} Y_j$, $0 \leq i \leq n$, where $x_{i,j} \in \overline{k}$. Put $\varepsilon_{a^*,Y_i,r} = \delta_{a^*,L_i,r}$ for $s+2 \leq i \leq n$; $\varepsilon_{a^*,Y_{s+1},r} = Z \delta_{a^*,r}$, $\varepsilon_{a^*,Y_0,r} = \delta_{a^*,r}$, $\varepsilon_{a^*,Y_i,r} = t_i \delta_{a^*,r}$ for $1 \leq i \leq s$. Finally, put $G_{a^*,s,r,i} = \sum_{0 \leq i \leq n} x_{i,j}^{p^r} \varepsilon_{a^*,s,r,j}$ for $0 \leq i \leq n$.

Now we are going to construct all polynomials $G_{a^*,j}$, $G_{a^*,j,i}$ for $0 \leq i \leq n$. Put

$$\varphi_{a^*,j} = \Phi_{a^*,j}(Z, 1, t_1, \ldots, t_s, Y).$$

So, $\deg_Y \varphi_{a^*,j} = e_1$ and $\varphi_{a^*,j,0} = \mathrm{lc}_Y \varphi_{a^*,j} \in k_{a^*}$. Let $\deg_Z \varphi_{a^*,j} = e_4$, $\deg_Z G_{a^*,X_i,r} = e_{6,i}$, and $e_{6,4} = \max_i\{e_{6,i} - e_4 + 1, 0\}$. Using Lemma 2 from [6], we can write

$$\varphi_{a^*,j,0}^{e_{6,4}} G_{a^*,X_i,r}(t_1, \ldots, t_s, Y) = Q_{a^*,i} \varphi_{a^*,j} + R_{a^*,i},$$

where $Q_{a^*,i}, R_{a^*,i} \in k_{a^*}[Z, t_1, \ldots, t_s, Y]$ and $\deg_Y R_{a^*,i} < \deg_Y \varphi_{a^*,j}$. Let $\deg_Z R_{a^*,i} = e_{7,i}$ and $e_{7,3} = \max_i\{e_{7,i} - e_3 + 1, 0\}$. Again using Lemma 2 from [6], we can write

$$(\mathrm{lc}_Z H_{a^*,j})^{e_{7,3}} R_{a^*,i} = Q'_{a^*,i} \varphi_{a^*,j} + R'_{a^*,i},$$

where $Q'_{a^*,i}, R'_{a^*,i} \in k_{a^*}[Z, t_1, \ldots, t_s, Y]$, $\deg_Y R'_{a^*,i} < \deg_Y \varphi_{a^*,j}$, $\deg_Z R'_{a^*,i} < \deg_Z H_{a^*,j}$. Put

$$G_{a^*,j} = \delta_{a^*,r} \varphi_{a^*,j}^{e_{6,4}} (\mathrm{lc}_Z H_{a^*,j,0})^{e_{7,3}}, \quad G_{a^*,j,i} = R'_{a^*,i}.$$

Now properties (x)′ and (ix)′ are fulfilled for $W_{a^*,s} = V_{a^*,s}$ (note that one should replace everywhere in (x) (and hence also in (x)′) max by $\max_i$; it is a small correction, see the introduction to the second part).

Let $s \le n-2$ and $j \in J_{a^*,s,r}$. Now we are going to construct all the polynomials Ψ_{a^*,j,i_1,i_2}. Let $Y^{(i_1)} \in \mathcal{L}'_s$, $0 \le i_1 \le \varkappa_{2,s}$, see the introduction. Let i_2 be an integer, $s+2 \le i_2 \le n$. Consider the resultant

$$\varphi_{a^*,j,i_1,i_2} = \mathrm{Res}_Y \left(\delta_{a^*,r} Z_1 - ((Y^{(i_1)} + tX_{i_2})^{\nabla}|_{Z=Y}),\ \varphi_{a^*,r}(Z, t_1, \dots, t_s, Y) \right).$$

Then $\varphi_{a^*,j,i_1,i_2} \in k_{a^*}[Z, t, t_1, \dots, t_s, Z_1]$. For every $\xi \in \Xi_{a^*,j}$, the element

$$\varphi_{a^*,r,i_1,i_2}(t^{p^r}, t_1^{p^r}, \dots, t_s^{p^r}, Z^{p^r})|_{Z=\xi}$$

coincides with

$$\delta_{a^*,r}^{e_1}(t_1^{p^r}, \dots, t_s^{p^r}) \prod_{\eta \in \iota_{5,6}(W_{a^*,j,\xi})} \left(Z_1 - ((Y^{(i_1)}/Y_0) + t(X_{i_2}/Y_0))(\eta) \right)^{p^r}$$

up to a nonzero factor from k_{a^*}. Hence

$$(\varphi_{a^*,j,i_1,i_2}|_{Z=\xi}) / \delta_{a^*,r}^{e_1} = \varphi_{a^*,j,i_1,i_2,\xi} \in \overline{k}[t, t_1, \dots, t_s, Z_1], \quad \xi \in \Xi_{a^*,j}. \tag{52}$$

Let us write

$$\varphi_{a^*,j,i_1,i_2} / \delta_{a^*,r}^{e_1} = \sum_{0 \le i < e_3} \varphi_{a^*,j,i_1,i_2,i} Z^i$$

where $\varphi_{a^*,j,i_1,i_2,i} \in k_{a^*}(t_1, \dots, t_s)[t, Z_1]$. Solving the linear system

$$\sum_{0 \le i < e_3} U_i \xi^i = \varphi_{a^*,j,i_1,i_2,\xi}, \quad \xi \in \Xi_{a^*,j},$$

with respect to the unknowns $U_0, \dots, U_{e_3-1}$, we deduce that $\varphi_{a^*,j,i_1,i_2,i} \in k_{a^*}[t, t_1, \dots, t_s, Z_1]$. Therefore, $\varphi_{a^*,j,i_1,i_2} / \delta_{a^*,r}^{e_1} \in k_{a^*}[Z, t, t_1, \dots, t_s,\ Z_1]$.

Using Lemma 2 from [6] and Noether normalization, we compute the polynomial

$$\psi_{a^*,j,i_1,i_2} \in k_{a^*}[Z, t, t_1, \dots, t_s, Z]$$

coinciding with $\varphi_{a^*,j,i_1,i_2} / \delta_{a^*,r}^{e_3}$ up to a nonzero factor from k_{a^*}. Now $\mathrm{lc}_Z \psi_{a^*,j,i_1,i_2} \in k_{a^*}$, and $\psi_{a^*,j,i_1,i_2}(t^{p^r}, t_1^{p^r}, \dots, t_s^{p^r},\ Z^{p^r})|_{Z=\xi}$ coincides with

$$\prod_{\eta \in \iota_{5,6}(W_{a^*,j,\xi})} \left(Z - ((Y^{(i_1)}/Y_0) + t(X_{i_2}/Y_0))(\eta) \right)^{p^r}$$

up to a nonzero factor from k_{a^*} for every $\xi \in \Xi_{a^*,j}$. Set

$$\Psi_{a^*,j,i_1,i_2} = Y_0^{e_3} \psi_{a^*,j,i_1,i_2}(Z, t, Y_1/Y_0, \dots, Y_s/Y_0, Z_1/Y_0).$$

Then $\Psi_{a^*,j,i_1,i_2} \in k_{a^*}[Z, t, Y_0, \dots, Y_s, Z_1]$ is a homogeneous polynomial in $Y_0, \dots,\ Y_s, Z_1$. Furthermore, by Lemma 8, all the assertions of (xii)$'$ for $W_{a^*,s} = V_{a^*,s}$ (see the beginning of the section) hold true.

Thus, now properties (iv)$'$–(xii)$'$ with $s = q$ are fulfilled for $W_{a^*,s} = V_{a^*,s}$. It remains to construct the polynomial $h_{a^*,n-s+1}$ and all polynomials $q_{a^*,n-s+1,w}$, $n-s+1 \le w \le m-1$, if $V'''_{a^*,s} \ne \varnothing$ and $s = q > c' - 1$. The variety $V'''_{a^*,s}$ is not empty if and only if $\deg_Z g^{(6)}_{a^*,s,r} \ne 0$ for some r where $0 \le r \le \rho$. Hence, in this case $s \ge n - m + 1$. Now put

$$\delta^{(6)}_{a^*,r} = \mathrm{Res}_Z \left(g^{(6)}_{a^*,r}, (\widetilde{h}_{a^*,n-s+1})^{\nabla} \right).$$

So, $0 \ne \delta^{(6)}_{a^*,r} \in k_{a^*}[t, t_1, \dots, t_s]$. Put $N_1 = \sum\limits_{0 \le r \le \rho} \deg_t \delta^{(6)}_{a^*,r}$. We enumerate the elements of $\mathcal{I}_{N_1}$ and find $t' \in \mathcal{I}_{N_1}$ such that $\left(\prod\limits_{0 \le r \le \rho} \delta^{(6)}_{a^*,r} \right) \Big|_{t=t'} \ne 0$. Put $t_{a^*,n-s+1} = t'$ and $q_{a^*,n-s+1,w} =$

$q_{n-s+1,w}|_{t=t'}$ for $n-s+1 \le w \le m-1$; $q_{a^*,n-s+1,n-s} = 1$ and $q_{a^*,n-s+1,w} = 0$ for $0 \le w \le n-s-1$. Set $h_{a^*,n-s+1} = \widetilde{h}_{a^*,n-s+1}|_{t=t'}$, see Sec. 4. Then, obviously, $\dim(V'''_{a^*,s} \cap \mathcal{Z}(h_{a^*,n-s+1})) = s-1$.

If $V'''_{a^*,s} \ne \varnothing$ and $s = q > c'-1$, we proceed to the next, $(q-1)$th, recursive step.

If $V'''_{a^*,s} = \varnothing$ or $s = q = c'-1$, then the qth step is the final one. We set $c'_{a^*} = s = q$. In this case, we have $V_{a^*,s_2} = \varnothing$ and $V_{a^*,s_2,r_2} = \varnothing$ for $0 \le s_2 \le q-1$, $0 \le r_2 \le \rho_{s_2}$. We put $\Phi_{a^*,s_2,r_2} = 1$, $\lambda_{a^*,s_2,r_2,0} = \lambda_{a^*,s_2,r_2,1} = 1$, $J_{a^*,s_2,r_2} = \varnothing$, and $\Psi_{a^*,s_2,r_2,i_1,i_2} = 1$ for $0 \le s_2 \le q-1$, $0 \le r_2 \le \rho_{s_2}$, $0 \le i_1 \le \varkappa_{2,s_2}$, $s_2+2 \le i_2 \le n$.

Put

$$\begin{aligned}\mathcal{Q}_{a^*} = \Big(&\{(\Phi_{a^*,s,r}, \lambda_{a^*,s,r,0}, \lambda_{a^*,s,r,1})\}_{\forall s,r},\ \{G_{a^*,s,r,i}\}_{\forall s,r,i},\ \{\Psi_{a^*,s,r,i_1,i_2}\}_{\forall s,r,i_1,i_2},\\ &\{(H_{a^*,j}, \Delta_{a^*,j}, \Phi_{a^*,j})\}_{j\in J_{a^*,s,r},\forall s,r},\ \{\Psi_{a^*,j,i_1,i_2}\}_{j\in J_{a^*,s,r},\forall i_1,i_2,\forall s,r},\\ &\{h_{a^*,i}\}_{1\le i\le n-c'_{a^*}},\ \{q_{a^*,i,i_1}\}_{1\le i\le n-c'_{a^*},\,0\le i_1\le m-1}\Big).\end{aligned}$$

So, $\mathcal{Q}_{a^*}$ is a 7-tuple of some families. Elements of these families are defined above.

Now, similarly to Sec. 3, under condition ($\mathfrak{g}$) the described construction defines a multivalued function (or a binary relation)

$$\mathfrak{F} : \bigcup_{n,d_0,\ldots,d_{m-1}} \overline{k}^{\gamma_0+\ldots+\gamma_{m-1}} \to \mathcal{K}, \quad a^* \mapsto \mathcal{Q}_{a^*},$$

which is an algorithm corresponding to a multivalued computation forest

$$T_1 = \{T_{1,n,d_0,\ldots,d_{m-1}}\}_{\forall n,d_0,\ldots,d_{m-1}}$$

in the sense of Sec. 2. So, $\mathfrak{F} = \mathfrak{F}(T_1)$. The values of this function depend on the choice of linear forms $Y_{s,0},\ldots,Y_{s,s+1}$ for $c' \le s \le \min\{c, n-1\}$.

Let v be a vertex of the tree $T_{1,n,d_0\ldots d_{m-1}}$. Then the quasiprojective algebraic variety

$$\mathcal{W}_v = \mathcal{Z}(\psi_{v,1},\ldots,\psi_{v,\mu_{v,1}}) \setminus \mathcal{Z}(\psi_{v,\mu_{v,1}+1},\ldots,\psi_{v,\mu_{v,2}}) \subset \mathcal{U}_c \tag{53}$$

corresponds to v.

Take $A_{\mathfrak{g}} = L(T_{1,n,d_0\ldots d_{m-1}})$ to be the set of leaves of the tree $T_{1,n,d_0\ldots d_{m-1}}$. Let $\alpha \in A_{\mathfrak{g}}$. Then (see (53) with $v = \alpha$) all polynomials $\psi_{\alpha,j} \in k[a_1,\ldots,a_\nu]$ have degrees bounded from above by $D_{n-c'}^{O(1)}$ with an absolute constant in $O(1)$. Note also that each leaf α is of level $\mathrm{l}(\alpha) = D_{n-c'}^{O(1)}$. We have $\bigcup_{\alpha\in A_{\mathfrak{g}}} \mathcal{W}_\alpha = \mathcal{U}_c$, i.e., we get a covering of the set $\mathcal{U}_c$.

Furthermore, the 7-tuple

$$\begin{aligned}\mathcal{Q}_{\alpha} = \Big(&\{(\Phi_{\alpha,s,r}, \lambda_{\alpha,s,r,0}, \lambda_{\alpha,s,r,1})\}_{\forall s,r},\ \{G_{\alpha,s,r,i}\}_{\forall s,r,i},\ \{\Psi_{\alpha,s,r,i_1,i_2}\}_{\forall s,r,i_1,i_2},\\ &\{(H_{\alpha,j}, \Delta_{\alpha,j}, \Phi_{\alpha,j})\}_{j\in J_{\alpha,s,r},\forall s,r},\ \{\Psi_{\alpha,j,i_1,i_2}\}_{j\in J_{\alpha,s,r},\forall i_1,i_2,\forall s,r},\\ &\{h_{\alpha,i}\}_{1\le i\le n-c'_{\alpha}},\ \{q_{\alpha,i,i_1}\}_{1\le i\le n-c'_{\alpha},\,0\le i_1\le m-1}\Big)\end{aligned}$$

corresponds to α. Here, all the objects from the right-hand side are defined in the introduction, and for them conditions (iv)–(xiii) hold true. Besides, for $c' \le s \le \min\{c, n-1\}$, linear forms $Y_{s,0},\ldots,Y_{s,s+1}$ correspond to α.

Now, for every $a^* \in \mathcal{W}_\alpha$ we have $\mathcal{Q}_{a^*} = \mathcal{Q}_\alpha|_{a_1=a_1^*,\ldots,a_\nu=a_\nu^*}$. In particular, $c'_\alpha = c'_{a^*}$, and we identify $J_{\alpha,s,r}$ with $J_{a^*,s,r}$ and $\Xi_{a^*,j}$ with Ξ_{j,a^*} for all $j \in J_{\alpha,s,r}$ and for all s, r. The linear forms $Y_{s,0},\ldots,Y_{s,s+1}$ corresponding to α coincide with the ones appearing in the main recursion for the element $a^* \in \mathcal{W}_\alpha$, see Sec. 6 and the present section.

From the description of the main recursion and the results of [6], it follows immediately that assertions (b), (c), and (d) of Theorem 1 are fulfilled with $A_{\mathfrak{g}}$ in place of A.

Moreover, let s be fixed. Put

$$\mathcal{Q}_{\alpha,s} = \Big(\{(\Phi_{\alpha,s,r}, \lambda_{\alpha,r,0}, \lambda_{\alpha,s,r,1})\}_{\forall r},\ \{G_{\alpha,s,r,i}\}_{\forall r,i},\ \{\Psi_{\alpha,s,r,i_1,i_2}\}_{\forall r,i_1,i_2},$$
$$\{(H_{\alpha,j}, \Delta_{\alpha,j}, \Phi_{\alpha,j})\}_{j\in J_{\alpha,s,r},\forall r},\ \{\Psi_{\alpha,j,i_1,i_2}\}_{j\in J_{\alpha,r},\forall i_1,i_2,\forall r}\Big).$$

Then, from the description of the main recursion and the results of [6], it follows that all the objects from the left-hand side of the last equality are computed already at some vertex v of the tree $T_{1,n,d_0,\dots,d_{m-1}}$ with level $\mathrm{l}(v) = D_{n-s}^{O(1)}$. The leaf α is a descendant of v. Furthermore (see (53)), all polynomials $\psi_{v,j} \in k[a_1,\dots,a_\nu]$ have degrees bounded from above by $D_{n-s}^{O(1)}$ with an absolute constant in $O(1)$.

Assume that condition ($\mathfrak{g}$) does not necessarily hold. Denote by f the family of coefficients from $k[a_1,\dots,a_\nu]$ of all the polynomials $f_0,\dots,f_{m-1}$. Then, by Theorem 3 applied to the tree $T_{1,d_0,\dots,d_{m-1}}(f)$ (see the definition of this tree in Sec. 2), we get an irredundant subtree $T'_{1,d_0,\dots,d_{m-1}}(f)$ of the tree $T_{1,d_0,\dots,d_{m-1}}(f)$ such that $\mathcal{S}(T'_{1,d_0,\dots,d_{m-1}}(f) = \mathcal{S}(T_{1,d_0,\dots,d_{m-1}}(f)$. Put $A = L(T'_{1,d_0,\dots,d_{m-1}}(f)$. Now, all assertions of the modified Theorem 1 hold true. Thus, the modified Theorem 1 is proved.

Put $\Gamma = A$ and assume that A is not used in any notation introduced earlier (i.e., we change the notation). Finally, applying Lemma 3 to the covering $\mathcal{U}_c = \bigcup_{\gamma\in\Gamma} \mathcal{W}_\gamma$, we prove Theorem 1.

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