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A REMARK ON THE ISOMORPHISM BETWEEN THE BERNOULLI SCHEME AND THE PLANCHEREL MEASURE

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We formulate a theorem of Romik and Śniady that establishes an isomorphism between the Bernoulli scheme and the Plancherel measure. Then we derive several combinatorial results as corollaries. The first one is related to measurable partitions; the other two are related to the Knuth equivalence. We also give several examples and one conjecture belonging to A. Vershik. Bibliography: 7 titles.

1. Introduction

The RSK algorithm is a well-known correspondence that maps a permutation of length n to a pair of Young tableaux of size n with the same shape. A. Vershik and S. Kerov [1] extended the definition of the RSK correspondence to the space $\mathcal{A}^n$ where $\mathcal{A}$ is an arbitrary lineary ordered set. In this case, the insertion tableau (P) contains elements from $\mathcal{A}$ and the recording tableau (Q) contains the numbers $1, 2, \dots, n$. Thus, we consider the map RSK : $\mathcal{A}^n \to \mathbb{YT}_n(\mathcal{A}) \times \mathbb{YT}_n$ where $\mathbb{YT}_n$ is the set of standard Young tableaux of size n and $\mathbb{YT}_n(\mathcal{A})$ is the set of semistandard Young tableaux filled with elements from $\mathcal{A}$. Let us for the moment "forget" about the first coordinate and consider this transformation as a map from $\mathcal{A}^n$ to $\mathbb{YT}_n$. In this case, as observed by Vershik and Kerov, we can pass to the limit and obtain a map $\mathrm{RSK} : \mathcal{A}^{\mathbb{N}} \to \mathbb{YT}_\infty$. Here $\mathbb{YT}_\infty$ is the set of infinite Young tableaux.

Vershik and Kerov [1] proved that for every central ergodic measure μ on $\mathbb{YT}_\infty$ there is an alphabet $\mathcal{A}$ and a measure m on $\mathcal{A}$ such that the map $\mathrm{RSK} : (\mathcal{A}, m)^{\mathbb{N}} \to (\mathbb{YT}_\infty, \mu)$ is a homomorphism of measure spaces. In particular, if μ is the Plancherel measure, then the corresponding alphabet $\mathcal{A}$ is the interval $[0, 1]$ with the one-dimensional Lebesgue measure m. D. Romik and P. Śniady, in the recent paper [2], proved that in this case $\mathrm{RSK} : ([0, 1]^{\mathbb{N}}, \mathrm{Leb}_\infty) \to (\mathbb{YT}_\infty, \mathrm{Planch})$ is an isomorphism of measure spaces, where $\mathrm{Leb}_\infty = \bigotimes_{i=1}^{\mathbb{N}} \mathrm{Leb}_1([0, 1])$ and Planch is the Plancherel measure. Later, Śniady extended this result to a wider class of homomorphisms constructed by Vershik and Kerov in [3].

In this paper, we will discuss several corollaries of the theorem of Romik and Śniady suggested by A. Vershik. We will consider only the case where the alphabet $\mathcal{A}$ is the interval $[0, 1]$ with the Lebesgue measure Leb_1 and the central measure on $\mathbb{YT}_\infty$ is the Plancherel measure.

2. The isomorphism theorem

Here we state the result of Romik and Śniady ([2, Theorem 1.4]) in a convenient form.

Theorem 1. *The homomorphism defined in* [1] *from* $([0, 1]^{\mathbb{N}}, \mathrm{Leb}_\infty)$ *to the space of infinite Young tableaux equipped with the Plancherel measure is an isomorphism of measure spaces which maps the left shift to the Schützenberger transformation. This means that almost every realization of the Bernoulli scheme corresponds to a single infinite Young tableau.*

Below we discuss several corollaries of this theorem.

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3. Measurable partitions

3.1. The general case. In this subsection, we discuss the following general problem. Consider the infinite-dimensional cube $[0,1]^{\mathbb{N}}$ with the product measure Leb_∞. Assume that for every n there is a partition (up to a set of zero measure) $\widetilde{\xi}_n$ of the cube $[0,1]^n$ into a finite collection of measurable sets. Moreover, assume that this sequence of partitions is

(1) increasing, i.e., the projection of every element from $\widetilde{\xi}_n$ onto the first $n-1$ coordinates lies in some element from $\widetilde{\xi}_{n-1}$,
(2) stationary, i.e., the projection of every element from $\widetilde{\xi}_n$ onto the last $n-1$ coordinates lies in some element from $\widetilde{\xi}_{n-1}$.

Fix n, and for every element $a \in \widetilde{\xi}_n$ consider the set $a \times [0,1]^{\mathbb{N}}$. The collection of all such sets forms a partition of $[0,1]^n \times [0,1]^{\mathbb{N}} = [0,1]^{\mathbb{N}}$, which we denote by ξ_n. Note that the increasing property implies that elements of ξ_n are disjoint unions of elements of ξ_{n+1}, and the stationarity implies that the left shift maps ξ_n to a partition that is a refinement of ξ_{n-1}.

Definition 1. We say that such a sequence of partitions converges to the partition into separate points and write $\xi_n \to \epsilon$ if almost every point x of the cube $[0,1]^{\mathbb{N}}$ is uniquely determined by the sequence of sets $a_n \in \xi_n$ containing x.

Consider two examples.

Example 1. Let $\mathcal{X}_n$ be the partition of the interval $[0,1]$ into 2^n intervals of the form $[m/2^n, (m+1)/2^n]$, $m \in \mathbb{Z}$. Set $\widetilde{\xi}_n = \otimes_{k=1}^n \mathcal{X}_n$. We claim that $\xi_n \to \epsilon$.

Proof. Indeed, let $x = (x_1, x_2, \ldots) \in [0,1]^{\mathbb{N}}$ and fix $k \in \mathbb{N}$. For every $n > k$, the element of the partition ξ_n that contains x determines the coordinate x_k up to an error of $1/2^n$. Thus, as $n \to \infty$, all coordinates of x can be recovered. □

Remark 1. It is easy to see that the number of elements in $\widetilde{\xi}_n$ equals 2^{n^2}.

Example 2. Now consider the partition $\widetilde{\xi}_n$ of the cube into the simplices of the form

$$\left\{ y \in [0,1]^n \mid y_{\sigma(1)} \le y_{\sigma(2)} \le \ldots \le y_{\sigma(n)} \right\}$$

indexed by the permutations $\sigma \in S_n$. Then, again, $\xi_n \to \epsilon$.

Proof. Again, take $x = (x_1, x_2, \ldots) \in [0,1]^{\mathbb{N}}$ and fix $k \in \mathbb{N}$. Note that the element of ξ_n containing x uniquely determines the order of the numbers $x_1, x_2, \ldots, x_n$. It follows that for every $n > k$ we know the value $|\{l \le n \mid x_l < x_k\}|$, which is the number of elements among $x_1, x_2, \ldots, x_n$ that are smaller than x_k. The law of large numbers says that for almost every x

$$x_k = \lim_{n \to \infty} \frac{|\{l \le n \mid x_l < x_k\}|}{n},$$

which means that in the limit every coordinate can be determined almost surely. □

Remark 2. In this case, the number of elements in $\widetilde{\xi}_n$ equals $n!$.

3.2. A corollary of the isomorphism theorem. We can formulate the first corollary in terms of measurable partitions. Here RSK is a map from $[0,1]^n$ to $\mathbb{YT}_n$.

Corollary 1. *Consider the partition $\widetilde{\xi}_n$ of the cube $[0,1]^n$ into the sets of the form $\mathrm{RSK}^{-1}(\tau)$, where τ is an arbitrary Young tableau of size n. In other words, two points $y_1, y_2 \in [0,1]^n$ are in the same block of the partition if and only if their recording tableaux coincide. Then $\xi_n \to \epsilon$.*

Proof. Indeed, the infinite Young tableau that is the RSK image of a sequence $(x_1, x_2, \ldots)$ is exactly the limit of the sequence of the finite tableaux Q_n obtained by applying RSK to the n-tuples $(x_1, x_2, \ldots, x_n)$. The claim follows from Theorem 1. □

Remark 3. The number of elements in the partition $\widetilde{\xi}_n$ is equal to the number of different Young tableaux of size n. This number a_n is well studied: for example, it equals the number of involutions in the symmetric group S_n. Asymptotically, as $n \to \infty$,

$$a_n \sim \sqrt{n!}e^{\sqrt{n}}(8\pi en)^{-1/4}.$$

This formula was obtained in [5, p. 583].

The following open question was posed by A. Vershik.

Problem. What is the minimum possible asymptotics for the number of elements in an increasing stationary sequence of partitions that converges to the partition into separate points?

4. The Knuth equivalence

4.1. Dual Knuth partitions. Now consider the classical version of RSK, which is a map from S_n to $\mathbb{YT}_n \times \mathbb{YT}_n$. It is well known that certain transpositions of consecutive numbers in a permutation, called Knuth transformations, preserve the P-tableau of the permutation (see, e.g., [4]). It is also well known that if $\mathrm{RSK}(\sigma) = (P, Q)$, then $\mathrm{RSK}(\sigma^{-1}) = (Q, P)$. Thus, it is natural to define the dual Knuth transformations as the images of the Knuth transformations under the map $\sigma \mapsto \sigma^{-1}$. One can also give an explicit description of these transformations similar to the definition of the usual Knuth transformations. Obviously, the dual transformations preserve the Q-tableau.

A collection of permutations that can be obtained from each other using Knuth transformations is called a Knuth class and corresponds to a single P-tableau. A dual Knuth class, corresponding to a single Q-tableau, is defined in a similar way.

Now consider RSK as a map from $[0,1]^n$ to $\mathbb{YT}_n([0,1]) \times \mathbb{YT}_n$. The definition of the Knuth transformations can be extended to this case in an obvious way, and all the points that can be obtained from a given point constitute a Knuth class, a collection of points in $[0,1]^n$ with the same P-tableau. However, since on $[0,1]^n$ there is no group structure respected by RSK, we have no good definition of a dual Knuth transformation. Nevertheless, we can define a dual Knuth class as the set of points corresponding to a single Q-tableau. The set of dual Knuth classes can be endowed with the Plancherel measure in a natural way (note that the measure of a Knuth class coincides with Lebesgue measure of this class regarded as a set in $[0,1]^n$).

Corollary 2. *For every $k \in \mathbb{N}$ and every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for every $n > N$ the following holds true: there exists a set of dual Knuth classes with Plancherel measure at least $1 - \epsilon$ on which the dispersion of the kth element is less than ε.*

Proof. This follows directly from the description of the map inverse to RSK obtained by Romik and Śniady ([2, Theorem 1.5]). ☐

Remark 4. It would be natural to ask whether the kth element actually stabilizes as n goes to infinity. The answer is negative: numerical simulations show that asymptotically it takes values from an interval of length 1/2.

4.2. Knuth partitions. Let us return to the language of partitions from Sec. 3.1. Let $\widetilde{\eta}_n$ be a partition of the cube $[0,1]^n$ whose all elements are finite. For every set $b \in \widetilde{\eta}_n$ and for every point $y = (y_{n+1}, y_{n+2}, \dots) \in [0,1]^{\mathbb{N}}$, consider the set $a \times \{y\} \in [0,1]^{\mathbb{N}}$. The collection of all such sets forms a partition of the cube $[0,1]^{\mathbb{N}}$, which we denote by η_n. Note that all elements in η_n are finite. Assume that the sequence of partitions η_n decreases, i.e., every element in η_{n+1} is a disjoint union of elements in η_n.

Definition 2. Let $x \in [0,1]^{\mathbb{N}}$, and for each n find the set $b_n(x) \in \eta_n$ such that $x \in b(x)$. Assume that for almost all x there is no measurable set $B(x) \subset [0,1]^{\mathbb{N}}$ such that

$$\bigcup_{n=1}^{\infty} b_n(x) \subset B(x) \quad \text{and} \quad 0 < \mathrm{Leb}_\infty(B(x)) < 1.$$

In this case, we say that the partition η_n converges to the trivial partition (or that this partition is ergodic) and write $\eta_n \to \nu$.

Theorem 2. *Consider a stationary increasing sequence of partitions* $\widetilde{\xi}_n$ *and a decreasing sequence of partitions* $\widetilde{\eta}_n$*. Assume that for every* n *the partitions* $\widetilde{\xi}_n$ *and* $\widetilde{\eta}_n$ *are the independent complements of each other.*

(1) *Assume that* $\xi_n \to \epsilon$*. It is not hard to prove that* $\eta_n \to \nu$*.*
(2) *The converse is not necessarily true. The first counterexample was constructed in* [7], *see also* [6, p. 18].

Let us give some examples.

Example 3. Consider the partition $\widetilde{\xi}_n$ from Example 1. Set

$$x = (x_1, x_2, \dots) \sim_n y = (y_1, y_2, \dots)$$

if

$$\begin{cases} 2^n(x_k - y_k) \in \mathbb{Z}, & 1 \le k \le n, \\ x_k = y_k, & k > n. \end{cases}$$

Then the independent complement $\widetilde{\eta}_n$ to the partition $\widetilde{\xi}_n$ consists exactly of the equivalence classes induced by the relation $\sim_n$.

Example 4. Consider the partition $\widetilde{\xi}_n$ from Example 2. Then the independent complement $\widetilde{\eta}_n$ consists of the orbits of the action of S_n on $[0,1]^n$. Theorem 2 implies that the action of the infinite symmetric group S_∞ on $[0,1]^{\mathbb{N}}$ is ergodic.

Applying Theorem 2 to Corollary 1 and using the description of partition blocks from Sec. 4.1, we obtain the following statement.

Corollary 3. *The Knuth transformations act ergodically on the* k*th element of the Bernoulli scheme* $([0,1], \mathrm{Leb}_1)^{\mathbb{N}}$ *for every* k*. This means the following. Take a random infinite sequence* $x = (x_1, x_2, \dots, x_n, \dots)$*. Let* x^n *be the finite sequence consisting of the first* n *elements of* x*. Let* γ_n *be the empirical distribution of the* k*th element in the Knuth class of* x^n*. Then* γ_n *tends to the uniform distribution on* $[0,1]$ *almost surely.*

5. Conclusion

Although Corollaries 2 and 3 are purely combinatorial, it is not known how to prove them directly. Numerical simulations carried out by the author and others show that the convergence to the delta-like and uniform distributions is very slow. It would be of interest to find a proof of these statements that does not use complicated constructions related to the limit form, Schützenberger transformation, theorem of Romik and Śniady, etc. Some observations on this topic, several of which were discussed above, can be found in a forthcoming paper by A. Vershik, to whom the author is grateful for information and guidance.

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