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CONTACT PROBLEM FOR A RIGID PUNCH AND AN ELASTIC HALF SPACE AS AN INVERSE PROBLEM

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We solve a contact problem of indentation of a punch into an elastic half space with regard for the friction and in the presence of the zones of adhesion, sliding, and separation. The applied approach is based on the statement of the problem in the form of the inverse problem in which the Coulomb law of friction is used as an additional condition in the regions with friction. In the formulation of the inverse problem, we take into account the presence of the zones of adhesion whose sizes are unknown. The correctness of the solution of the inverse problem is analyzed. The proposed approach, in combination with the procedure of discretization, enables us to determine the zones of microsliding alternating with the zones of adhesion and separation.

Introduction

We consider a problem of indentation of a rigid punch into the elastic half space in the presence of friction, adhesion, and separation. For the first time, the problem of indentation of a punch in the presence of friction and adhesion was approximately solved by Galin. Then the analogous solution of the problem was obtained with the help of equations from the Fuks class [6]. Significant difficulties were connected with the necessity of determination of additional constants, which required the introduction of special conditions and restricted the possibilities to solve the problem in the formulated statement. In the subsequent studies of the problem with one-sided connections, the researchers applied certain variational inequalities based of the Signorini problem [3]. The numerical methods used for the investigation of variational inequalities make it possible to guarantee the convergence of the iterative process only with respect to the direct variable in finding of the saddle point of the Lagrangian functional [3, 4, 8]. In order to find the solution, it is necessary to use a modified Lagrangian functional, which requires the evaluation of additional constants. Moreover, the mentioned statement does not enable one to solve the problem under the assumption of existence not only of the regions of sliding and separation but also the regions of adhesion.

Both the finite-element method [1, 2, 7, 10] and some other numerical methods [11] were extensively used for the solution of problems of this kind.

In the present work, we propose an approach to the solution of contact problems with regard for the presence of adhesion, friction, and separation based on the statement of the problem as an inverse problem in which, as an additional condition, we use the validity of Coulomb's law in the regions with friction with regard for the presence of the zones of adhesion whose sizes are unknown in the inverse problem.

1. Mathematical Model

The system of equations of the theory of elasticity for a half space $\Omega = \{x\colon\ x = (x_1, x_2, x_3) \in \mathbb{R}^3\}$ with piecewise smooth boundary Γ formed by the union of disjoint open sets Γ_U, Γ_σ, and Γ_k, has the form

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$$L(U,F) \equiv (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,U + \mu\nabla^2 U + F = 0\,, \tag{1}$$

where $\lambda = \nu E/\big((1+\nu)(1-2\nu)\big)$, $\mu = E/(2+2\nu)$ are the Lamé coefficients; E is the elasticity modulus; ν is Poisson's ratio; $F = \{F_1, F_2, F_3\}$ is the vector of loads, $U = \{U_1, U_2, U_3\}$ is the vector of displacements along the axes x_1, x_2, and x_3, respectively; Γ_U and Γ_σ are, respectively, the regions, where the displacements and stresses are given; Γ_k is the region occupied by the punch formed by the zones of adhesion, separation, and friction with unknown boundaries. Thus, we get

$$U(x) = U_0(x), \quad x \in \Gamma_U\,, \qquad \sigma_{ij}(U)\,n_j = f_i(x), \quad x \in \Gamma_\sigma\,, \tag{2}$$

where $n = \{n_1, n_2, n_3\}$ is the unit vector of normal to the boundary Γ_σ and σ_{ij} are stresses, $i, j = 1, 2$. For the plane problem, in the zone of adhesion Γ_k^C, we find

$$\begin{gathered} U_2 \le \delta_1\,, \qquad \sigma_{22} \le 0\,, \qquad U_2\sigma_{22} = 0 \\ |\sigma_{12}| < k|\sigma_{22}|\,, \qquad U_1' = 0\,, \quad k \ge 0\,, \end{gathered} \tag{3}$$

where k is the friction coefficient and δ_1 is the indentation of the punch.

If $|\sigma_{12}| > k|\sigma_{22}|$, then the zone of friction Γ_k^T is present. Thus, the following conditions should be satisfied:

$$k|\sigma_{22}| - |\sigma_{12}| = 0 \qquad \text{for} \quad \sigma_{22} \le 0\,, \quad U_2 = \delta_1\,. \tag{4}$$

If $\sigma_{22} > 0$, then we observe the formation of the zone of separation Γ_k^O. In this case, we get the following conditions:

$$\sigma_{22} = 0\,, \qquad \sigma_{12} = 0 \qquad \text{for} \quad U_2 > \delta_1\,. \tag{5}$$

Note that the values of $\pm x_1^T$ and $\pm x_1^O$ specifying the boundaries of the sections Γ_k^C, Γ_k^T, and Γ_k^O are unknown (x_1^T and x_1^O are measured from $x_1 = 0$ and, in addition, $\Gamma_k = \Gamma_k^C \cup \Gamma_k^T \cup \Gamma_k^O$).

2. Statement of the Inverse Problem

We now represent the solution of problem (1)–(5) in the form of an iterative sum of two states:

$$\begin{gathered} U^{(m)}(x) = U^0(x) + \tilde{U}^{(m)}(x), \qquad U^{(m)}(x) \in H^2_\Omega\,, \\ \tilde{U}^{(m)}(x) \in H^2_\Omega\,, \quad m = 1, \ldots, M\,, \end{gathered} \tag{6}$$

where the vector function $U^0(x)$ describes the state corresponding to the complete adhesion (1), (3), while

$\tilde{U}^{(m)}(x)$ describes the additional state formed due to the presence of the zones of sliding and separation, i.e., such that

$$\tilde{U}^{(m)}(x)\Big|_{\Gamma_k} = \tilde{U}^{(m)}_{\Gamma_k}$$

satisfies conditions (4) or/and (5), and H^2_Ω is the Sobolev space. The vector function $\tilde{U}^{(m)}(x)$ is uniquely determined from Eqs. (1) and the given boundary conditions

$$\tilde{U}^{(m)}(x)\Big|_{\Gamma_k} = \tilde{U}^{(m)}_{\Gamma_k}.$$

It is easy to see [5, 9] that the vector function $\tilde{U}^{(m)} \in H^2_\Omega$ such that $\tilde{U}^{(m)}\Big|_{\Gamma_k} = \tilde{U}^{(m)}_{\Gamma_k}$ exists iff

$$\tilde{U}^{(m)}_{\Gamma_k} \in H^{(3/2)}_{\Gamma_k}, \qquad \frac{\partial U^{(m)}_{\Gamma_k}}{\partial n} \in H^{(1/2)}_{\Gamma_k}.$$

We now introduce the set of admissible functions

$$Q = \left\{ \tilde{U}^{(m)}_{\Gamma_k} \in C^2_\Gamma,\ x \in \Gamma_k,\ L(U,F) = 0,\ \tilde{U}^{(m)}_{\Gamma_k} \in H^{(3/2)}_{\Gamma_k}, \right.$$

$$\left. \frac{\partial \tilde{U}^{(m)}_{\Gamma_k}}{\partial n} \in H^{(1/2)}_{\Gamma_k},\ \underline{U} \le \tilde{U}^{(m)}_{\Gamma_k} \le \bar{U} \right\}, \tag{7}$$

where $[\underline{U}, \bar{U}]$ is the range of $\tilde{U}^{(m)}_{\Gamma_k}$ and n is the normal to the surface.

We now formulate the problem of determination of the function $\tilde{U}^{(m)}_{\Gamma_k}$ as an inverse problem. In this case, we consider conditions (4) and (5) as additional information and choose the function $\tilde{U}^{(m)}_{\Gamma_k}$ as a quasi-solution.

Consider the case where condition (5) is not satisfied. Then $\tilde{U}^{(m)}_{\Gamma_k}$ is defined on the set Q as a solution of the problem

$$\tilde{U}^{(m)}_{\Gamma_k} = \arg\min J\left(\tilde{U}^{(m)}_{\Gamma_k}\right), \qquad \tilde{U}^{(m)}_{\Gamma_k} \in Q, \tag{8}$$

where

$$J\left(\tilde{U}^{(m)}_{\Gamma_k}\right) = \int\limits_{\Gamma_k} \left(\left|\sigma_{12}\left(\tilde{U}^{(m)}_{\Gamma_k}\right)\right| - k\left|\sigma_{22}\left(\tilde{U}^{(m)}_{\Gamma_k}\right)\right| \right)^2 d\Gamma.$$

We now introduce a function

$$\mu^{(m)}(x) = \begin{cases} 1, & \sigma_{12}^{(m)}\left(\tilde{U}_{\Gamma_k}^{(m)}\right) \geq \sigma_{22}^{(m)}\left(\tilde{U}_{\Gamma_k}^{(m)}\right), \quad x \in \Gamma_k^T, \\ 0, & \sigma_{12}^{(m)}\left(\tilde{U}_{\Gamma_k}^{(m)}\right) < \sigma_{22}^{(m)}\left(\tilde{U}_{\Gamma_k}^{(m)}\right), \quad x \in \Gamma_k^C, \end{cases}$$

where $\sigma_{12}^{(m)}\left(\tilde{U}_{\Gamma_k}^{(m)}\right)$, $\sigma_{22}^{(m)}\left(\tilde{U}_{\Gamma_k}^{(m)}\right)$ is a solution of the direct problem (1) with the boundary conditions

$$\tilde{U}^{(m)}(x)\Big|_{\Gamma_k} = \tilde{U}_{\Gamma_k}^{(m)}.$$

Then the functional J takes the form

$$J^{(m)}\left(\tilde{U}_{\Gamma_k}^{(m)}\right) = \int_{\Gamma_k} \left(\sigma_{12}^{(m)}\left(\tilde{U}_{\Gamma_k}^{(m)}\right) - k\sigma_{22}^{(m)}\left(\tilde{U}_{\Gamma_k}^{(m)}\right)\right)^2 \mu^{(m)}(x)\, d\Gamma.$$

As the solution of the direct problem, we consider a triple $\left(x_1^T, x_1^O, U(x)\right)$, where x_1^T and x_1^O are the points of the boundary Γ_k specifying the regions Γ_k^T and Γ_k^O according to conditions (4) and (5):

$$\sigma_{12}^{(m)}(x_1^T) - k\sigma_{22}^{(m)}(x_1^T) \geq 0 \qquad \text{for} \qquad \Gamma_k^T,$$

$$\sigma_{22}^{(m)}(x_1^0) > 0, \qquad \sigma_{12}^{(m)}(x_1^0) = 0 \qquad \text{for} \qquad \Gamma_k^O.$$

3. Analysis of Correctness of the Solution of the Inverse Problem

The set of admissible functions specified by relation (7) is a compact set in the space $H_\Gamma^{3/2}$. Therefore, from any sequence $\left\{\tilde{U}_{\Gamma_k}^{(m)}\right\} \subset Q$, we can choose at least one sequence $\left\{\tilde{U}_{\Gamma_k}^{(m_n)}\right\} \subset \left\{\tilde{U}_{\Gamma_k}^{(m)}\right\}$ convergent in the norm of the space $H_\Gamma^{3/2}$ to an element $\tilde{U}_{\Gamma_k}^0 \in Q$.

The analysis of correctness of the solution of inverse problem is connected with the determination of the continuity of the functional $J\left(\tilde{U}_{\Gamma_k}\right)$, i.e., with the dependence $\sigma_{ij}\left(\tilde{U}_{\Gamma_k}\right)$, $i, j = 1, 2$.

If we consider the integral relation for the generalized solution of problem (1), (2) for the mth and $(m-1)$th states in the presence of the unknown friction force, then we can prove that

$$\left\|\tilde{U}^{(m)} - \tilde{U}^{(m-1)}\right\|_{H_\Omega^2} \leq \gamma(\delta), \qquad \gamma(\delta) \to 0,$$

for

$$\left\|\tilde{U}_{\Gamma_k}^{(m)} - \tilde{U}_{\Gamma_k}^{(m-1)}\right\|_{H_\Omega^{3/2}} \leq \delta, \quad \delta \to 0.$$

This means that there exists a sequence of solutions such that $\tilde{U}\left(x, \tilde{U}_{\Gamma_k}^{(m)}\right) \to \tilde{U}\left(x, \tilde{U}_{\Gamma_k}^{0}\right)$ in H_Ω^2. Hence,

$$\sigma_{ij}\left(\tilde{U}_{\Gamma_k}^{(m)}, x\right) \to \sigma_{ij}(U_{\Gamma_k}, x) \qquad \text{in} \qquad H_\Omega^2 .$$

We now determine the traces of the functions $\sigma_{ij}(U(x))$, $x \in \Omega$, on the boundary $\Gamma_k \subset \partial\Omega$. It is known that, for the function $\sigma_{ij} \in H_\Omega$, which has the derivative

$$\frac{\partial \sigma_{ij}}{\partial x_2} \in H_{\Omega_\delta}^2 ,$$

where

$$\Omega_\delta = \Omega_\delta(\Gamma_k) = \{x \in \Omega ,\ 0 \le x_2 \le \delta ,\ (x_1, x_2) \in \Gamma_k\} ,$$

its trace exists ; it is defined as an element of H_Γ, $\Gamma_k = \partial\Omega$, and is continuous on $x_2 \in [0, \delta]$ in the norm of the space H_Γ. Hence, the trace of the functions $\sigma_{ij}\left(\tilde{U}_{\Gamma_k}, x\right)$ exists on Γ_k, belongs to H_{Γ_k}, and is continuous on $x_2 \in [0, \delta]$. Therefore, we get the convergence of the functional $J^{(m)}$:

$$\int_{\Gamma_k} \left(\sigma_{12}\left(\tilde{U}_{\Gamma_k}^{(m)}\right) - k\sigma_{22}\left(\tilde{U}_{\Gamma_k}^{(m)}\right)\right)^2 \mu^{(m)}(x)\, d\Gamma \;\to\; \int_{\Gamma_k} \left(\sigma_{12}\left(\tilde{U}_{\Gamma_k}\right) - k\sigma_{22}\left(\tilde{U}_{\Gamma_k}\right)\right)^2 \mu(x)\, d\Gamma$$

as $m \to \infty$. In addition, the functional $J(\tilde{U}_{\Gamma_k})$ is finite if $\tilde{U}_{\Gamma_k}^{(m)}(x) \in H_{\Gamma_k}^{3/2}$. If we take into account the fact that the set Q is compact in $H_\Gamma^{3/2}$, then it follows from the Weierstrass theorem that the problem of minimization of the functional $J(\tilde{U}_{\Gamma_k})$ on the set Q has at least one solution and any minimizing sequence converges to the set

$$Q_* = \left\{\tilde{U}_{\Gamma_k}^{(m)} \in Q:\ J(\tilde{U}_{\Gamma_k}) = J_* ,\ J_* = \min J(\tilde{U}_{\Gamma_k}),\ \tilde{U}_{\Gamma_k} \in Q_*\right\}$$

in the norm of the space $H_{\Gamma_k}^{3/2}$.

By using the topological lemma and the fact that any sequence $\sigma_{12}\left(\tilde{U}_{\Gamma_k}^{(m)}\right) - k\sigma_{22}\left(\tilde{U}_{\Gamma_k}^{(m)}\right)$ converges to a certain limit, we conclude that $\tilde{U}_{\Gamma_k}^{(m)} \in Q$ converges to an element $\tilde{U}_{\Gamma_k} \in Q_*$ in the norm of the space $H_{\Gamma_k}^{3/2}$.

4. Procedure of Solution

To solve the problem of infinite-dimensional optimization (8), we pass to a finite-dimensional problem by approximating the vector function $U(x)$ by the finite-element method.

To describe the unknown functions of the direct and inverse problems, we introduce a grid X_S, where $X_S = \{x_{1S}, x_{2S}\}$, $S = 1, \ldots, N$, and present the unknown functions $U(x)$, $\sigma_{ij}(x)$, $\mu(x)$, and $\tilde{U}_{\Gamma_k}^{(m)}(x)$ in the form of vectors whose components are the nodal values of the functions of the problem:

$$\tilde{U}^{(m)} = \left\{\tilde{U}_{1S}^{(m)}, \tilde{U}_{2S}^{(m)}\right\}, \quad U^0 = \left\{U_{1S}^0, U_{1S}^0\right\},$$

$$\sigma_{ij}^{(m)} = \left\{\sigma_{ijS}^{(m)}\right\}, \quad i, j = 1, 2, \quad S = 1, \ldots, N.$$

The nodes that belong to the boundary Γ_k are enumerated as follows:

$$j = \{j_1, j_2, \ldots, j_p\} \quad \text{for} \quad X_j = \left\{x_{1j}, x_{2j}\right\}.$$

In this case, $X_j \in \Gamma_k$, and we can form the sets

$$j^C = \left\{j_{m1}^C, j_{m2}^C, \ldots, j_{m\ell}^C\right\}, \quad j^O = \left\{j_{k1}^O, j_{k2}^O, \ldots, j_{kd}^O\right\}, \quad \text{and} \quad j^T = \left\{j_{r1}^T, \ldots, j_{rq}^T\right\}, \quad j^T \cup j^O \cup j^C = j,$$

that describe the coordinates of the zones of adhesion, friction, and separation,

$$\mu = \{\mu_j\}, \quad j = \{j_1, j_p\}, \quad \text{and} \quad \tilde{U}_{\Gamma_k} = \left\{\tilde{U}_{1\Gamma_k}^j, \tilde{U}_{2\Gamma_k}^j\right\}.$$

After necessary transformations, problem (1)–(3) is reduced to a system of linear algebraic equations of the form

$$KU^{(m)} = R^{(m)}\left(\tilde{U}_{\Gamma_k}^{(m)}\right),$$

where K is the stiffness matrix preserved in the process of optimization,

$$U^{(m)} = \left\{U_1^{1(m)}, U_2^{1(m)}, \ldots, U_1^{N(m)}, U_2^{N(m)}\right\}^\top$$

is the vector of nodal displacements, and $R^{(m)}\left(\tilde{U}_{\Gamma_k}^{(m)}\right)$ is the vector of nodal values of the right-hand sides.

In the discrete form, problem (8) takes the form

$$\tilde{U}_{\Gamma_k}^{(m)} = \arg\min \Delta^{(m)}\left(\tilde{U}_{\Gamma_k}\right)\Delta^{(m)T}\left(\tilde{U}_{\Gamma_k}\right), \quad \tilde{U}_{\Gamma_k}^{(m)} \in Q, \tag{9}$$

where

$$\Delta^{(m)} = D\left(\left|\sigma_{12}^{(m)}\right| - k\left|\sigma_{22}^{(m)}\right|\right),$$

$D = \operatorname{diag}\left\{\mu_j^{(m)}, \ j = j_1, \ldots, j_p\right\}$ is a diagonal matrix and

$$\sigma_{k\ell}^{(m)} = \left\{\sigma_{k\ell j_1}^{(m)}, \sigma_{k\ell j_2}^{(m)}, \ldots, \sigma_{k\ell j_p}^{(m)}\right\}^\top, \quad k, \ell = 1, 2.$$

By using the Newton method, we determine the functions $\tilde{U}^{(m)}_{\Gamma_k}$ from (9) as follows:

$$\tilde{U}^{(m)}_{1\Gamma_k} = \tilde{U}^{(m-1)}_{1\Gamma_k} - \left[W\left(\tilde{U}^{(m-1)}_{1\Gamma_k}\right)\right]^{-1} \cdot G\left(\tilde{U}^{(m-1)}_{1\Gamma_k}\right), \tag{10}$$

where

$$\left[W\left(\tilde{U}^{(m-1)}_{1}\right)\right] = [A][A^T], \qquad G = [A]\Delta^{(m-1)},$$

$$A = \left\{ \frac{\partial \Delta_i}{\partial \tilde{U}_{\Gamma_k j}} \right\}_{\tilde{U}_{\Gamma_k} = \tilde{U}^{(m-1)}_{\Gamma_k}}, \quad i, j = 1, \ldots, j_p .$$

The matrix A can be calculated as the Frechet matrix, where m is the number of step of the iterative process. Conditions (4) and (5) are verified by applying the following algorithm (Fig. 1):

Step 1. Set $U_2 = \delta_1$, $U_1|_{\Gamma_k} = 0$, $\sigma_{12}|_{\Gamma_\sigma} = 0$, $\sigma_{22}|_{\Gamma_\sigma} = 0$, the friction coefficient k, the accuracy ε, ζ, $n = 0$, and a small quantity α_n.

Step 2. Form the sets j, define the vector $U^{(0)}$, and compute $\sigma^{(0)}_{12}$ and $\sigma^{(0)}_{22}$ on the set j.

Step 3. Determine $k^0 \approx \max_i \left[\left|\sigma^{j0}_{12}\right| / \left|\sigma^{j0}_{22}\right| \right]$.

Step 4. Determine $\Delta^{(m)}_j = \left|\sigma^{(m)}_{12j}\right| - k^{(n)} \left|\sigma^{(m)}_{22j}\right|$, $j = 1, \ldots, p$.

Step 5. Determine the sets j^O.

Step 6. Determine the sets j^T and μ.

Step 7. Form the vector $\tilde{U}^{(m)}_{\Gamma_k} = \left\{ U^{(m)}_{1\Gamma_k}, \ldots, U^{(m)}_{p\Gamma_k} \right\}$ by using elements of the sets j^T and j^O.

Step 8. Determine the vector $\tilde{U}^{(m)}_{\Gamma_k}$ by the Newton method and verify the condition $\left\| \tilde{U}^{(m)}_{\Gamma_k} - \tilde{U}^{(m-1)}_{\Gamma_k} \right\| \le \varepsilon$.

Step 9. If $\left|k^{(n)} - k\right| \le \zeta$, then the program is terminated. Otherwise, go to *Step 10*.

Step 10. Set $n = n + 1$.

Step 11. Find $k^{(n)} = k^{(n-1)} - \alpha_n k^{(n-1)}$.

Step 12. Go to *Step 4*.

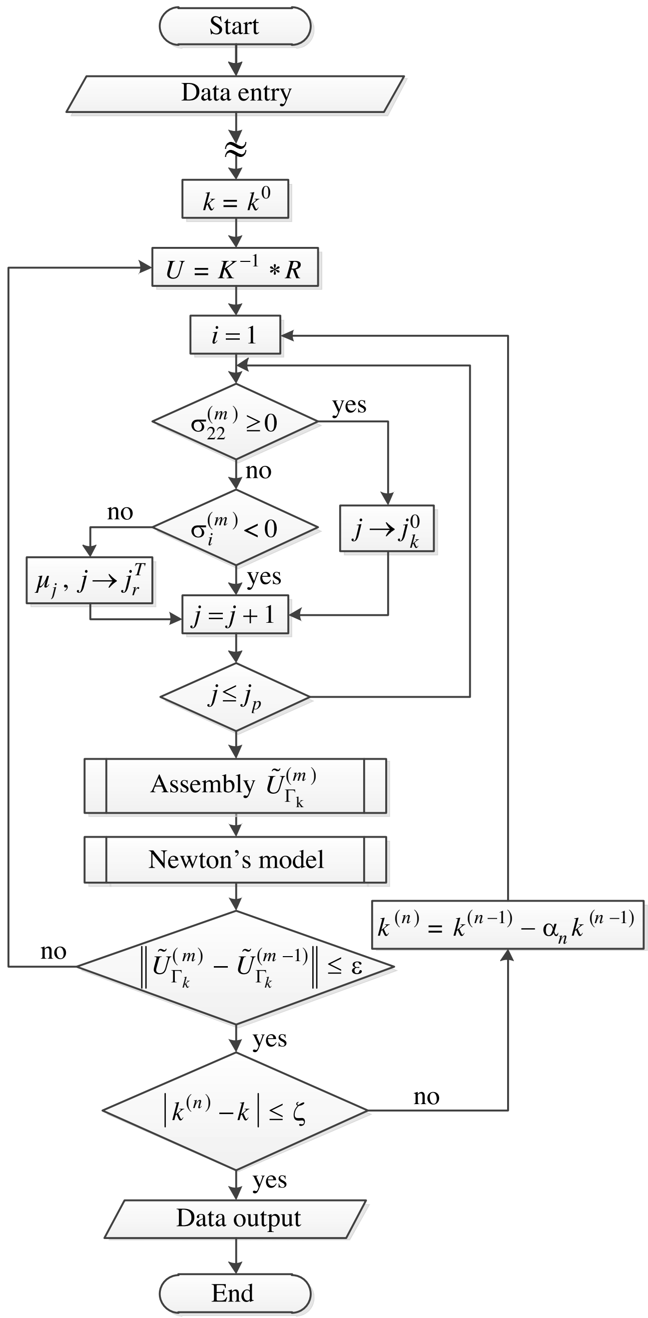

Fig. 1

5. Analysis of the Results

With the use of the presented algorithm, we solve the classical problems of indentation (with shift) of a rigid punch into the elastic half space with the mechanical characteristics $E = 200$ GPa and $\nu = 0.3$. The loading was realized kinematically, as a result of determination of the depth of indentation $\delta_1 = 4 \cdot 10^{-3}$ mm and the shift $\delta_2 = 10^{-3}$ mm. For the finite-element analysis, we applied a COSMOS/M-type package.

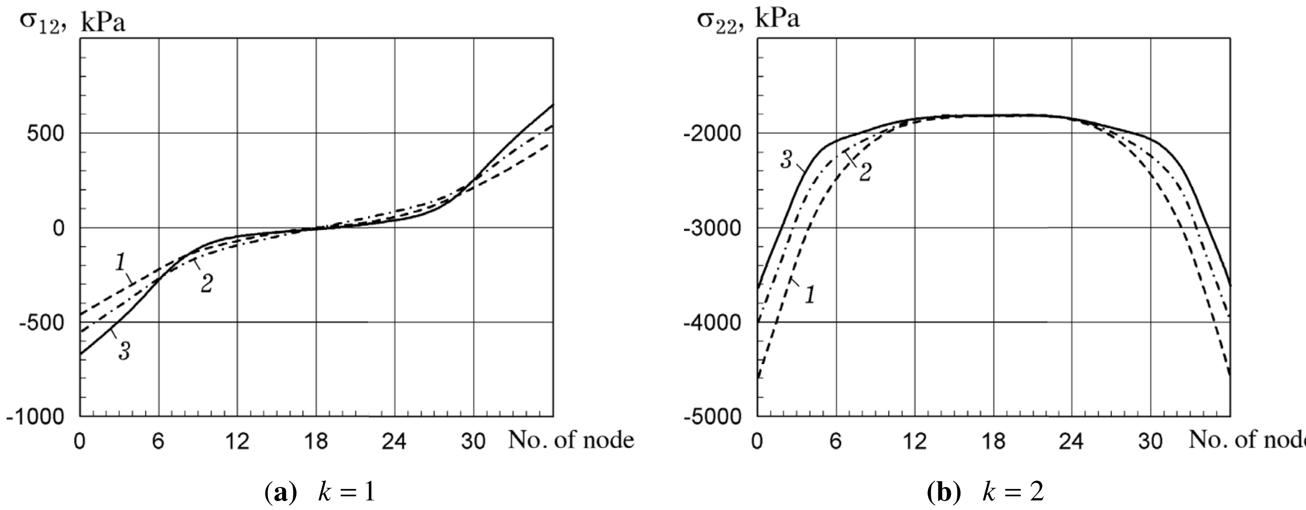

Fig. 2. Distributions of normal and tangential stresses: $\sigma_{k2}^{(1)}$ — curves *1*, $\sigma_{k2}^{(2)}$ — curves *2*, $\sigma_{k2}^{(3)}$ — curves *3*.

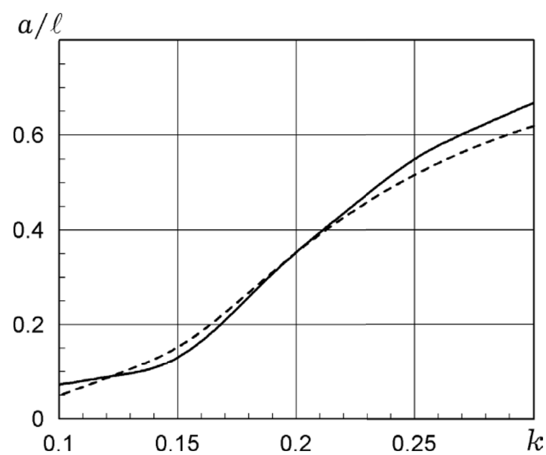

Fig. 3. Dependences of the size of the zone of adhesion on k .

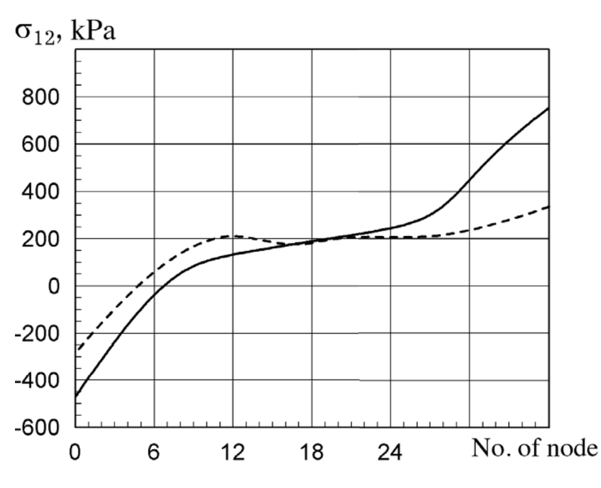

Fig. 4. Distributions of tangential stresses in the case of combined influence.

The finite-element partition was realized by the automatic choice of the step of a grid with an aim to attain the required relative error equal to 0.02. The total number of elements was equal to 1800 and the contact zone contained 36 elements. The size ratio was equal to $\ell/H = 0.3$, where H is the size that characterizes the finite-element model of the half space and ℓ is the size of the contact zone.

In Fig. 2, we show the results of the iterative process (6) for $\delta_1 = 4 \cdot 10^{-3}$ mm. The process of friction was observed on the boundaries of the contact zone.

We see that the presence of friction decreases the values of tangential stresses and increases the equivalent force. The convergence is attained after three iterations of Newton's method.

In Fig. 3, we present the dependence of the size of the zone of adhesion a on the friction coefficient. It almost coincides with the results obtained in [6] (dashed curve).

We also studied the case of combined influence for $\delta_1 = 4 \cdot 10^{-3}$ mm and $\delta_2 = 10^{-3}$ mm. In Fig. 4, we present the tangential stresses under the conditions of complete adhesion (solid line) and with regard for friction (dashed line). The normal stresses remain practically invariable.

CONCLUSIONS

The contact problem of the theory of elasticity with regard for the friction and separation can be solved as an inverse problem in which the role of unknown function is played by the deviation of tangential displacements from their values in the case of complete adhesion. In combination with the procedure of discretization of the problem, this approach guarantees a sufficiently high accuracy and does not require significant computational costs, unlike the method of successive approximations used for the solution of variational inequalities. In addition, the proposed approach makes it possible to determine the zones of microsliding, adhesion, and separation.

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