DOI 10.1007/s10958-019-04338-2

Journal of Mathematical Sciences, Vol. 240, No. 1, July, 2019

AXIALLY SYMMETRIC VIBRATIONS OF ELASTIC ANNULAR BASES AND A PERFECT TWO-LAYER LIQUID IN A RIGID ANNULAR CYLINDRICAL VESSEL

Yu. M. Kononov, V. P. Shevchenko, and Yu. O. Dzhukha UDC 533.6.013.42

We deduce a frequency equation for the natural coupled axially symmetric vibrations of elastic bases (in the form of annular plates) and a heavy two-layer incompressible perfect liquid in a rigid annular cylindrical vessel. We consider different limiting cases: the case of degeneration of annular plates into membranes, the case of absolutely rigid or circular plates, and the case of absence of the upper plate (liquid with free surface). For a broad range of parameters of the analyzed mechanical system, we investigate the frequency spectra and obtain a series of mechanical effects in the problem of hydroelasticity.

Introduction

The problem of vibration of an incompressible perfect liquid placed in a rigid cylindrical vessel with two elastic bases can be regarded as a generalization of the problem of vibration of liquid in a rigid cylindrical vessel with an elastic membrane (or a plate) on the free surface. The intense investigations of the last mentioned problem were originated more than 35 years ago [6, Sec. 5] and, at present, there are sufficiently many available publications dealing with problems of this kind (see, e.g., [2, 15]). The problem of vibration of a perfect liquid in a rigid cylindrical vessel with elastic bases was, apparently, first considered from the viewpoint of functional analysis in [1, 13 (pp. 167–178)] and later in [5, 7–9, 11, etc.]. A large number of works was devoted to the vibration of liquids with free surface in rigid cylindrical vessels with elastic bottoms (see the surveys in [9–11]). The interest in the problem of axially symmetric vibrations of the elastic bottom and liquids in cylindrical vessels is connected with the necessity of taking into account the static deflections of the bottom and longitudinal vibrations of the liquid as a whole. The axially symmetric vibrations of elastic bases and a single-layer perfect liquid in a rigid cylindrical vessel were investigated in [11]. In [9], the results obtained in [11] were generalized to the case of a coaxial cylindrical vessel. The axially symmetric vibrations of a two-layer liquid were studied in [3, 4, 14] published relatively recently, as applied to the problem of capillary phase separators. Among the works published in English, we should especially mention [15–18].

In the present work, we generalize the problem studied in [9] to the case of a two-layer perfect liquid where the contours of the plate are fixed. The aim of the present work is to deduce a frequency equation and analyze the frequency spectrum of the mechanical system under consideration. We consider the case of degeneration of the plates into membranes, the case of absolutely rigid plates, the case of circular plates, and also the case of liquids with free surfaces. We performed numerical investigations for wide ranges of the parameters of mechanical systems: elastic characteristics of the plates, densities, depths of filling, as well as for the zero-gravity case. The obtained results can be used to determine the natural forms of vibrations and for the analysis of forced vibrations in the "solid–liquid–elastic plates" system.

Stus Donets’k National University, Vinnytsya, Ukraine.

Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 60, No. 1, pp. 85–95, January–March, 2017. Original article submitted November 25, 2016.

 1072-3374/19/2401–0098

1. Statement of the Problem

We consider coupled axially symmetric vibrations of elastic bases and a heavy two-layer incompressible perfect liquid with densities ρ_1 and ρ_2 that completely fills a straight annular cylindrical vessel with rigid lateral surface of the outer radius a and inner radius $a\varepsilon$, $0 \le \varepsilon < 1$. The bases of annular cylindrical vessel are annular isotropic plates with flexural stiffnesses D_i subjected to the action of tensile forces T_i in the median plane, $i = 1,2$. The subscript $i = 1$ corresponds to the upper base and upper liquid with density ρ_1, whereas the subscript $i = 2$ corresponds to the lower base and lower liquid with density ρ_2. We introduce a cylindrical coordinate system $Or\theta z$ so that the plane $Or\theta$ coincides with the interface of the liquids, and the Oz-axis is directed along the axis of the cylinder in the direction opposite to the vector of gravitational acceleration $\mathbf{g}$. We consider the linear statement of the problem under the assumption that the motion of liquid is potential and the joint vibrations of the plates and liquid are not separated.

The equations of the motion of the analyzed mechanical system take the form [9, 11, 12]

$$k_{01}\frac{\partial^2 W_1}{\partial t^2} + D_1\Delta_2^2 W_1 - T_1\Delta_2 W_1 + \rho_1 g W_1 = \rho_1\left(Q_1 - \left.\frac{\partial \Phi_1}{\partial t}\right|_{z=h_1} - g h_1\right), \tag{1}$$

$$k_{02}\frac{\partial^2 W_2}{\partial t^2} + D_2\Delta_2^2 W_2 - T_2\Delta_2 W_2 - \rho_2 g W_2 = -\rho_2\left(Q_2 - \left.\frac{\partial \Phi_2}{\partial t}\right|_{z=-h_2} + g h_2\right), \tag{2}$$

$$\Delta\Phi_1 = 0, \qquad \Delta\Phi_2 = 0.$$

We solve Eqs. (1) and (2) with regard for boundary conditions

$$\left.\frac{\partial \Phi_1}{\partial r}\right|_{r=a,\,r=a\varepsilon} = \left.\frac{\partial \Phi_2}{\partial r}\right|_{r=a,\,r=a\varepsilon} = 0,$$

$$\left.\frac{\partial \Phi_1}{\partial z}\right|_{z=h_1} = \frac{\partial W_1}{\partial t}, \qquad \left.\frac{\partial \Phi_2}{\partial z}\right|_{z=-h_2} = \frac{\partial W_2}{\partial t},$$

$$\rho_1\left(Q_1 - \left.\frac{\partial \Phi_1}{\partial t}\right|_{z=0} - g\zeta\right) = \rho_2\left(Q_2 - \left.\frac{\partial \Phi_2}{\partial t}\right|_{z=0} - g\zeta\right), \tag{3}$$

$$\left.\frac{\partial \Phi_1}{\partial z}\right|_{z=0} = \left.\frac{\partial \Phi_2}{\partial z}\right|_{z=0} = \frac{\partial \zeta}{\partial t},$$

$$\int_S W_1\, dS = \int_S \zeta\, dS = \int_S W_2\, dS, \tag{4}$$

$$W_i\big|_{\gamma_j} = 0, \qquad \left.\frac{\partial W_i}{\partial r}\right|_{\gamma_j} = 0, \qquad i, j = 1,2. \tag{5}$$

Here, $k_{0i} = \rho_{0i}\delta_{0i}$; W_i, ρ_{0i}, and δ_{0i} are the deflection, density, and thickness of ith plate, respectively; Φ_i is the velocity potential of the ith liquid; $z = \zeta(r,t)$ is the equation of interface between the liquids (inner surface); Q_i are arbitrary functions of time;

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \quad \text{and} \quad \Delta = \Delta_2 + \frac{\partial^2}{\partial z^2}$$

are the two- and three-dimensional axially symmetric Laplace operators, respectively, and S is an annular domain. Here, for the sake of convenience, we introduce an additional subscript j and the notation of the contours. The subscript $j = 1$ corresponds to the outer contour γ_1 and $j = 2$ corresponds to the inner contour sγ_2.

2. Procedure of Solution

In view of the axial symmetry of the function Φ_i, we represent W_i and ζ in the form of generalized Fourier series in the eigenfunctions $\psi_n(r)$ as follows [9, 11, 12]:

$$\Phi_i(r,z,t) = a_{0i}(t) + a_{1i}(t)z + \sum_{n=1}^{\infty}[A_{in}(t)e^{k_n z} + B_{in}(t)e^{-k_n z}]\psi_n(r), \tag{6}$$

$$W_i(r,t) = W_{i0}(t) + \sum_{n=1}^{\infty} W_{in}(t)\psi_n(r), \tag{7}$$

$$\zeta(r,t) = \zeta_0(t) + \sum_{n=1}^{\infty} \zeta_n(t)\psi_n(r), \tag{8}$$

where

$$W_{i0} = \frac{2}{(1-\varepsilon^2)a^2}\int_{a\varepsilon}^{a} rW_i\,dr, \qquad W_{in} = \frac{1}{N_n^2}\int_{a\varepsilon}^{a} rW_i\psi_n\,dr,$$

$$\zeta_0 = \frac{2}{(1-\varepsilon^2)a^2}\int_{a\varepsilon}^{a} r\zeta\,dr, \qquad \zeta_n = \frac{1}{N_n^2}\int_{a\varepsilon}^{a} r\zeta\psi_n\,dr, \qquad N_n^2 = \int_{a\varepsilon}^{a} r\psi_n^2\,dr.$$

In the case of axial symmetry, the eigenfunctions $\psi_n(r)$, which form, together with an arbitrary constant, a complete orthogonal system of functions on the segment $[a, a\varepsilon]$ are given by the formulas

$$\psi_n(r) = J_0(k_n r) + \gamma_n Y_0(k_n r),$$

where $\gamma_n = -J_1(\xi_n)/Y_1(\xi_n)$, J_0, J_1, and Y_0, Y_1 are Bessel functions of the first and second kind of orders zero and one, $\nu k_n = \xi_n / a$ are eigenvalues, and ξ_n are the roots of the equations

$$J_1(\xi_n)Y_1(\xi_n\varepsilon) - J_1(\xi_n\varepsilon)Y_1(\xi_n) = 0$$

for $\varepsilon \neq 0$ and $J_1(\xi_n) = 0$ for $\varepsilon = 0$.

Substituting expressions (6)–(8) in the boundary conditions (3) and using the orthogonality of the functions $\psi_n(r)$, we find

$$A_{1n} = \frac{\dot{W}_{1n} - \dot{\zeta}_n e^{-\kappa_{1n}}}{2k_n \sinh\kappa_{1n}}, \qquad B_{1n} = \frac{\dot{W}_{1n} - \dot{\zeta}_n e^{\kappa_{1n}}}{2k_n \sinh\kappa_{1n}},$$

$$A_{2n} = -\frac{\dot{W}_{2n} - \dot{\zeta}_n e^{\kappa_{2n}}}{2k_n \sinh\kappa_{2n}}, \qquad B_{2n} = -\frac{\dot{W}_{2n} - \dot{\zeta}_n e^{-\kappa_{2n}}}{2k_n \sinh\kappa_{2n}}, \qquad \kappa_{in} = k_n h_i,$$

$$a_{11} = a_{12} = \dot{W}_{10} = \dot{W}_{20} = \dot{\zeta}_0, \qquad \rho_1(Q_1 - \dot{a}_{01} - g\zeta_0) = \rho_2(Q_2 - \dot{a}_{02} - g\zeta_0).$$

Equations (1) and (2), as well as the equation of the interface of liquids (inner surface), take the form

$$k_{01}\frac{\partial^2 W_1}{\partial t^2} + D_1\Delta_2^2 W_1 - T_1\Delta_2 W_1 + \rho_1 g W_1$$

$$= \rho_1\left(Q_1 - \dot{a}_{01} - h_1(\ddot{\zeta}_0 + g) - \sum_{n=1}^{\infty}\frac{\ddot{W}_{1n}\cosh\kappa_{1n} - \ddot{\zeta}_n}{k_n \sinh\kappa_{1n}}\psi_n\right), \tag{9}$$

$$k_{02}\frac{\partial^2 W_2}{\partial t^2} + D_2\Delta_2^2 W_2 - T_2\Delta_2 W_2 - \rho_2 g W_2$$

$$= -\rho_2\left(Q_2 - \dot{a}_{02} + h_2(\ddot{\zeta}_0 + g) + \sum_{n=1}^{\infty}\frac{\ddot{W}_{2n}\cosh\kappa_{2n} - \ddot{\zeta}_n}{k_n \sinh\kappa_{2n}}\psi_n\right), \tag{10}$$

$$\ddot{\zeta}_n + \sigma_n^2\zeta_n - \frac{1}{a_n}(b_{1n}\ddot{W}_{1n} + b_{2n}\ddot{W}_{2n}) = 0, \tag{11}$$

where $\sigma_n^2 = gk_n\Delta\rho/a_n$ is the squared frequency of vibration of the inner surface for the rigid bases; $a_n = \rho_1\coth\kappa_{1n} + \rho_2\coth\kappa_{2n}$, $\Delta\rho = \rho_2 - \rho_1$, and $b_{in} = \rho_i/\sinh\kappa_{in}$.

3. Derivation of the Frequency Equation

Consider the problem of natural joint vibrations of elastic plates and a liquid. For this purpose, we set

$$W_i(r,t) = e^{i\omega t}w_i(r) + W_i^{st}(r), \qquad \rho_1(Q_1 - \dot{a}_{01}) = \tilde{Q}e^{i\omega t}, \qquad \text{and} \qquad \zeta_0 = we^{i\omega t}.$$

Here, W_i^{st} is the static deflection of the plates considered in [13]. In this case, by virtue of (11), Eqs. (9) and (10) take the form

$$D_i\Delta_2^2 w_i - T_i\Delta_2 w_i - [k_{0i}\omega^2 + (-1)^i\rho_i g]w_i$$
$$= (-1)^{i+1}\tilde{Q} + [\rho_i h_i\omega^2 + (\delta_{i1}-1)g\Delta\rho]w + \rho_i\omega^2\sum_{n=1}^{\infty}\tilde{w}_{in}\psi_n, \qquad i = 1,2, \tag{12}$$

where, in view of (4),

$$\tilde{w}_{1n} = \frac{w_{1n}(\cosh\kappa_{1n} - \omega^2\tilde{b}_{1n}) - \omega^2\tilde{b}_{2n}w_{2n}}{k_n\sinh\kappa_{1n}},$$

$$\tilde{w}_{2n} = \frac{w_{2n}(\cosh\kappa_{2n} - \omega^2\tilde{b}_{2n}) - \omega^2\tilde{b}_{1n}w_{1n}}{k_n\sinh\kappa_{2n}},$$

$$\tilde{b}_{in} = \frac{b_{in}}{a_n(\omega^2 - \sigma_n^2)}, \tag{13}$$

$$w_{in} = \frac{1}{N_n^2}\int_{a\varepsilon}^{a} rw_i\psi_n dr,$$

$$w = \frac{2}{a^2(1-\varepsilon^2)}\int_{a\varepsilon}^{a} rw_1 dr = \frac{2}{a^2(1-\varepsilon^2)}\int_{a\varepsilon}^{a} rw_2 dr, \tag{14}$$

and δ_{i1} is the Kronecker delta.

We seek the solution of each equation in (12) as the sum of the general solution of homogeneous equation and a partial solution of the inhomogeneous equation [6, 7, 9–11]:

$$w_i = \sum_{k=1}^{4} w_{ik}^0 A_{ik}^0 + \rho_i\omega^2\sum_{n=1}^{\infty}\frac{\tilde{w}_{in}}{d_{in}}\psi_n + \tilde{k}_{0i}\{(-1)^{i+1}\tilde{Q} + [\rho_i h_i\omega^2 + (\delta_{i1}-1)g\Delta\rho]w\}, \qquad i = 1,2. \tag{15}$$

Here, $\tilde{k}_{0i} = -1/(k_{0i}\omega^2 + (-1)^i\rho_i g)$; $d_{in} = (D_i k_n^2 + T_i)k_n^2 - [k_{0i}\omega^2 + (-1)^i\rho_i g]$, and A_{ik}^0, w_{in}, $\tilde{Q}$, and w are unknown constants, $i = 1,2$, $k = 1,2,3,4$.

Substituting (15) in (14), we get two equations for A_{ik}^0, $\tilde{Q}$, and $\tilde{w}$:

$$\sum_{k=1}^{4}(\tilde{w}_{1k}^0 A_{1k}^0 - \tilde{w}_{2k}^0 A_{2k}^0) + (\tilde{k}_{01} + \tilde{k}_{02})\tilde{Q} + \left[(\rho_1 h_1\tilde{k}_{01} - \rho_2 h_2\tilde{k}_{02})\omega^2 + \tilde{k}_{02}g\Delta\rho\right]w = 0, \tag{16}$$

$$\sum_{k=1}^{4}\tilde{w}_{2k}^0 A_{2k}^0 - \tilde{k}_{02}\tilde{Q} + \tilde{k}_2 w = 0, \tag{17}$$

where $\tilde{k}_2 = \tilde{k}_{02}(\rho_2 h_2 \omega^2 - g\Delta\rho) - 1$ and

$$\tilde{w}_{ik}^0 = \frac{2}{(1-\varepsilon^2)a^2} \int_{a\varepsilon}^{a} r w_{ik}^0 dr .$$

Substituting (15) in (13) and solving the system of two linear equations for w_{1n} and w_{2n}, we arrive at the final expressions for plate deflections:

$$w_1 = \sum_{k=1}^{4} \left[\left(w_{1k}^0 + \sum_{n=1}^{\infty} a_{11n} E_{1kn}^0 \psi_n \right) A_{1k}^0 + \left(\sum_{n=1}^{\infty} a_{12n} E_{2kn}^0 \psi_n \right) A_{2k}^0 \right] + \tilde{k}_{01}(\tilde{Q} + \rho_1 h_1 \omega^2 w),$$

$$w_2 = \sum_{k=1}^{4} \left[\left(\sum_{n=1}^{\infty} a_{21n} E_{1kn}^0 \psi_n \right) A_{1k}^0 + \left(w_{2k}^0 + \sum_{n=1}^{\infty} a_{22n} E_{2kn}^0 \psi_n \right) A_{2k}^0 \right] + \tilde{k}_{02}\left[-\tilde{Q} + (\rho_2 h_2 \omega^2 - g\Delta\rho) w \right]. \quad (18)$$

Here,

$$a_{11n} = \frac{1}{\tilde{\Delta}_n} \omega^2 b_{1n} (a_{1n} d_{2n} + \omega^2 b_{2n} c_n), \qquad a_{12n} = -\frac{1}{\tilde{\Delta}_n} \omega^4 k_n b_{1n} b_{2n} d_{2n},$$

$$a_{21n} = -\frac{1}{\tilde{\Delta}_n} \omega^4 k_n b_{1n} b_{2n} d_{1n}, \qquad a_{22n} = \frac{1}{\tilde{\Delta}_n} \omega^2 b_{2n} (a_{2n} d_{1n} + \omega^2 b_{1n} c_n),$$

$$\tilde{a}_n = a_n(\omega^2 - \sigma_n^2), \qquad b_n = b_{1n} \cosh \kappa_{2n} + b_{2n} \cosh \kappa_{1n},$$

$$c_n = \omega^2 b_n - \tilde{a}_n \cosh \kappa_{1n} \cosh \kappa_{2n}, \quad (19)$$

$$\tilde{\Delta}_n = k_n^2 \tilde{a}_n d_{1n} d_{2n} - \omega^2 (a_{1n} b_{1n} d_{2n} + a_{2n} b_{2n} d_{1n} + \omega^2 b_{1n} b_{2n} c_n),$$

$$a_{in} = k_n (\tilde{a}_n \cosh \kappa_{in} - \omega^2 b_{in}), \qquad E_{ikn}^0 = \frac{1}{N_n^2} \int_{a\varepsilon}^{a} r w_{ik}^0 \psi_n \, dr .$$

In view of the conditions of fastening of the plates (5), the expressions for deflections of the plates (18), and the additional equations (16) and (17), we arrive at the following frequency equation for the natural joint axisymmetric vibrations of the elastic bases and two-layer liquid:

$$\left| \| C_{qr} \|_{q,r=1}^{10} \right| = 0, \tag{20}$$

where

$$C_{i+j-1,k} = B_{ijk} + \sum_{n=1}^{\infty} a_{i1n} E_{1kn}^0 B_{jn}^*, \quad C_{i+j-1,k+4} = \sum_{n=1}^{\infty} a_{i2n} E_{2kn}^0 B_{jn}^*,$$

$$C_{i+j-1,9} = \tilde{k}_{01}, \quad C_{i+j-1,10} = \tilde{k}_{01} \rho_1 h_1 \omega^2, \quad C_{i+j,k} = C_{ijk},$$

$$C_{i+j,k+4} = 0, \quad C_{i+j,10} = 0, \quad i = 1, \quad j = 1, \quad k = 1,2,3,4,$$

$$C_{i+j,k} = B_{ijk} + \sum_{n=1}^{\infty} a_{i1n} E_{1kn}^0 B_{jn}^*, \quad C_{i+j,k+4} = \sum_{n=1}^{\infty} a_{i2n} E_{2kn}^0 B_{jn}^*,$$

$$C_{i+j,9} = \tilde{k}_{01}, \quad C_{i+j,10} = \tilde{k}_{01} \rho_1 h_1 \omega^2, \quad C_{i+j+1,k} = C_{ijk}, \quad C_{i+j+1,k+4} = 0,$$

$$C_{i+j+1,9} = 0, \quad C_{i+j+1,10} = 0, \quad i = 1, \quad j = 2, \quad k = 1,2,3,4,$$

$$C_{i+j+2,k} = \sum_{n=1}^{\infty} a_{i1n} E_{1kn}^0 B_{jn}^*, \quad C_{i+j+2,k+4} = B_{ijk} + \sum_{n=1}^{\infty} a_{i2n} E_{2kn}^0 B_{jn}^*,$$

$$C_{i+j+2,9} = -\tilde{k}_{02}, \quad C_{i+j+2,10} = \tilde{k}_{02} (\rho_2 h_2 \omega^2 - g \Delta\rho),$$

$$C_{i+j+3,k} = 0, \quad C_{i+j+3,k+4} = C_{ijk}, \quad C_{i+j+3,9} = 0, \tag{21}$$

$$C_{i+j+3,10} = 0, \quad i = 2, \quad j = 1, \quad k = 1,2,3,4,$$

$$C_{i+j+3,k} = \sum_{n=1}^{\infty} a_{i1n} E_{1kn}^0 B_{jn}^*, \quad C_{i+j+3,k+4} = B_{ijk} + \sum_{n=1}^{\infty} a_{i2n} E_{2kn}^0 B_{jn}^*,$$

$$C_{i+j+3,9} = -\tilde{k}_{02}, \quad C_{i+j+3,10} = \tilde{k}_{02} (\rho_2 h_2 \omega^2 - g \Delta\rho),$$

$$C_{i+j+4,k} = 0, \quad C_{i+j+4,k+4} = C_{ijk},$$

$$C_{i+j+4,9} = 0, \quad C_{i+j+4,10} = 0, \quad i = 2, \quad j = 2, \quad k = 1,2,3,4,$$

$$C_{9,k} = \tilde{w}_{1k}^0, \quad C_{9,k+4} = -\tilde{w}_{2k}^0, \quad C_{9,9} = \tilde{k}_{01} + \tilde{k}_{02},$$

$$C_{9,10} = (\rho_1 h_1 \tilde{k}_{01} - \rho_2 h_2 \tilde{k}_{02})\omega^2 + \tilde{k}_{02} g \Delta\rho, \quad C_{10,k} = 0,$$

$$C_{10,k+4} = \tilde{w}_{2k}^0, \quad C_{10,9} = -\tilde{k}_{02}, \quad C_{10,10} = \tilde{k}_2, \quad k = 1,2,3,4.$$

Here,

$$B_{ijk} = w_{ik}^0\Big|_{\gamma_j}, \ C_{ijk} = \frac{dw_{ik}^0}{dr}\Bigg|_{\gamma_j}, \ B_{jn}^* = Z_0\left(\frac{r}{a}\right)\Bigg|_{\gamma_j},$$

and

$$Z_m(x) = J_m(\xi_n x) + \gamma_n Y_m(\xi_n x).$$

Equation (20) describes joint natural vibrations of elastic annular plates and a two-layer perfect liquid in an annular cylinder under the conditions of rigid fastening of the outer and inner contours of the plates. It can be expected that the frequency spectrum consists of four sets of frequencies corresponding to the vibrations of the upper and lower elastic bases, liquid column as a whole, and the inner interface of the liquids. For a single-layer liquid, the frequency spectrum contains three sets of frequencies.

4. Special Cases of the Frequency Equation for Joint Natural Vibrations of Elastic Bases and a Liquid

The obtained equation (20) is fairly general and covers numerous special cases that are of independent interest.

Upper plate degenerates into a membrane. In this case, it is necessary to delete the second and fourth rows and the second and fourth columns in the determinant of Eq. (20) and set $D_1 = 0$ in relations (19).

Lower plate degenerates into a membrane. By analogy with the previous case, it is necessary delete the sixth and eights rows and the sixth and eights columns in the determinant of Eq. (20) and set $D_2 = 0$ in relations (19). In this case, Eq. (20) takes the form

$$\left| \left\| C_{qr} \right\|_{q,r=5,7,9,10}^{5,7,9,10} \right| = 0,$$

where

$$C_{5,5} = B_{211} + \sum_{n=1}^{\infty} a_{22n} E_{21n}^0 B_{1n}^*, \quad C_{5,7} = B_{212} + \sum_{n=1}^{\infty} a_{22n} E_{22n}^0 B_{1n}^*,$$

$$C_{5,9} = -\tilde{k}_{02}, \quad C_{5,10} = \tilde{k}_{02}(\rho_2 h_2 \omega^2 - g\Delta\rho),$$

$$C_{7,5} = B_{221} + \sum_{n=1}^{\infty} a_{22n} E_{21n}^0 B_{2n}^*, \quad C_{7,7} = B_{222} + \sum_{n=1}^{\infty} a_{22n} E_{22n}^0 B_{2n}^*,$$

$$C_{7,9} = -\tilde{k}_{02}, \quad C_{7,10} = \tilde{k}_{02}(\rho_2 h_2 \omega^2 - g\Delta\rho),$$

$$C_{9,5} = -\tilde{w}_{21}^0, \quad C_{9,7} = -\tilde{w}_{22}^0, \quad C_{9,9} = \tilde{k}_{01} + \tilde{k}_{02},$$

$$C_{9,10} = (\rho_1 h_1 \tilde{k}_{01} - \rho_2 h_2 \tilde{k}_{02})\omega^2 + \tilde{k}_{02} g\Delta\rho, \quad C_{10,5} = \tilde{w}_{21}^0,$$

$$C_{10,7} = \tilde{w}_{22}^0, \quad C_{10,9} = -\tilde{k}_{02}, \quad C_{10,10} = \tilde{k}_2.$$

Lower and upper plates degenerate into membranes. In this case, it is necessary to delete the second, fourth, sixth, and eighth rows and the second, fourth, sixth, and eighth columns in the determinant of Eq. (20) and set $D_1 = D_2 = 0$ in relations (19).

The case of presence of free surface on the liquid. This case is realized if the upper plate is absent. In this case, it is necessary to delete the first, second, third, and fourth rows and the first, second, third, and fourth columns in the determinant of Eq. (20) and set $k_{01} = 0$, $T_1 = 0$, and $D_1 = 0$ in relations (19).

Either the lower plate or the upper plate is absolutely rigid. If the upper or lower plate becomes absolutely rigid, then $w_1 \equiv 0$ $(\tilde{w}_{1k}^0 \equiv 0)$ or $w_2 \equiv 0$ $(\tilde{w}_{2k}^0 \equiv 0)$. Passing to the limit in Eq. (20) as $T_1 \to \infty$ $(a_{11n} \to 0$, $a_{12n} \to 0$, $w_{1k}^0 = 0)$ or as $T_2 \to \infty$ $(a_{21n} \to 0$, $a_{22n} \to 0$, $w_{2k}^0 = 0)$, we get the following frequency equations:

— in the first case (as $T_1 \to \infty$),

$$\left| \| C_{qr} \|_{q,r=5}^{10} \right| = 0,$$

— in the second case (as $T_2 \to \infty$),

$$\left| \| C_{qr} \|_{q,r=1,2,3,4,9,10}^{1,2,3,4,9,10} \right| = 0.$$

The coefficients C_{qr} are given by relations (21).

Degeneration of the annular cylinder into a circular cylinder ($\boldsymbol{\varepsilon = 0}$). In this case, the frequency equation (20) takes the form

$$\left| \| C_{qr} \|_{q,r=1}^{6} \right| = 0, \tag{22}$$

where

$$C_{i,k} = B_{ik} + \sum_{n=1}^{\infty} a_{i1n} E_{1kn}^0 B_n^*, \quad C_{i,k+2} = \sum_{n=1}^{\infty} a_{i2n} E_{2kn}^0 B_n^*,$$

$$C_{i,5} = \tilde{k}_{01}, \quad C_{i,6} = \tilde{k}_{01}\rho_1 h_1\omega^2, \quad C_{i+1,k} = C_{ik},$$

$$C_{i+1,k+2} = 0, \quad C_{i+1,5} = 0, \quad C_{i+1,6} = 0, \quad i=1, \quad k=1,2,$$

$$C_{i+1,k} = \sum_{n=1}^{\infty} a_{i1n}E^0_{1kn}B^*_n, \quad C_{i+1,k+2} = B_{ik} + \sum_{n=1}^{\infty} a_{i2n}E^0_{2kn}B^*_n,$$

$$C_{i+1,5} = -\tilde{k}_{02}, \quad C_{i+1,6} = \tilde{k}_{02}(\rho_2 h_2\omega^2 - g\Delta\rho), \quad C_{i+2,k} = 0,$$

$$C_{i+2,k+2} = C_{ik}, \quad C_{i+2,5} = 0, \quad C_{i+2,6} = 0, \quad i=2, \quad k=1,2,$$

$$C_{5,k} = \tilde{w}^0_{1k}, \quad C_{5,k+2} = -\tilde{w}^0_{2k}, \quad C_{5,5} = \tilde{k}_{01} + \tilde{k}_{02},$$

$$C_{5,6} = (\rho_1 h_1\tilde{k}_{01} - \rho_2 h_2\tilde{k}_{02})\omega^2 + \tilde{k}_{02}g\Delta\rho, \quad C_{6,k} = 0, \quad C_{6,k+2} = \tilde{w}^0_{2k},$$

$$C_{6,5} = -\tilde{k}_{02}, \quad C_{6,6} = \tilde{k}_2, \quad k=1,2,$$

$$B_{ik} = w^0_{ik}\Big|_{r=a}, \quad C_{ik} = \frac{dw^0_{ik}}{dr}\Big|_{r=a}, \quad B^*_n = J_0(\xi_n).$$

For the nonstratified liquid $(\rho_1 = \rho_2)$, this case was investigated in detail in [11]. It is worth noting that Eq. (22) differs from Eq. (20) as $\varepsilon \to 0$. Note that, as $\varepsilon \to 0$, Eq. (20) describes the vibrations of the analyzed mechanical system in the case of immobile (fastened) centers. This is a new problem of the axially symmetric vibrations of liquid and elastic circular plates with fixed centers, which follows from the problem analyzed above.

Suppose that the upper and lower bases are perfectly elastic $(T_1 = T_2 = 0)$ and $k_{0i}\omega^2 + (-1)^i\rho_i g > 0$. In this case, we get

$$w^0_{i1} = J_0(\mu_i r), \quad w^0_{i2} = Y_0(\mu_i r), \quad w^0_{i3} = I_0(\mu_i r), \quad w^0_{i4} = K_0(\mu_i r),$$

$$\tilde{w}^0_{i1} = 2\frac{J_1(\tilde{\mu}_i) - \varepsilon J_1(\varepsilon\tilde{\mu}_i)}{\tilde{\mu}_i(1-\varepsilon^2)}, \quad \tilde{w}^0_{i2} = 2\frac{Y_1(\tilde{\mu}_i) - \varepsilon Y_1(\varepsilon\tilde{\mu}_i)}{\tilde{\mu}_i(1-\varepsilon^2)},$$

$$\tilde{w}^0_{i3} = 2\frac{I_1(\tilde{\mu}_i) - \varepsilon I_1(\varepsilon\tilde{\mu}_i)}{\tilde{\mu}_i(1-\varepsilon^2)}, \quad \tilde{w}^0_{i4} = -2\frac{K_1(\tilde{\mu}_i) - \varepsilon K_1(\varepsilon\tilde{\mu}_i)}{\tilde{\mu}_i(1-\varepsilon^2)},$$

$$E^0_{i1n} = 2\tilde{\mu}_i\frac{J_1(\tilde{\mu}_i)Z_0(1) - \varepsilon J_1(\varepsilon\tilde{\mu}_i)Z_0(\varepsilon)}{(\tilde{\mu}_i^2 - \xi_n^2)\tilde{N}_n^2}, \tag{23}$$

$$E^0_{i2n} = 2\tilde{\mu}_i\frac{Y_1(\tilde{\mu}_i)Z_0(1) - \varepsilon Y_1(\varepsilon\tilde{\mu}_i)Z_0(\varepsilon)}{(\tilde{\mu}_i^2 - \xi_n^2)\tilde{N}_n^2},$$

$$E^0_{i3n} = 2\tilde{\mu}_i \frac{I_1(\tilde{\mu}_i)Z_0(1) - \varepsilon I_1(\varepsilon\tilde{\mu}_i)Z_0(\varepsilon)}{(\tilde{\mu}_i^2 + \xi_n^2)\tilde{N}_n^2},$$

$$E^0_{i4n} = -2\tilde{\mu}_i \frac{K_1(\tilde{\mu}_i)Z_0(1) - \varepsilon K_1(\varepsilon\tilde{\mu}_i)Z_0(\varepsilon)}{(\tilde{\mu}_i^2 + \xi_n^2)\tilde{N}_n^2},$$

where

$$\mu_1^4 = \frac{1}{D_1}(k_{01}\omega^2 - \rho_1 g), \qquad \mu_2^4 = \frac{1}{D_2}(k_{02}\omega^2 + \rho_2 g), \qquad \tilde{N}_n^2 = Z_0^2(1) - \varepsilon^2 Z_0^2(\varepsilon),$$

$\tilde{\mu}_i = a\mu_i$, and J_p, Y_p, I_p, and K_p are Bessel functions of the first and second kinds of real and imaginary arguments. We now introduce dimensionless variables as follows:

$$\Omega^2 = \frac{\omega^2 a}{g}, \qquad \tilde{D}_i = \frac{D_i}{\rho_2 g a^4}, \qquad \tilde{T}_i = \frac{T_i}{\rho_2 g a^2},$$

$$k^*_{0i} = \frac{k_{0i}}{\rho_2 a}, \qquad \tilde{h}_i = \frac{h_i}{a}, \qquad \rho_{12} = \frac{\rho_1}{\rho_2}.$$

The dimensionless quantities take the following form:

$$\kappa_{in} = \xi_n \tilde{h}_i, \qquad A_n = \rho_{12}\coth\kappa_{1n} + \coth\kappa_{2n},$$

$$\tilde{\sigma}_n^2 = \frac{1}{A_n}\xi_n(1 - \rho_{12}), \qquad \tilde{b}_{1n} = \rho_{12}\frac{1}{\sinh\kappa_{1n}}, \qquad \tilde{b}_{2n} = \frac{1}{\sinh\kappa_{2n}}$$

$$\tilde{b}_n = b_{1n}\cosh\kappa_{2n} + b_{2n}\cosh\kappa_{1n}, \qquad \tilde{A}_n = A_n(\Omega^2 - \sigma_n^2),$$

$$\tilde{d}_{1n} = (\tilde{D}_1\xi_n^2 + \tilde{T}_1)\xi_n^2 - (k^*_{01}\Omega^2 - \rho_{12}), \qquad \tilde{d}_{2n} = (\tilde{D}_2\xi_n^2 + \tilde{T}_2)\xi_n^2 - (k^*_{02}\Omega^2 + 1),$$

$$\tilde{a}_{in} = \xi_n(\tilde{A}_n\cosh\kappa_{in} - \Omega^2\tilde{b}_{in}), \qquad \tilde{c}_n = \Omega^2\tilde{b}_n - \tilde{A}_n\cosh\kappa_{1n}\cosh\kappa_{2n},$$

$$\tilde{\Delta}^*_n = \xi_n^2\tilde{A}_n\tilde{d}_{1n}\tilde{d}_{2n} - \Omega^2(\tilde{a}_{1n}\tilde{b}_{1n}\tilde{d}_{2n} + \tilde{a}_{2n}\tilde{b}_{2n}\tilde{d}_{1n} + \Omega^2\tilde{b}_{1n}\tilde{b}_{2n}\tilde{c}_n),$$

$$\tilde{k}^*_{01} = -(k^*_{01}\Omega^2 - \rho_{12})^{-1}, \qquad \tilde{k}^*_{02} = -(k^*_{02}\Omega^2 + 1)^{-1},$$

$$a_{11n} = \frac{1}{\tilde{\Delta}^*_n}\Omega^2\tilde{b}_{1n}(\tilde{a}_{1n}\tilde{d}_{2n} + \Omega^2\tilde{b}_{2n}\tilde{c}_n), \qquad a_{12n} = -\frac{1}{\tilde{\Delta}^*_n}\Omega^4\xi_n\tilde{b}_{1n}\tilde{b}_{2n}\tilde{d}_{2n},$$

$$a_{21n} = -\frac{1}{\tilde{\Delta}^*_n}\Omega^4\xi_n\tilde{b}_{1n}\tilde{b}_{2n}\tilde{d}_{1n}, \qquad a_{22n} = \frac{1}{\tilde{\Delta}^*_n}\Omega^2\tilde{b}_{2n}(\tilde{a}_{2n}\tilde{d}_{1n} + \Omega^2\tilde{b}_{1n}\tilde{c}_n),$$

$$\tilde{k}_2 = \tilde{k}_{02}^*(\Omega^2 \tilde{h}_2 - 1 + \rho_{12}) - 1,$$

$$\tilde{\mu}_1^4 = \frac{k_{01}^* \Omega^2 - \rho_{12}}{\tilde{D}_1}, \qquad \tilde{\mu}_2^4 = \frac{k_{02}^* \Omega^2 + 1}{\tilde{D}_2}.$$

If $\varepsilon = 0$ and $\rho_{12} = 1$, then expressions (23) can be rewritten as follows [11]:

$$w_{i1}^0 = J_0(\tilde{\mu}_i \frac{r}{a}), \quad w_{i2}^0 = I_0(\tilde{\mu}_i \frac{r}{a}), \quad \tilde{w}_{i1}^0 = \frac{2}{\tilde{\mu}_i} J_1(\tilde{\mu}_i), \quad \tilde{w}_{i2}^0 = \frac{2}{\tilde{\mu}_i} I_1(\tilde{\mu}_i),$$

$$E_{i1n}^0 = \frac{2\tilde{\mu}_i J_1(\tilde{\mu}_i)}{(\tilde{\mu}_i{}^2 - \xi_n^2) J_0(\xi_n)}, \qquad E_{i2n}^0 = \frac{2\tilde{\mu}_i I_1(\tilde{\mu}_i)}{(\tilde{\mu}_i{}^2 + \xi_n^2) J_0(\xi_n)}.$$

5. Numerical Investigations and Conclusions

In view of the complexity of the analyzed problem, we performed numerical investigations for two most interesting cases: the absence of the upper base (the presence of free surface of the liquid) and the case of weightless $(g = 0)$ homogeneous liquid $(\rho_1 = \rho_2)$. Despite a significant number of available publications, these cases are studied quite poorly.

In the absence of the upper base $(k_{01} = 0,\ T_1 = 0,$ and $D_1 = 0)$, for the absolutely elastic bottom $(T_2 = 0)$ and the introduced dimensionless variables, we find:

$$\tilde{d}_{1n} = \rho_{12}, \qquad \tilde{d}_{2n} = \tilde{D}_2 \xi_n^4 - k_{02}^* \Omega^2 - 1, \qquad \text{and} \qquad \tilde{k}_{01}^* = \frac{1}{\rho_{12}}.$$

The functions w_{2k}^0 and the expressions $\tilde{w}_{2k}^0$ and E_{2kn}^0 are given by relations (23).

In the absence of gravitation $(g = 0)$, we introduce different dimensionless variables:

$$\Omega^2 = \frac{1}{D_2} \omega^2 \rho_2 a^5, \qquad D_{12} = \frac{D_1}{D_2}, \qquad \tilde{\mu}_1^4 = \frac{1}{D_{12}} k_{01}^* \Omega^2, \qquad \tilde{\mu}_2^4 = k_{02}^* \Omega^2.$$

If $T_i \neq 0$ and $k_{01} = k_{02} = 0$, i.e., in the case of inertialess plates, we obtain

$$w_{i1}^0 = 1, \quad w_{i2}^0 = I_0\left(\gamma_{0i} \frac{r}{a}\right), \quad w_{i2}^0 = \ln\left(\frac{r}{a}\right), \quad w_{i2}^0 = K_0\left(\gamma_{0i} \frac{r}{a}\right),$$

$$\gamma_{01}^2 = \frac{1}{D_1} T_1 a^2, \qquad \gamma_{02}^2 = \frac{1}{D_2} T_2 a^2. \tag{24}$$

As follows from relations (15)–(17) and (24) and the numerical results, for the inertialess plates and also in the case where one of the plates is absolutely rigid $(T_i \to \infty)$ and the other plate is inertialess, the frequency

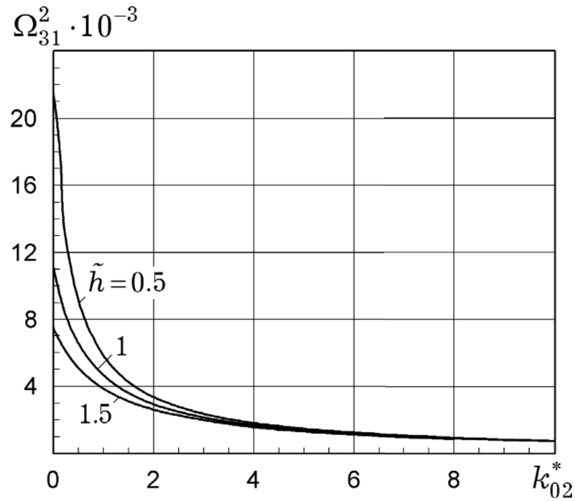

Fig. 1. Dependences of Ω_{31}^2 on k_{02}^* for different $\tilde{h}$.

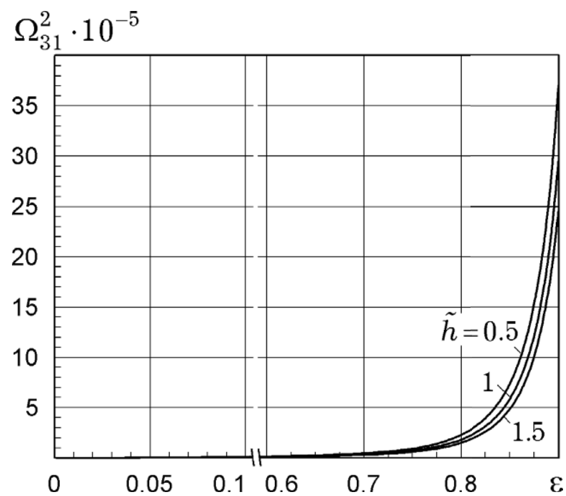

Fig. 2. Dependences of Ω_{31}^2 on ε for different $\tilde{h}$.

equation has no real roots. In addition, the frequency equation has no real roots also for the inertial plates. Hence, in the absence of gravitation, there are no axially symmetric vibrations if one of the plates is absolutely rigid.

Numerical investigations of the frequency equation (20) in the presence of bulk forces were carried out for the following values of the dimensionless parameters: $\varepsilon = 0.001 - 0.9$, $\tilde{h} = 0.5 - 1.5$, $k_{02}^* = 0 - 10$, $\tilde{D}_2 = 1$, $\tilde{T}_2 = 0$, and $\rho_{12} = 1$, where $\tilde{h} = \tilde{h}_1 + \tilde{h}_2$. In this case, three sets of frequencies are preserved and they correspond to vibrations of the upper base or of the free surface, of the elastic bottom, and of the liquid column. The calculations were performed by taking into account five terms in the series in Eq. (20), $n = 1, 2, \ldots, 5$, in the case of the free surface. We present the dependences of the squared first dimensionless frequency from the third set (frequency of vibration of the liquid column as a whole) Ω_{31}^2 on the mass characteristic of the bottom $k_{02}^* = 0 - 10$ in Fig. 1 and on $\varepsilon = 0.001 - 0.9$ in Fig. 2.

The results of our analytic and numerical investigations enable us to make the following conclusions:

1. The frequency spectrum consists of four sets of frequencies corresponding to the vibrations of the upper and lower elastic bases, of the liquid column as a whole, and of the interface of liquids. For a single-layer liquid, the frequency spectrum contains three sets of frequencies.

2. In the absence of gravitation, there are no axially symmetric vibrations if one of the plates is absolutely rigid.

3. In the presence of free surface of the two-layer liquid, the frequency spectrum consists of four sets of frequencies corresponding to vibrations of the free surface, elastic bottom, liquid column, and inner surface. In a broad range of the parameters of mechanical system, we observe weak variations of frequencies from the first and fourth sets and noticeable changes in the frequencies from the second and third sets; moreover,

— the dependence of the squared first frequency from the third set on the dimensionless stiffness is almost linear in most cases;

— as the depth of filling increases, we observe an insignificant decrease in the frequencies from the first and fourth sets and a significant decrease in the frequencies from the second and third sets;

— a significant increase in the frequencies from the third set takes place as the depth of filling decreases.

4. The series of frequency equations converge fairly rapidly. As a rule, it is sufficient to take two or three terms in series in order to attain the accuracy acceptable for practical calculations. If the mass characteristics of the plates are taken into account, then the period of solution of the frequency equations noticeably increases.

In our opinion, it is reasonable to devote subsequent investigations in this field to the generalization of the analyzed problem for different conditions of fastening of the contours of annular plates, to the determination of the natural forms of vibrations, and to the analysis of forced longitudinal vibrations of a cylindrical vessel with elastic bases and a two-layer liquid.

The present paper contains some results of investigations supported by the grant of the State Foundation for Fundamental Research within the framework of the Competitive Project F71/80–2016.

REFERENCES

1. A. V. Andronov, "On small vibrations of a perfect liquid in the vessel with elastic bottoms," *Simferopol' Univ., Deposited at Ukr. NIINTI 30.12.83,* No. 1478 (1983).
2. R. I. Bohun and V. A. Trotsenko, "Free vibrations of a fluid in a cylindrical container with arbitrary axisymmetric bottom and elastic elements on the free surface of the fluid," *Nelin. Kolyv.,* **13**, No. 4, 461–482 (2010); ***English translation:*** *Nonlin. Oscillat.,* **13**, No. 4, 493–514 (2011).
3. D. A. Goncharov, "Dynamics of two-layer liquid separated by an elastic partition with regard for the forces of surface tension," *Nauka Obraz.: Elektron. Nauch.-Tekh. Izd.,* No. 11, Bauman MGTU, Moscow (2013); http://technomag.bmstu.ru /doc/619258.html (DOI: 10.7463/1113.0619258).
4. D. A. Goncharov, "Axially symmetric vibrations of a two-density liquid in a cylindrical vessel," *Nauka Obraz.: Elektron. Nauch.-Tekh. Izd.,* No. 4, Bauman MGTU, Moscow (2012); http://technomag.bmstu.ru/doc/362856.html.
5. N. K. Didok and Yu. N. Kononov, "Dynamics and stability of vibrations of a cylindrical vessel with perfect liquid and elastic bases," *Trudy IPMM Nats. Akad. Nauk Ukrainy,* **27,** 122–131 (2013).
6. L. V. Dokuchaev, *Nonlinear Dynamics of Flying Vehicles with Deformable Elements* [in Russian], Mashinostroenie, Moscow (1987).

7. A. Yu. Karnaukh, “Vibrations of an elastic plate separating liquids in a cylindrical vessel with elastic bases,” *Izv. Vuzov. Sev.-Kavk. Region. Estestv. Nauki,* No. 2, 33–36 (2013).
8. Yu. Kononov and Yu. Dzhukha, “Axially symmetric vibrations of elastic bases and a two-layer perfect liquid in a rigid annular cylindrical vessel,” in: *Konf. Molod. Uchen. “Pidstryhach Readings-2016” (May 25–27, 2016),* IPPMM NAN Ukrainy, Lviv (2016); http://iapmm.lviv.ua/chyt2016/theses/Kononov.pdf.
9. Yu. N. Kononov and Yu. A. Dzhukha, “Axially symmetric vibrations of elastic bases and a perfect liquid in a rigid annular cylindrical vessel,” *Visn. Zaporiz. Nats. Univ., Ser. Fiz.-Mat. Nauky,* No. 1, 103–115 (2016).
10. Yu. N. Kononov, N. K. Didok, and Yu. A. Dzhukha, “On the solution of generalized inhomogeneous biharmonic equations in the problems of hydroelasticity,” *Visn. Donets'k Nats. Univ., Ser A: Pryrod. Nauky,* No. 1, 64–69 (2014).
11. Yu. N. Kononov, V. F. Rusakov, and Yu. A. Dzhukha, “Axially symmetric vibrations of elastic bases and a perfect liquid in a rigid cylindrical vessel,” *Visn. Zaporiz. Nats. Univ., Ser. Fiz.-Mat. Nauky,* No. 2, 106–115 (2015).
12. Yu. N. Kononov, V. P. Shevchenko, and Yu. A. Dzhukha, “Axially symmetric vibrations of a two-layer perfect liquid with free surface in a rigid circular cylindrical vessel with elastic bottom,” *Visn. Donets'k Nats. Univ., Ser A: Pryrod. Nauky,* No. 1-2, 116–125 (2015).
13. N. D. Kopachevskii, S. G. Krein, and N. Z. Kan, *Operator Methods in Linear Hydrodynamics: Evolutionary and Spectral Problems* [in Russian], Nauka, Moscow (1989).
14. A. A. Pozhalostin and D. A. Goncharov, “Free axially symmetric vibrations of a two-layer liquid with elastic separator between layers in the presence of forces of surface tension,” *Inzh. Zh: Nauka Innov.,* No. 12, Bauman MGTU, Moscow (2013); http://engjournal.ru/catalog/eng/teormach/1147.html.
15. H. F. Bauer and M. Chiba, “Axisymmetric vibration of a viscous liquid covered by an elastic structure,” *J. Sound Vibrat.,* **281**, No. 3-5, 835–847 (2005).
16. Z. Ding, “Free bending vibration of annular cylindrical tank partially filled with liquid in consideration of surface wave,” *Appl. Math. Mech.,* **15**, No. 9, 831–839 (1994).
17. J.-W. Jang, A. Alaniz, L. Yang, J. Powers, and C. Hall, “Mechanical slosh models for rocket-propelled spacecraft,” in: *AlAA Guidance: Navigation and Control Conference* (Aug. 19–22, 2013, Boston, USA), Amer. Inst. Aeronautics Astronautics, Reston (2013). http://ntrs.nasa.gov/ archive/nasa/casi.ntrs.nasa.gov/20140002967.pdf.
18. M. J. Jhung, W. T. Kim, and Y. H. Ryu, “Dynamic characteristics of cylindrical shells considering fluid-structure interaction,” *Nuclear Eng. Technol.,* **41**, No. 10, 1333–1346 (2009).