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ON THE CONVERGENCE RATE OF THE CONTINUOUS NEWTON METHOD

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ABSTRACT. In this paper we study the convergence of the continuous Newton method for solving nonlinear equations with holomorphic mappings in complex Banach spaces. Our contribution is based on recent progress in the geometric theory of spiral-like functions. We prove convergence theorems and illustrate them by numerical simulations.

CONTENTS

Introduction

Consider the classical problem of finding an approximate solution to a nonlinear equation

$$f(z) = 0$$

in a domain D in the complex plane $\mathbb{C}$, where $f : D \to \mathbb{C}$ is a holomorphic function in D. To this end one uses diverse modifications of the recurrence formula

$$z_{n+1} = z_n - \lambda_n \frac{f(z_n)}{f'(z_n)} \tag{0.1}$$

for $n = 0, 1, \dots$, where z_0 is an initial approximation in D and $\lambda_n > 0$. For a suitably chosen sequence $\{\lambda_n\}$, formula (0.1) is often called the damped Newton method while for $\lambda_n \equiv 1$ it is called the classical Newton method, see [7].

We focus on the classical Newton method. The convergence of (0.1) is widely explored and depends on the specific choice of the initial point $z_0 \in D$.

The recurrence formula (0.1) displays immediately the initial boundary value problem

$$\begin{cases} \dot{z} &= -\dfrac{f(z)}{f'(z)}, \quad \text{if} \quad t > 0, \\ z(0) &= z_0, \end{cases} \tag{0.2}$$

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for a curve $z = z(t)$ in D starting at z_0 and leading to a solution $a \in D$ of $f(z) = 0$. In fact, if $\lambda \neq 0$, then for $f(a)$ to vanish it is necessary and sufficient that

$$-\lambda \frac{f(a)}{f'(a)} = 0,$$

which is equivalent to

$$a - \lambda \frac{f(a)}{f'(a)} = a.$$

The standard successive approximations for solving this equation look like

$$z_{n+1} = z_n - \lambda \frac{f(z_n)}{f'(z_n)}$$

for $n = 0, 1, \dots$. On writing $z_n = z(n)$ and passing to a continuous parameter $t \in [0, \infty)$ we get

$$\frac{z(t + \Delta t) - z(t)}{\Delta t} = -\frac{f(z(t))}{f'(z(t))}$$

with $\Delta t = \lambda$. Taking the limit as $\Delta t \to 0$ yields (0.2), as desired.

As defined in (0.2), the continuous version of the Newton method was probably first introduced in [4]. Later this method and its diverse modifications were studied mostly within the framework of real analysis, see for instance [7, 13]. In [1] and [2] the continuous Newton method is developed in Hilbert spaces. The recent paper [9] studied the convergence rate of the continuous Newton method for holomorphic functions in the unit disk of the complex plane. Milano [11] applied and compared the continuous Newton method and a robust version of this method with known results for the power flow problem. In [12, Chap. 22] are presented interesting actual problems regarding the continuous Newton method and semigroups in Banach spaces.

It is worth pointing out that the vector field on the right-hand side of (0.2) just amounts to (minus) the logarithmic derivative of f. The first integral of (0.2) is

$$f(z(t)) = f(z_0)e^{-t}$$

for $t \geq 0$, as is easy to check. Hence it follows that $f(z(t)) \to 0$, when $t \to \infty$. Notice that $f(z(t))$ describes the discrepancy of the approximate solution $z(t)$, provided that $z(t)$ converges to $a \in D$ as $t \to \infty$.

Summarizing we conclude that the study of the continuous version of the Newton method consists of two main steps. The first of the two is to describe those initial data $z_0 \in D$ for which the initial boundary value problem (0.2) has a solution $z(t)$ defined for all $t \geq 0$. It is generally seen that the solution is unique whenever it exists. The second step consists in studying the asymptotic behavior of the global solution $z(t)$, as $t \to \infty$. This is precisely what the present paper is aimed at.

1. Spiral-like Mappings

Throughout this paper by D is meant a domain in a complex Banach space X endowed with a norm $\|\cdot\|$. We denote by $\mathrm{Hol}(D, X)$ the space of all holomorphic (i.e., Fréchet differentiable) mappings on D with values in X. For $f \in \mathrm{Hol}(D, X)$ we denote by $f'(x)$ the Fréchet derivative of f at a point $x \in X$. By definition, this is a bounded linear operator in X.

As usual, X^* stands for the dual space of X. By the Hahn–Banach theorem, for each $x \in X$ there is a functional $l_x \in X^*$ with the property that $l_x(x) = \|l_x\| \, \|x\|$. On normalizing l_x one obtains a functional whose norm is $\|x\|$. Write x^* for any functional $l \in X^*$ satisfying $\mathrm{Re}\, l(x) = \|x\|^2 = \|l\|^2$, and $*x$ for the set of all functionals l with this property (cf. the Hodge star operator). Such a functional x^* is in general not unique. However, if X is a Hilbert space, then the element x^* is unique and it can be identified with x, which is due to the Riesz representation theorem.

A mapping $f \in \mathrm{Hol}(D, X)$ is said to be locally biholomorphic if for each $x \in D$ there are neighborhoods $U \subset D$ of x and V of $f(x)$, such that $f|_U$ is a bijective mapping of U onto V and its inverse is holomorphic. It is well known that $f \in \mathrm{Hol}(D, X)$ is locally biholomorphic on D if and only if, for each $x \in D$, the Fréchet derivative $f'(x)$ is a bijective mapping of X. By the inverse mapping theorem of Banach, the bijectivity of $f'(x)$ implies readily the boundedness of its inverse. For a finite dimensional space X, a mapping $f \in \mathrm{Hol}(D, X)$ is locally biholomorphic if and only if it is locally one-to-one. However, this fact no longer holds for general infinite dimensional spaces X (see for instance [6]).

A bounded linear operator A in X is called strongly accretive if there is a constant $k > 0$ with the property that $\mathrm{Re}\,\langle Ax, x^*\rangle \geq k\,\|x\|^2$ for all $x \in X$ and $x^* \in *x$. The following assertion characterizes those bounded linear operators in X which have spectrum in the open right half-plane $\mathrm{Re}\,\lambda > 0$.

Lemma 1.1. *Suppose $A : X \to X$ is a bounded linear operator. The following are equivalent:*

(1) *The spectrum of the operator A lies in the open right half-plane* $\mathrm{Re}\,\lambda > 0$.
(2) *The linear semigroup* $\exp(-tA)$ *converges to* 0 *in the operator norm, as $t \to \infty$.*
(3) *There is an equivalent norm on X, such that A is strongly accretive with respect to the corresponding sesquilinear form.*

Proof. The equivalence of (1) and (2) is actually a consequence of the spectral mapping theorem. However, we will need some addition details.

Denote by $\chi(A)$ the lower exponential index of A, that is,

$$0 < \chi(A) := \inf_{\lambda \in \mathrm{sp}\,A} \mathrm{Re}\,\lambda = \lim_{t\to\infty} \frac{\log \|\exp(-tA)\|}{-t}, \tag{1.1}$$

where $\mathrm{sp}\,A$ stands for the spectrum of A (see [3]). Then for any $\lambda \in (0, \chi(A))$ there is $C > 0$ such that

$$\|\exp(-tA)\| \leq C\,\exp(-\lambda\|A\|)$$

for all $t \geq 0$. On setting

$$\|x\|_1 := \sup_{t\geq 0} \|\exp\left(-t(A - \lambda I)\right) x\|$$

we conclude that $\|x\| \leq \|x\|_1 \leq C\,\|x\|$ for all $x \in X$, which is due to (1.1), and

$$\|\exp(-tA)x\|_1 \leq \exp(-\lambda\|A\|)\,\|x\|_1 \tag{1.2}$$

for $t \geq 0$. Hence it follows that

$$\mathrm{Re}\,\langle Ax, x^*\rangle_1 \geq \lambda\,\|x\|_1^2$$

for all $x \in X$. Using the Hille–Yosida exponential formula (see [19]) one proves that the last estimate implies (1.2) (and so (2)), which completes the proof. □

Definition 1.1. Let A be a bounded linear operator in X with spectrum in the open right half-plane and D a convex domain in X containing the origin. A mapping $f \in \mathrm{Hol}(D, X)$ is called A*-spiral-like* with respect to the origin if $\exp(-tA)f(x) \in f(D)$ for all $x \in D$ and $t \geq 0$.

For $A = \lambda I$ with $\mathrm{Re}\,\lambda > 0$ we say for short that f is λ-spiral-like. If $A = I$ in the above definition, then f is called starlike with respect to the origin.

2. General Results for Banach Spaces

Suppose $f \in \mathrm{Hol}(D, X)$ is a locally biholomorphic mapping of D, such that the origin belongs to the closure of $f(D)$. When looking for an approximate solution of the nonlinear equation $f(x) = 0$ in D, one can exploit similarly to (0.2) a continuous analogue of the classical Newton method

$$\begin{cases} \dot{x} + (f'(x))^{-1} f(x) = 0, & \text{if } t > 0, \\ x(0) = x_0, & \end{cases} \tag{2.1}$$

where $x_0 \in D$ is an initial approximation. We slightly generalize it by considering

$$\begin{cases} \dot{x} + (f'(x))^{-1} A\, f(x) = 0, & \text{if } t > 0, \\ x(0) = x_0, & \end{cases} \tag{2.2}$$

where A is a bounded linear operator in X.

Definition 2.1. The method (2.2) is called *well defined* on D if for any data $x_0 \in D$ the initial value problem has a unique solution $x = x(t)$, such that $x(t) \in D$ for all $t > 0$ and the discrepancy $f(x(t))$ tends to zero as $t \to \infty$.

The following theorem gives a criterion for the continuous version of the Newton method to be well defined.

Theorem 2.1. *Suppose that f is a biholomorphic mapping on a domain $D \subset X$ and A satisfies one of the equivalent conditions of Lemma* 1.1. *Then method* (2.2) *is well defined if and only if f is A-spiral-like in X.*

Proof. Let the method defined by (2.2) is well defined. Given any $x_0 \in D$, the initial value problem (2.2) has a unique solution $x = x(t)$ with values in D and $f(x(t)) \to 0$ as $t \to \infty$. Set $y(t) = f(x(t))$. From the differential equation we get

$$\begin{aligned} \dot{y} &= f'(x)\,\dot{x} \\ &= -f'(x)\left(f'(x)\right]^{-1} A\, f(x) \\ &= -Ay \end{aligned}$$

for all $t > 0$. On the other hand, under our assumptions on A, the initial value problem

$$\begin{cases} \dot{y} + Ay = 0, & \text{if} \quad t > 0, \\ y(0) = y_0 \end{cases}$$

has a unique solution $y(t) = \exp(-tA)y_0$ for each $y_0 \in f(D)$. Hence it follows that $\exp(-tA)y_0 = f(x(t)) \in f(D)$ for all $t > 0$. Thus, f is A-spiral-like.

Conversely, if f is A-spiral-like, then, for each $x_0 \in D$, the trajectory $x(t) = f^{-1}(\exp(-tA)f(x_0))$ with $t \geq 0$ is well defined and does not go beyond the domain D. A direct calculation shows that $x = x(t)$ satisfies the initial value problem (2.2). Moreover, $f(x(t)) = \exp(-tA)f(x_0)$ tends to zero uniformly with respect to x_0 on each ball inside D, as desired. □

One can show that, if f is a locally biholomorphic mapping vanishing at a point $a \in D$ and A a linear operator in X satisfying one of the equivalent conditions of Lemma 1.1, then f is actually biholomorphic provided the method given by (2.2) is well defined. In particular, f is A-spiral-like.

3. A Nevanlinna Type Condition

Denote by $D = \mathbb{D}$ the unit disk around the origin in $\mathbb{C}$. In the one-dimensional case $X = \mathbb{C}$ a criterion for a mapping $f \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ to be starlike with respect to the origin is given by the familiar Nevanlinna condition

$$\mathrm{Re}\left(z\frac{f'(z)}{f(z)}\right) > 0$$

for all $z \in \mathbb{D}$. However, verifying such a condition might be hard because of its computational complexity. The following sufficient condition simplifies the use of Theorem 2.1.

Theorem 3.1. *Let f be a holomorphic function in $\mathbb{D}$ vanishing at the origin and satisfying*

$$\begin{aligned} f'(z) &\neq 0, \\ \mathrm{Re}\,\frac{f(z)f''(z)}{(f'(z))^2} &< 1 \end{aligned} \tag{3.1}$$

for all $z \in \mathbb{D}$. Then f is starlike on $\mathbb{D}$.

Proof. It suffices to show that (3.1) implies the Nevanlinna condition, i.e.,

$$\mathrm{Re}\, g(z) > 0$$

for all $z \in \mathbb{D}$, where

$$g(z) := \frac{1}{z}\frac{f(z)}{f'(z)}.$$

To do this we consider the function $zg(z)$ and note that condition (3.1) is equivalent to

$$\operatorname{Re}(zg)'(z) > 0$$

for all $z \in \mathbb{D}$, for

$$(zg)'(z) = \frac{(f'(z))^2 - f(z)f''(z)}{(f'(z))^2}$$

and our claim is obvious.

Thus, we have to show that $\operatorname{Re}(g(z) + zg'(z)) > 0$ implies $\operatorname{Re} g(z) > 0$. Setting $z = re^{i\varphi}$ with $r \in [0, 1)$ and $\varphi \in [0, 2\pi)$, we get

$$zg'(z) = r\frac{\partial}{\partial r}g$$

and thus

$$\operatorname{Re}\big(g(z) + zg'(z)\big) = \operatorname{Re} g(re^{i\varphi}) + \operatorname{Re}\Big(r\frac{\partial}{\partial r}g\Big) > 0. \tag{3.2}$$

We first show that from (3.2) it follows that $\operatorname{Re} g(z) \geq 0$ for all $z \in \mathbb{D}$. Suppose, contrary to our claim, that there is $z_0 = r_0e^{i\varphi_0}$ in $\mathbb{D}$, such that $\operatorname{Re} g(z_0) < 0$. From (3.2) we get $\operatorname{Re} g(0) > 0$. Hence, there is $r_1 \in (0, r_0)$ such that

$$\begin{aligned} \operatorname{Re} g(r_1e^{i\varphi_0}) &= 0, \\ \operatorname{Re} g(r_0e^{i\varphi_0}) &< 0, \end{aligned}$$

and thus one can find $r_2 \in (r_1, r_0)$ with the property that $\operatorname{Re} g(r_2e^{i\varphi_0}) < 0$ and

$$\operatorname{Re}\Big(\frac{\partial}{\partial r}g\Big)(r_2e^{i\varphi_0}) < 0$$

which contradicts (3.2). We thus conclude that $\operatorname{Re} g(z) \geq 0$ everywhere in $\mathbb{D}$.

If we assume that $\operatorname{Re} g(z_0) = 0$ for some $z_0 \in \mathbb{D}$, then it follows by the maximum principle for holomorphic functions that $g(z) = ic$ for all $z \in \mathbb{D}$, where c is a real constant. Hence, $\operatorname{Re}(g(z) + zg'(z)) = 0$, which is impossible. □

Example 3.1. Let f be a holomorphic function in $\mathbb{D}$ determined by the equation

$$f(z) = -f'(z)\,(z + 2\log(1 - z)).$$

In this case we have

$$\operatorname{Re}\Big(z\frac{f'(z)}{f(z)}\Big) = -\operatorname{Re}\frac{z}{z + 2\log(1 - z)}$$

and it is not clear how to see if the Nevanlinna condition holds. On the other hand, since

$$\operatorname{Re}\frac{f(z)f''(z)}{(f'(z))^2} = \operatorname{Re}\Big(1 - \Big(\frac{f(z)}{f'(z)}\Big)'\Big),$$

one easily verifies that

$$\operatorname{Re}\frac{f(z)f''(z)}{(f'(z))^2} = \operatorname{Re}\Big(1 - \frac{1+z}{1-z}\Big) < 1,$$

and thus the continuous Newton method is well defined.

4. A Canonical Reduction

To clarify the remark after Theorem 2.1 we first consider a more general version of the continuous Newton method. Namely, let g be a holomorphic mapping of D to X (not necessarily locally biholomorphic) and let $h \in \mathrm{Hol}(D, X)$ have invertible total derivative $h'(x)$ at each point $x \in D$.

We study the behavior of the solution $x = x(t)$ (if there is any) to the initial value problem

$$\begin{cases} \dot{x} + (h'(x))^{-1} A\, g(x) = 0, & \text{if } t > 0, \\ x(0) = x_0 \end{cases} \tag{4.1}$$

for large t, where $x_0 \in D$ is an initial approximation. If g is locally biholomorphic, one can choose $h = g =: f$, thus recovering the continuous Newton method of (2.2). In a sense the converse assertion holds also true.

Theorem 4.1. *Let g and h be holomorphic mappings on D and $h'(x)$ be invertible at each point $x \in D$. Suppose* (4.1) *is well defined on D, with $g(a) = 0$ and $A = h'(a)\,(g'(a))^{-1}$ for some $a \in D$. Then there is a biholomorphic mapping f on D, such that the method* (2.1) *is well defined on D and the solutions of* (4.1) *and* (2.1) *are the same and converge to a as $t \to \infty$.*

Proof. Given any $x_0 \in D$, let $x = x(t, x_0)$ be the solution of (4.1). We define the mapping f by

$$f(x_0) = \lim_{t\to\infty} e^t\,(x(t, x_0) - a)\,. \tag{4.2}$$

First we show that this limit exists for each $x_0 \in D$. For simplicity we set $a = 0$. Consider the mapping $Q \in \mathrm{Hol}(D, X)$ given by the formula $Q(x) := (h'(x))^{-1} A g(x)$. Since $Q(0) = 0$ and $Q'(0) = I$, the Taylor expansion of Q looks like

$$Q(x) = x + \sum_{k=k_0}^{\infty} P_k(x)$$

for x in a ball $B_r \subset D$ of radius $r > 0$ with center at the origin, where $k_0 \geq 2$ and P_k are homogenous polynomials of degree k on X. By the Schwarz lemma,

$$\|Q(x) - x\| \leq \frac{M}{r^{k_0}}\,\|x\|^{k_0},$$

where $M = \sup\limits_{x\in D} \|Q(x) - x\|$ (see for instance [14]).

A simple calculation shows that $\mathrm{Re}\,\langle Q(x), x^*\rangle > 0$ for all $x \neq 0$ satisfying

$$\|x\| < \min\left\{\left(\frac{M}{r^{k_0}}\right)^{\frac{1}{k_0-1}}, r\right\} = r_1.$$

This means that the ball B_{r_1} is invariant for the solution $x(t, \cdot)$ of (4.1), i.e., $\|x(t, x_0)\| < r_1$ for all $t \geq 0$ and $x_0 \in B_{r_1}$. Without loss of generality we can assume that $r_1 = 1$. Then it follows from [14, Corollary 9.1] that

$$\|x(t, x_0)\| \leq e^{-t}\,\frac{\|x_0\|}{(1 - \|x_0\|)^2},$$

and thus

$$\begin{aligned} \|e^t\,(Q(x(t, x_0)) - x(t, x_0))\,\| &\leq e^t\,\frac{M}{r^{k_0}}\,\|x(t, x_0)\|^{k_0} \\ &\leq e^{(1-k_0)t}\,\frac{M}{r^{k_0}}\,\frac{\|x_0\|^{k_0}}{(1 - \|x_0\|)^{2k_0}} \\ &\to 0 \end{aligned}$$

since $k_0 \geq 2$. Setting now $y(t, x_0) = e^t\,x(t, x_0)$, we get

$$\dot{y}(t, x_0) = e^t\,(x(t, x_0) - Q(x(t, x_0))) \to 0$$

as $t \to \infty$ for each $x_0 \in B_1$. Thus the limit (4.2)

$$\lim_{t\to\infty} e^t\, x(t, x_0) = \lim_{t\to\infty} y(t, x_0) =: f(x_0)$$

exists for all $x_0 \in B_1$.

The global convergence for all $x_0 \in D$ follows now from the fact that one can find a sufficiently large $T > 0$ with the property that $x(T, x_0) \in B_1$. Therefore, using the semigroup property one concludes that

$$\begin{aligned} \lim_{t\to\infty} e^{T+t}\, x(T+t, x_0) &= e^T \lim_{t\to\infty} e^t\, x(t, x(T, x_0)) \\ &= e^T f(x(T, x_0)). \end{aligned}$$

We have actually proved that

$$e^{-s} f(x_0) = f(x(s, x_0)) \in D$$

for any $s \geq 0$, which means that f is a starlike mapping. Moreover, on differentiating the latter equality in $s \geq 0$ we see that $x(s, x_0)$ satisfies (2.1), as desired. □

5. A Local Continuous Newton Method

In this section, we study the following problem. Let $f \in \mathrm{Hol}(D, X)$ be a locally biholomorphic mapping satisfying $f(0) = 0$. A general question is whether there is a ball B_r in D such that the continuous Newton method is well defined on B_r. For example, in the one-dimensional case a well-known result due to Grunksy says that each univalent function f on the unit disk $\mathbb{D}$ is starlike on D_r with $0 < r \leq \tanh(\pi/4)$, see [5]. However, this is no longer true in higher dimensions, and thus additional conditions are required. In [17, 18] it was shown that a holomorphic mapping on the open unit ball $\mathbb{B} := \{x \in X : \|x\| < 1\}$ with $f(0) = 0$ is starlike if and only if $\mathrm{Re}\, \langle (f'(x))^{-1} f(x), x^* \rangle \geq 0$ for all $x^* \in *x$.

We consider a weaker condition on f, namely

$$\mathrm{Re}\, \langle (f'(x))^{-1} f(x), x^* \rangle \geq -m\, \|x\|^2 \tag{5.1}$$

for all $x^* \in *x$, where m is a nonnegative constant. We show that the answer to the above question is affirmative.

Other local problems are described as follows. Let λ be a complex number satisfying $\mathrm{Re}\, \lambda > 0$ and $\arg \lambda \in (0, \pi/2)$. Suppose $f : \mathbb{B} \to X$ is a locally biholomorphic mapping on $\mathbb{B}$, such that $f(0) = 0$ and the generalized continuous Newton method with $A = \lambda I$ is well defined. We ask whether the continuous Newton method is well defined on a possibly smaller ball. Conversely, if the continuous Newton method is well defined, is there a number $r \in (0, 1)$ depending on λ, such that the generalized continuous Newton method is well defined on the ball B_r?

To answer these question, we replace (5.1) by a more general condition. More precisely,

$$\mathrm{Re}\, \langle e^{i\varphi} (f'(x))^{-1} f(x), x^* \rangle \geq -m\, \|x\|^2 \tag{5.2}$$

for all $x^* \in *x$.

Theorem 5.1. *Let f be a locally biholomorphic mapping on $\mathbb{B}$ satisfying $f(0) = 0$. Suppose that condition* (5.2) *is fulfilled with some $m \geq 0$ and $-\pi/2 < \varphi < \pi/2$. Then, for each $0 < r < r_0$, the continuous Newton method given by* (2.1) *is well defined on B_r and it converges to the origin, where $r_0 = r_0(\varphi) \leq 1$ is the unique root of the quadratic equation*

$$(1 - r^2) - 2r(1 - r\cos\varphi)(m + \cos\varphi) = 0 \tag{5.3}$$

in $(0, 1]$.

Proof. Denote $g(x) := (f'(x))^{-1}f(x)$. By assumption,

$$\mathrm{Re}\,\langle e^{i\varphi}g(x), x^*\rangle \ge -m\,\|x\|^2$$

for all $x^* \in *x$. Write $x = zv$ where $z \in \mathbb{C}$ and $\|v\| = \|v^*\| = 1$. Consider the function $h(z) = \langle g(zv), v^*\rangle$. We get

$$\mathrm{Re}\,\langle e^{i\varphi}g(zv), (zv)^*\rangle = \mathrm{Re}\; e^{i\varphi}h(z)\overline{z} \ge -m\,|z|^2.$$

From $h(0) = 0$ it follows that there is a holomorphic function Q on the disk $\mathbb{D}$, such that $h(z) = zQ(z)$. Then $h'(0) = Q(0) = 1$ and, by the above,

$$\mathrm{Re}(e^{i\varphi}\,|z|^2Q(z)) \ge -m\,|z|^2$$

or $\mathrm{Re}(e^{i\varphi}\,Q(z)) \ge -m$ whenever $|z| < 1$. On applying an inequality of [8] we calculate

$$\begin{aligned}
\mathrm{Re}\,(Q(z)-Q(0)) &= \mathrm{Re}\left(e^{-i\varphi}((e^{i\varphi}Q)(z)-(e^{i\varphi}Q)(0))\right)\\
&\ge \frac{2r(1-r\cos\varphi)}{1-r^2}\Big(\inf_{|\zeta|<1}\mathrm{Re}\,(e^{i\varphi}Q)(\zeta)-\mathrm{Re}\,(e^{i\varphi}Q)(0)\Big)\\
&\ge \frac{2r(1-r\cos\varphi)}{1-r^2}\,(-m-\cos\varphi)
\end{aligned}$$

for all $z \in B_r$ and $r \in (0,1)$.

Since $\mathrm{Re}\,Q(0) = 1$ we get

$$\mathrm{Re}\,Q(z) \ge 1 + \frac{2r(1-r\cos\varphi)}{1-r^2}\,(-m-\cos\varphi)\,,$$

which can be equivalently rewritten as

$$F(r,\varphi) := (1-r^2) - 2r(1-r\cos\varphi)(m+\cos\varphi) \ge 0.$$

By assumption, $-\pi/2 < \varphi < \pi/2$, and thus $F(0,\varphi) = 1 > 0$ and $F(1,\varphi) = -2(1-\cos\varphi)(m+\cos\varphi) \le 0$. Therefore, the equation $F(r,\varphi) = 0$ has a unique solution $r_0 = r_0(\varphi)$ in the interval $(0,1]$. It follows that $F(r,\varphi) \ge 0$ for all $r \in (0, r_0]$. Thus, f is starlike on the ball B_r for each $0 < r \le r_0$, as desired. □

For $\varphi = 0$ the formulation of Theorem 5.1 is especially simple.

Corollary 5.1. *Let f be a locally biholomorphic mapping on $\mathbb{B}$ vanishing at the origin. Assume that condition* (5.1) *holds for some $m \ge 0$. Then, for each $0 < r < 1/(1+2m)$, the continuous Newton method is well defined on B_r and it converges to the origin.*

Example 5.1. Let $f(z) = \dfrac{z}{1-z-k}$, where $k \in [0,1)$. In this case we have

$$(f'(z))^{-1}f(z) = \frac{1}{1-k}\,z(1-z-k).$$

Obviously,

$$\mathrm{Re}\,\langle (f'(z))^{-1}f(z), \overline{z}\rangle \ge -\frac{k}{1-k}\,|z|^2$$

which means that $m = k/(1-k)$ in (5.1). Thus, Theorem 5.1 applies, showing that the continuous Newton method with given f is well defined on B_r provided $r < r_0 = (1-k)/(1+k)$. Moreover, this method converges to the origin and the estimate

$$\frac{\|x(t)\|}{1-\|x(t)\|^2} \le e^{-t}\,\frac{\|x_0\|}{1-\|x_0\|^2}$$

holds for all initial data $x_0 \in B_{r_0}$.

A computer simulation shows that for x_0 away from the ball B_{r_0} the trajectory fails to converge to the origin, see Figure 1.

Choosing $m = 0$ in Corollary 5.1 and solving Eq. (5.3) we obtain in the same way

Corollary 5.2. *Let $f : \mathbb{B} \to X$ be a locally biholomorphic mapping on $\mathbb{B}$ satisfying $f(0) = 0$. Suppose that the generalized continuous Newton method corresponding to $A = \lambda I$, where $|\arg \lambda| < \pi/2$, is well defined. Then the continuous Newton method is well defined on the ball B_r whenever*

$$r \leq (\sqrt{2} \cos(\arg \lambda - \pi/4)^{-1} < 1.$$

Converse considerations lead us to the following result.

Theorem 5.2. *Assume that $f : \mathbb{B} \to X$ is a locally biholomorphic mapping on the unit ball, such that $f(0) = 0$ and the continuous Newton method is well defined. Then, for each $\varphi \in (-\pi/2, \pi/2)$ and r satisfying $0 < r \leq (1 - |\sin \varphi|)/\cos \varphi < 1$, the generalized continuous Newton method with $A = \lambda I$, where $\arg \lambda = \varphi$, is well defined on the smaller ball B_r.*

6. An Example

In this section, we consider an example mentioned in [15]. As usual, $\mathbb{D}$ stands for the open unit disk in the complex plane. Consider the function

$$f(z) = \frac{z}{1-z};$$

one verifies easily that

$$g(z) = \frac{f(z)}{f'(z)} = z(1-z).$$

Since $\operatorname{Re} g(z)\overline{z} \geq 0$ for all $z \in \mathbb{D}$, the continuous Newton method is well defined.

In Figure 3 we present several trajectories of the analytic solution along with approximation by the continuous Newton method. In addition in Fig. 2 we present the difference in norm between two successive iterations. If now we choose the same g but with $A = e^{-i(\pi/4)}$, we can see in Fig. 4 that the generalized continuous Newton method is not well defined. For instance, on taking $z_0 = (1+i)/\sqrt{2}$ we make certain that the solution is no longer invariant with respect to the whole unit disk. On the other hand, in Fig. 5 we observe that the solution is invariant for a small disc of radius $r_0 = \sqrt{2} - 1$.

7. A Convexity Condition

In the one-dimensional case one can show sufficient geometric conditions which guarantee not only the convergence of the continuous Newton method but also the existence of a unique solution. Convexity is a simple condition under which also the general continuous Newton method given by (4.1) with $A = I$ and $h(x) \equiv x$ works. It is based on the initial value problem

$$\begin{cases} \dot{z} + f(z) &= 0, \quad \text{if} \quad t > 0, \\ z(0) &= z_0, \end{cases} \tag{7.1}$$

where z_0 is an initial approximation. Moreover, even if $f(0) \neq 0$, one can point out a restriction on $f(0)$ which guaranties the existence of null point of f and the convergence of the trajectory of 7.1 to this point.

Theorem 7.1. *Let f be a holomorphic function in the disk $\mathbb{D}$, such that $f(0) < 1/2$, $f'(0) = 1$ and $f(\mathbb{D})$ is convex. Then,*

(1) *the equation $f(z) = 0$ has a unique solution a in $\mathbb{D}$;*
(2) *Cauchy problem (7.1) has a unique solution in $\mathbb{D}$ which converges to a for each $z_0 \in \mathbb{D}$;*
(3) *if in addition $f(0) = 0$ then we can also evaluate the convergence rate, to wit $|z(t, z_0)| \leq e^{-\varrho t}|z_0|$ for all $t \geq 0$, where $\varrho \in (0, 1/2]$.*

Proof. (1) Consider the mapping $g(z) := f(z) - f(0)$. Since $f(\mathbb{D})$ is convex, so is $g(\mathbb{D})$, and $g'(0) = f' = 1$. In addition, it follows from the convexity of $g(\mathbb{D})$ by [10, 16] that

$$\operatorname{Re} g(z)\overline{z} \geq \frac{1}{2}|z|^2.$$

For the original function f this reduces to

$$\operatorname{Re} f(z)\bar{z} \geq \frac{1}{2}|z|^2 - |f(0)||z|.$$

We want to show that the right-hand side of the inequality is greater than 0, that is

$$\frac{1}{2}|z|^2 - |f(0)||z| > 0. \tag{7.2}$$

Pick an arbitrary ε in the interval $(0,1)$ and set $r := 1-\varepsilon$; then

$$\frac{1}{2}|z| - |f(0)| = \frac{1}{2}(1-\varepsilon) - |f(0)|$$

for all z on the circle $|z| = r$. Hence, inequality (7.2) is fulfilled on the circle $|z| = r$ if and only if $|f(0)| < (1/2)(1-\varepsilon)$.

By assumption, $|f(0)| < 1/2$, and thus there is an $\varepsilon > 0$ with the property that $|f(0)| < \frac{1}{2}(1-\varepsilon)$. On setting $r = 1-\varepsilon$ we get

$$\operatorname{Re} f(z)\bar{z} \geq \varepsilon \tag{7.3}$$

for all $z \in \mathbb{D}$ satisfying $|z| = r$. By the Rouché principle, inequality (7.3) provides the existence and uniqueness of a null point of f in a disk of radius r around the origin. Since r can be chosen arbitrarily close to 1, this null point is actually unique in $\mathbb{D}$, which establishes (1).

On the other hand, inequality (7.3) implies that the Cauchy problem of (7.1) is solvable in $\mathbb{D}$. This proves the item (2).

Now, if $f(0) = 0$, then $g = f$, and from the convexity of $f(\mathcal{D})$ it follows by [10, 16] that

$$\operatorname{Re} \frac{f(z)}{z} > \frac{1}{2} \quad \text{or} \quad \operatorname{Re} f(z)\bar{z} > \frac{1}{2}|z|^2,$$

as is easy to see. Thus, we get an estimate

$$|z(t,z_0)| \leq e^{-\varrho t}|z_0|$$

with some $\varrho \in (0, 1/2]$ for the convergence rate; see [14]. This establishes (3). □

The applicability of Theorem 7.1 is illustrated by our next example, which can be solved immediately.

Example 7.1. Consider the equation $f(z) = 0$ for an unknown $z \in \mathbb{D}$, where

$$f(z) = \frac{1}{2}\Big(\frac{1+z}{1-z} - \frac{1}{2}\Big).$$

Since $f(0) = 1/4$, it follows that $|f(0)| < 1/2 - \varepsilon$ for all $\varepsilon \in (0, 1/4)$. In addition, we get

$$f'(z) = \frac{1}{(1-z)^2}$$

whence $f'(0) = 1$. All the hypotheses of the above theorem hold, and thus the continuous Newton method of (7.1) converges. The solution can be found via a simple calculation, and it is $a = -1/3 \in \mathbb{D}$.

In Fig. 6 we present several trajectories of the analytic solution along with its approximations by the Euler method. In Fig. 7 one sees the norm difference between any two successive iterations.

Theorem 7.1 extends directly to holomorphic mappings of the unit ball $\mathbb{B}$ around the origin in $\mathbb{C}^n$ into $\mathbb{C}^n$. The proof is actually the same, the only difference being that instead of [10, 16] we use a recent result of Jerry Muir concerning the accretivity of normalized convex holomorphic mappings of $\mathbb{B}$. He informed us in a private communication that the following is true and has to be published in the proceedings of the Cortona 2014 conference. Suppose $f : \mathbb{B} \to \mathbb{C}^n$ is a holomorphic mapping, such that $f(0) = 0$, $f'(0) = I$ and $f(\mathbb{B})$ is convex. Then $\operatorname{Re}\langle 2f(z) - z, \overline{z}\rangle > 0$, i.e., $\operatorname{Re}\langle f(z), \overline{z}\rangle > (1/2)|z|^2$ for all $z \in \mathbb{B}$. Note that the condition $\langle f(z), \overline{z}\rangle \neq 0$ for all z on the boundary of $\mathbb{B}$ implies that there is precisely one null point a of f in $\mathbb{B}$, which is due to the Rouché principle. The stronger condition

$\mathrm{Re}\,\langle f(z), \overline{z} \rangle > 0$ for all $z \in \partial\mathbb{B}$ implies that the trajectories $z = z(t, z_0)$ of $\dot{z} + f(z) = 0$ for $z_0 \in \mathbb{B}$ are repulsed from the boundary $\partial\mathbb{B}$ into the ball. However, they need not converge to a as $t \to \infty$ for all $z_0 \in \mathbb{B}$ unless the mapping f is accretive.

8. Figures

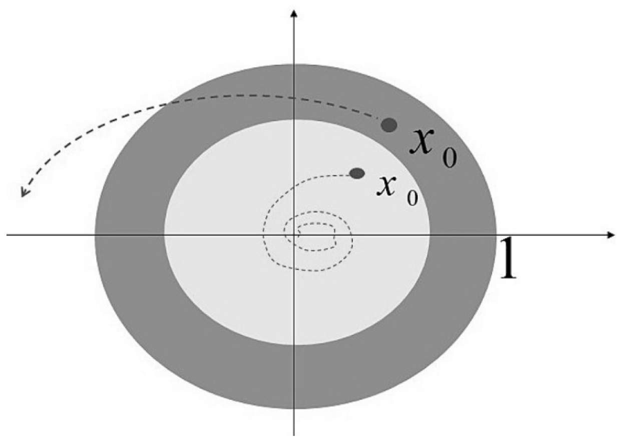

Fig. 1. The trajectory for two different x_0.

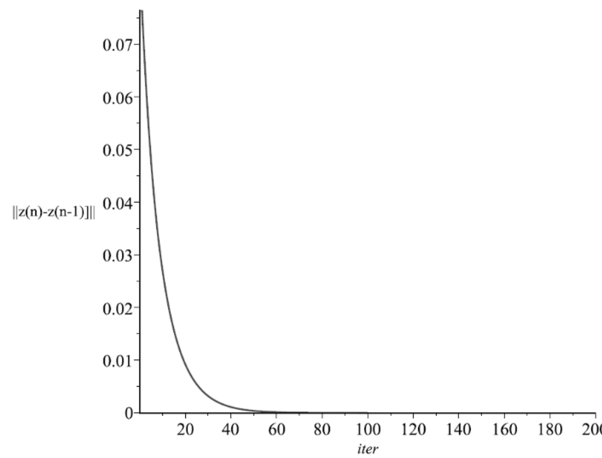

Fig. 2. Difference in norm between two successive iterations of the continuous Newton method.

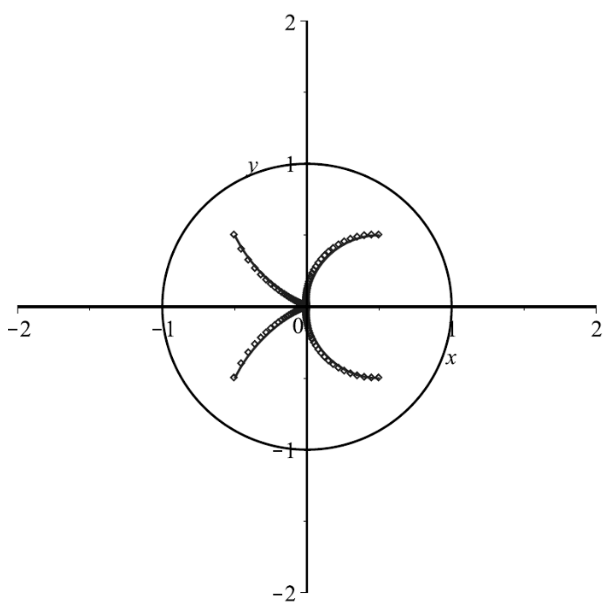

Fig. 3. Trajectories of the approximate solution (blue) starting from different x_0. In red are the exact solutions.

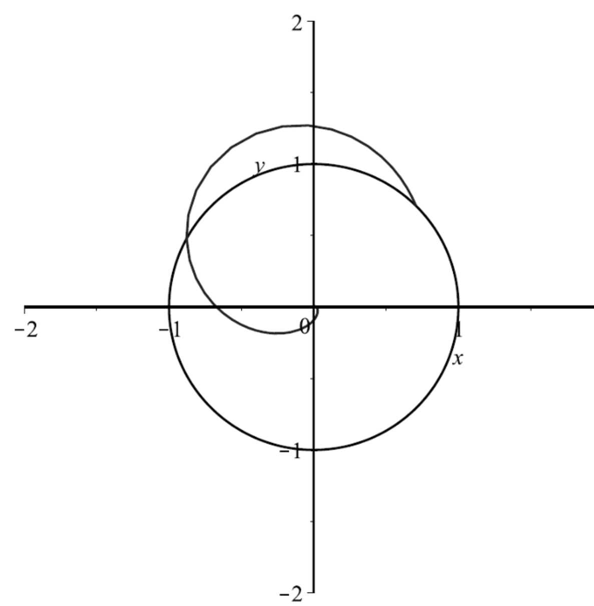

Fig. 4. The trajectory of the solution by the generalized continuous Newton method with $Ag(z) = e^{-i(\pi/4)}z(1 - z)$.

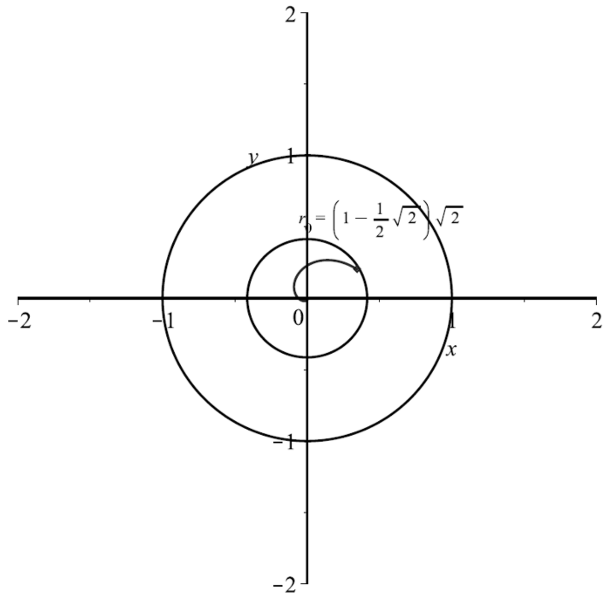

Fig. 5. The solution is invariant only for a small disk of radius $r_0 = \sqrt{2} - 1$.

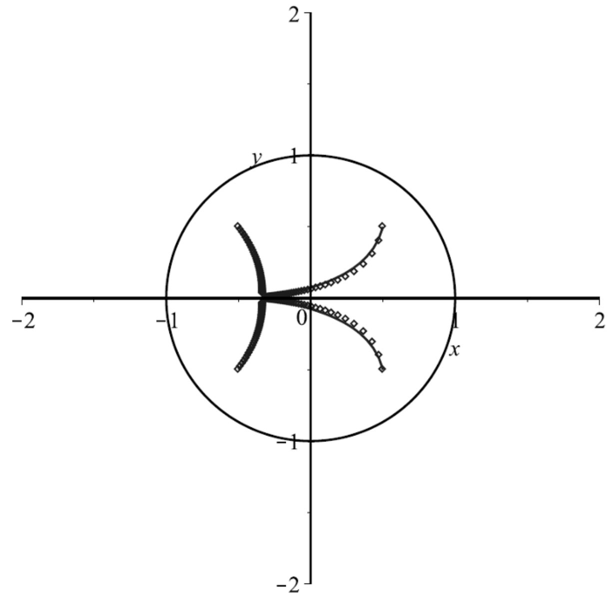

Fig. 6. In blue are the trajectories of the approximate solution starting from different z_0. In red are the exact solutions.

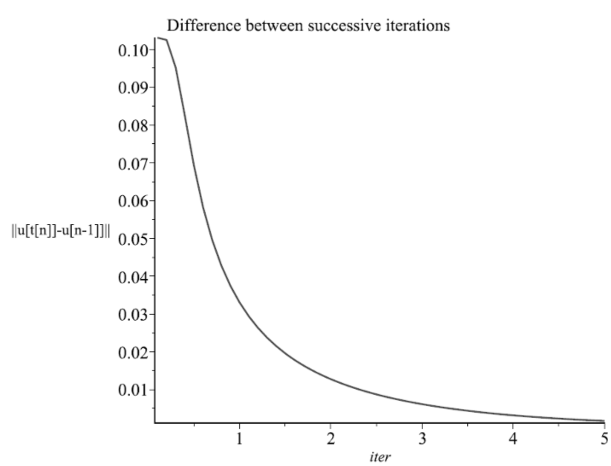

Fig. 7. Difference in norm between two successive iterations.

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