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DIFFERENTIAL OPERATORS OF INFINITE ORDER IN THE SPACE OF FORMAL LAURENT SERIES AND IN THE RING OF POWER SERIES WITH INTEGER COEFFICIENTS

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We study the Hurwitz product (convolution) in the space of formal Laurent series over an arbitrary field of zero characteristic. We obtain the convolution equation which is satisfied by the Euler series. We find the convolution representation for an arbitrary differential operator of infinite order in the space of formal Laurent series and describe translation invariant operators in this space. Using the p-adic topology in the ring of integers, we show that any differential operator of infinite order with integer coefficients is well defined as an operator from $\mathbb{Z}[[z]]$ to $\mathbb{Z}_p[[z]]$. Bibliography: 20 titles.

The classical Laplace–Borel transform sends an integer-valued function of exponential type to a Laurent series with a nontrivial domain of convergence (cf., for example, [1]). In this paper, we consider formal Laurent series in the space $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$, where F is an arbitrary field of zero characteristic regarded as the image of a formal power series under the formal Laplace transform (cf. Section 1). We note that formal Laurent series were used in the works on function theory (cf., for example, [2, 3]), vertex operator algebras and conformal field theory [4], combinatorics [5], and differential equations [6, 7]. The Hurwitz product of Laurent series is widely used in function theory and combinatorics (cf., for example, [8, 9]). An analog of the Hurwitz product (convolution) of the Euler series

$$\mathscr{E}_b(z) = \frac{1}{z} - \frac{1!b}{z^2} + \frac{2!b^2}{z^3} - \frac{3!b^3}{z^4} + \dots$$

and an arbitrary formal power series with integer coefficients was established in [7] with the help of the p-adic topology in $\mathbb{Z}$.

In this paper, we obtain the convolution equation satisfied by the Euler series $\sum\limits_{n=0}^{\infty} \frac{a^n n!}{z^{n+1}}$ (cf. Example 1.1). This equation can be regarded as a counterpart of the functional equation for

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the exponential function or geometric progression. In Section 2, we study differential operators of infinite order in the space of formal Laurent series $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$. We prove that such operators are convolution operators (Theorem 2.1). In Theorem 2.2, for the space $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$ we obtain an analog of the classical characterization of translation invariant linear operators (cf., for example, [10]–[14]). We note that all constructions and assertions in Section 2, except for Proposition 2.2 and Theorem 2.2, remain valid in the case where F is an arbitrary commutative ring with unit. More information about differential operators of infinite order and holomorphic functions can be found in [15, 16].

In Section 3, we discuss differential operators of infinite order in the ring $\mathbb{Z}[[z]]$ of formal power series with integer coefficients. We note that, in the space $F[[z]]$, where F is a field of zero characteristic, such operators are not well defined; for example, if

$$\varphi(z) = 1 + z + z^2 + z^3 + \dots, \quad g(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n,$$

then

$$\varphi\left(\frac{d}{dz}\right) g(z) = e^z + e^z + e^z + \dots .$$

Using the p-adic topology in the ring of integers, we show that any differential operator of infinite order with integer coefficients is well defined as an operator from $\mathbb{Z}[[z]]$ to $\mathbb{Z}_p[[z]]$, where $\mathbb{Z}_p$ is the ring of integer p-adic numbers (cf. Theorem 3.1).

1 Preliminaries

Let F be an arbitrary field of zero characteristic. We denote by $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$ the vector space of formal Laurent series

$$\sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}}, \quad c_n \in F,$$

and introduce the Krull topology in this space (cf. [17]).

Definition 1.1. For the power series

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n \in F[[z]]$$

we denote by $L(\varphi)$ or Φ its formal Laplace transform

$$L(\varphi)(s) = \Phi(s) = \sum_{n=0}^{\infty} \frac{n! a_n}{s^{n+1}}.$$

We note that L is an isomorphism between the vector spaces $F[[z]]$ and $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$.

Definition 1.2. Let V be a vector space over the field F, and let $V\left[\left[s, \frac{1}{s}\right]\right]$ be the vector space of all formal Laurent series with coefficients in V. For the series

$$h(s) = \sum_{n=-\infty}^{+\infty} b_n s^n \in V\left[\left[s, \frac{1}{s}\right]\right]$$

we introduce its $\operatorname{Res}(h(s)) = b_{-1}$.

In the space $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$, we introduce the convolution (the Hurwitz product). Let

$$g(z) = \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}} \in \frac{1}{z}F\left[\left[\frac{1}{z}\right]\right].$$

We regard $g(z-s)$ as an element of the space $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right][[s]]$, i.e., the space of formal power series in s with coefficients in $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$:

$$g(z-s) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \frac{c_n}{(1-\frac{s}{z})^{n+1}} = \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}} \left(1 + \frac{s}{z} + \frac{s^2}{z^2} + \cdots\right)^{n+1}. \tag{1.1}$$

Definition 1.3. For Laurent series $g_1, g_2 \in \frac{1}{s}F\left[\left[\frac{1}{s}\right]\right]$ we consider the product $g_1(s)g_2(z-s)$ as an element of $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]\left[\left[s, \frac{1}{s}\right]\right]$ and, based on Definition 1.2, introduce the *convolution*

$$(g_1 * g_2)(z) = \operatorname{Res}(g_1(s)g_2(z-s)) \in \frac{1}{z}F\left[\left[\frac{1}{z}\right]\right].$$

The following assertion shows that the convolution well agrees with the Laplace transform.

Proposition 1.1. *Let $\varphi, \psi \in F[[z]]$. Then*

$$L(\varphi\psi) = L(\varphi) * L(\psi). \tag{1.2}$$

Proof. Let

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \psi(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Since

$$\frac{n!}{(1-\frac{s}{z})^{n+1}} = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)\cdots(k+n)}{z^k} s^k$$

in the ring $F\left[\left[\frac{1}{z}\right]\right][[s]]$, from (1.1) it follows that

$$L(\psi)(z-s) = \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}} \left(\sum_{k=0}^{\infty} \frac{(k+1)(k+2)\cdots(k+n)}{z^k} s^k\right)$$
$$= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(k+1)(k+2)\cdots(k+n)b_n}{z^{n+k+1}}\right) s^k.$$

We compute $L(\varphi) * L(\psi)$:

$$\operatorname{Res}(L(\varphi)(s) \cdot L(\psi)(z-s)) = \operatorname{Res}\left(\sum_{k=0}^{\infty} \frac{k!a_k}{s^{k+1}} \cdot \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(k+1)(k+2)\cdots(k+n)b_n}{z^{n+k+1}}\right) s^k\right)$$

$$= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(k+n)!a_k b_n}{z^{n+k+1}}\right) = \sum_{k=0}^{\infty} \left(\sum_{j=k}^{\infty} \frac{j!a_k b_{j-k}}{z^{j+1}}\right) = \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} a_k b_{j-k}\right) \frac{j!}{z^{j+1}} = L(\varphi\psi)(z).$$

The required assertion is proved. □

Corollary 1.1. *The vector space* $\frac{1}{z}F\Big[\Big[\frac{1}{z}\Big]\Big]$ *with the convolution operation understood in the sense of Definition* 1.3 *is an associative commutative algebra with unit.*

Remark 1.1. Let

$$f, g \in \frac{1}{z}F\Big[\Big[\frac{1}{z}\Big]\Big], \quad f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}, \quad g(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}}.$$

Using the definition of convolution, we can show (cf. [6]) that

$$(f * g)(z) = \sum_{n=0}^{\infty} \frac{(-1)^n a_n}{n!} g^{(n)}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n b_n}{n!} f^{(n)}(z).$$

Remark 1.2. Assume that $F = \mathbb{C}$, $f, g \in d\frac{1}{z}\mathbb{C}\Big[\Big[\frac{1}{z}\Big]\Big]$, and the series $f(z)$ and $g(z)$ converge for $|z| > r$, where $r > 0$. Thus, $f(z)$ and $g(z)$ are holomorphic in the domain $\{z \in \mathbb{C} : |z| > r\}$. Using the integral Cauchy formula and Remark 1.1, we can show that

$$(f * g)(z) = \frac{1}{2\pi i} \oint\limits_{|s|=R} f(s)g(z-s)ds, |z| > R + r,$$

where $R > r$ (cf. also [11]).

Example 1.1. Let F be an arbitrary field of zero characteristic. For $a \in F$ we consider the exponential e^{az} as an element of the ring $F[[z]]$. The family of formal power series $\{e^{az} : a \in F\}$ satisfies the functional equation $e^{(a+b)z} = e^{az}e^{bz}$, $a, b \in F$. Now, we consider the family of geometric progressions

$$\Big\{\varphi_a(z) = \sum_{n=0}^{\infty} a^n z^n : a \in F\Big\}.$$

This family is characterized by the equality $(a - b)\varphi_a(z)\varphi_b(z) = a\varphi_a(z) - b\varphi_b(z)$, $a, b \in F$. The functional equations for formal power series connected with the classical equations were studied in many works. On the other hand, the series

$$\frac{1}{z}\varphi_a\Big(\frac{1}{z}\Big) = \sum_{n=0}^{\infty} \frac{a^n}{z^{n+1}}$$

is a result of the formal Laplace transform of the exponential e^{az} (cf. Definition 1.1). Making the change of variables $1/z \rightsquigarrow z$ and applying the formal Laplace transform, we obtain the series

$$\sum_{n=1}^{\infty} \frac{a^{n-1} n!}{z^{n+1}}$$

which almost coincides with the Euler series

$$f_a(z) = \sum_{n=0}^{\infty} \frac{a^n n!}{z^{n+1}}.$$

We show that for the family of series $\{f_a(z) : a \in F\}$ the following convolution equation holds:

$$(a - b)(f_a * f_b) = af_a - bf_b, \quad a, b \in F. \tag{1.3}$$

Let

$$\varphi_a(z) = (L^{-1} f_a)(z) = \sum_{n=0}^{\infty} a^n z^n.$$

Then $(a-b)\varphi_a(z)\varphi_b(z) = a\varphi_a(z) - b\varphi_b(z)$. Now, the equality (1.3) follows from Theorem 1.1.

2 Differential Operators of Infinite Order in $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$

Assume that

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n \in F[[z]].$$

Then the differential operator $\varphi\Big(\frac{d}{dz}\Big)$ of infinite order is well defined in the space $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$:

$$\varphi\left(\frac{d}{dz}\right) g(z) = \sum_{n=0}^{\infty} a_n g^{(n)}(z) \in \frac{1}{z}F\left[\left[\frac{1}{z}\right]\right],$$

where $g \in \frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$ since any power $\frac{1}{z^k}$ in the expression for $\varphi\Big(\frac{d}{dz}\Big)g(z)$ occurs only finitely many times. To transform the sum $\sum\limits_{n=0}^{\infty} a_n g^{(n)}(z)$, we need the following assertion which, in a sense, is similar to the integral Cauchy formula for unbounded domains.

Proposition 2.1. *Let* $g \in \frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$. *Then*

$$g^{(n)}(z) = -n!\operatorname{Res}\left(\frac{g(s)}{(s-z)^{n+1}}\right), \quad n \geqslant 0,$$

where

$$\frac{1}{(s-z)^{n+1}} \in \frac{1}{z}F\left[\left[\frac{1}{z}\right]\right][[s]]$$

and the product under the symbol Res is well defined as an element of the space of formal Laurent series $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]\left[\left[s, \frac{1}{s}\right]\right]$.

Proof. Let $g(z) = \sum\limits_{n=0}^{\infty} \frac{g_n}{z^{n+1}}$. Then for $n = 0$

$$\operatorname{Res}\left(\frac{g(s)}{z-s}\right) = \operatorname{Res}\left(\Big(\frac{g_0}{s} + \frac{g_1}{s^2} + \cdots\Big)\frac{1}{z}\Big(1 + \frac{s}{z} + \frac{s^2}{z^2} + \cdots\Big)\right) = \frac{g_0}{z} + \frac{g_1}{z^2} + \frac{g_2}{z^3} + \cdots = g(z).$$

For arbitrary $n \geqslant 1$ this equality is verified in a similar way. □

Now, for $g(s) = \sum\limits_{n=0}^{\infty} \frac{c_n}{s^{n+1}}$ we set

$$g_-(s) = \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{s^{n+1}}. \tag{2.1}$$

Using Proposition 2.1, we obtain the convolution representation for $\varphi\Big(\frac{d}{dz}\Big)$.

Theorem 2.1. *Let*

$$\varphi(z)=\sum_{n=0}^{\infty}a_nz^n\in F[[z]],\quad (z)\in\frac{1}{z}F\Big[\Big[\frac{1}{z}\Big]\Big].$$

Then

$$\varphi\left(\frac{d}{dz}\right)g=\Phi_-*g,$$

where Φ *is the Laplace transform of the power series* φ.

Proof. Since

$$\frac{n!}{(z-s)^{n+1}}=\frac{1}{z^{n+1}}\frac{n!}{(1-\frac{s}{z})^{n+1}}=\frac{1}{z^{n+1}}\sum_{k=0}^{\infty}\frac{(k+1)(k+2)\cdots(k+n)}{z^k}s^k\in\frac{1}{z^{n+1}}F\Big[\Big[\frac{1}{z}\Big]\Big][[s]],$$

from (1.1) and Proposition 2.1 it follows that

$$\sum_{n=0}^{\infty}a_ng^{(n)}(z)=-\sum_{n=0}^{\infty}a_nn!\,\mathrm{Res}\left(\frac{g(s)}{(s-z)^{n+1}}\right)=\sum_{n=0}^{\infty}\mathrm{Res}\left(\frac{(-1)^nn!a_ng(s)}{(z-s)^{n+1}}\right)$$

$$=\mathrm{Res}\left(\left(\sum_{n=0}^{\infty}\frac{(-1)^nn!a_n}{(z-s)^{n+1}}\right)g(s)\right)=\mathrm{Res}\;(\Phi_-(z-s)g(s))=(\Phi_-*g)(z).$$

The theorem is proved. □

Example 2.1. Assume that $a\in F$ and $g\in\frac{1}{z}F\Big[\Big[\frac{1}{z}\Big]\Big]$. Then

$$g(z)+ag'(z)+a^2g''(z)+a^3g'''(z)+\cdots=(\mathscr{E}_a*g)(z),$$

where

$$\varphi(z)=\sum_{n=0}^{\infty}a^nz^n,\quad \Phi_-(z)=\mathscr{E}_a(z)=\frac{1}{z}-\frac{1!a}{z^2}+\frac{2!a^2}{z^3}-\frac{3!a^3}{z^4}+\ldots.$$

Example 2.2. Let V be a vector space over the field F, and let $T:V\to V$ be an arbitrary linear operator. We consider the following implicit linear inhomogeneous differential equation in the space $\frac{1}{z}V\Big[\Big[\frac{1}{z}\Big]\Big]$:

$$Tw'(z)+g(z)=w(z),\tag{2.2}$$

where $g\in\frac{1}{z}V\Big[\Big[\frac{1}{z}\Big]\Big]$. Then the Laurent series $w(z)=\sum_{n=0}^{\infty}T^ng^{(n)}(z)$ is well defined and is a unique solution to Equation (2.2) in $\frac{1}{z}F\Big[\Big[\frac{1}{z}\Big]\Big]$. Thus,

$$w(z)=\varphi\Big(\frac{d}{dz}\Big)g(z),\quad \varphi(z)=\sum_{n=0}^{\infty}T^nz^n.$$

We note that Definition 1.3 and Theorem 2.1 are extended to the operator-vector case. Therefore, a unique solution to Equation (2.2) can be represented in the form of convolution: $w(z) = (\Phi_- * g)(z)$. By the last formula, we can interpret the series

$$\Phi_-(z) = \sum_{n=0}^{\infty} \frac{(-1)^n n! T^n}{z^{n+1}}$$

as the fundamental solution to Equation (2.2). Another approach to the notion of the fundamental solution of Equation (2.2) was considered in [18].

We fix an element h of the field F. The shift operator τ_h is well defined in the space $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$ by substituting $-h$ for s in (1.1):

$$(\tau_h g)(z) = g(z+h) = \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}} \left(1 - \frac{h}{z} + \frac{h^2}{z^2} - \cdots\right)^{n+1}.$$

Proposition 2.2 (Taylor expansion for formal Laurent series)**.** *In the space $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$, for any element $h \in F$*

$$\tau_h = e^{h\frac{d}{dz}}.$$

Proof. Let $g(z) = \sum\limits_{n=0}^{\infty} \frac{c_n}{z^{n+1}} \in \frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$. For $\varphi(z) = e^{hz}$ from Theorem 2.1 we find

$$e^{h\frac{d}{dz}} g = \Phi_- * g,$$

where

$$\Phi_-(z) = \sum_{n=0}^{\infty} \frac{(-1)^n h^n}{z^{n+1}} = \frac{1}{z+h}.$$

Therefore,

$$\Phi_-(z-s) = \sum_{n=0}^{\infty} \frac{s^n}{(z+h)^{n+1}} \in \frac{1}{z}F\left[\left[\frac{1}{z}\right]\right][[s]],$$

$$\text{Res}\,(\Phi_-(z-s)g(s)) = \text{Res}\left(\sum_{n=0}^{\infty} \frac{s^n}{(z+h)^{n+1}} \cdot \sum_{n=0}^{\infty} \frac{c_n}{s^{n+1}}\right) = \sum_{n=o}^{\infty} \frac{c_n}{(z+h)^{n+1}} = g(z+h).$$

The proposition is proved. □

The following assertion is an analog of the classical characterization of translation invariant operators for the space $\frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$.

Theorem 2.2. *Let F be a field of zero characteristic, and let*

$$A : \frac{1}{z}F\left[\left[\frac{1}{z}\right]\right] \to \frac{1}{z}F\left[\left[\frac{1}{z}\right]\right]$$

be a continuous linear operator in the Krull topology. Then the following conditions are equivalent:

(a) *A commutes with any shift operator τ_h, $h \in F$,*

(b) *A commutes with the differentiation operator $\dfrac{d}{dz}$,*

(c) *A is the convolution operator, i.e., there exists a Laurent series $\Psi \in \dfrac{1}{z}F\Big[\Big[\dfrac{1}{z}\Big]\Big]$ such that $A(g) = \Psi * g$,*

(d) *$A = \varphi\Big(\dfrac{d}{dz}\Big)$ for some formal power series $\varphi \in F[[z]]$.*

Proof. We first consider the implication (b) $\Rightarrow$ (d). We set $A\Big(\dfrac{1}{z}\Big) = \sum\limits_{n=1}^{\infty} \dfrac{a_n}{z^n}$. Then

$$A\Big(\frac{1}{z^2}\Big) = -\Big(A\frac{d}{dz}\Big)\Big(\frac{1}{z}\Big) = -\frac{d}{dz}A\Big(\frac{1}{z}\Big) = \sum_{n=1}^{\infty} \frac{na_n}{z^{n+1}}.$$

In a similar way, we can check that

$$A\Big(\frac{1}{z^k}\Big) = \sum_{n=1}^{\infty} C_{n+k-2}^{k-1} \frac{a_n}{z^{n+k-1}}$$

for all $k \geqslant 2$. Let $g(z) = \sum\limits_{k=1}^{\infty} \dfrac{c_k}{z^k}$. By the continuity of the operator A in the Krull topology,

$$\begin{aligned} A(g)(z) &= \sum_{k=1}^{\infty} c_k \sum_{n=1}^{\infty} C_{n+k-2}^{k-1} \frac{a_n}{z^{n+k-1}} = \sum_{n=1}^{\infty} a_n \sum_{k=1}^{\infty} \frac{(n+k-2)!}{(k-1)!(n-1)!} \frac{c_k}{z^{n+k-1}} \\ &= \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} \sum_{k=1}^{\infty} \frac{(n+k-1)!}{(k-1)!} \frac{c_k}{z^{n+k}} = \sum_{n=0}^{\infty} \frac{(-1)^n a_{n+1}}{n!} g^{(n)}(z) = \varphi\Big(\frac{d}{dz}\Big) g(z), \end{aligned}$$

where $\varphi(z) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n a_{n+1}}{n!} z^n$.

The inverse implication (d) $\Rightarrow$ (b) is obvious. The equivalence (c) $\Leftrightarrow$ (d) was proved in Theorem 2.1. Finally, the equivalence of (a) and (b) follows from Proposition 2.2. □

3 Differential Operators of Infinite Order in $\mathbb{Z}[[z]]$

Let p be a prime, and let $\mathbb{Z}_p$ denote the ring of integer p-adic numbers with the standard topology and norm $||\cdot||_p$ (cf. [19, Section 3]). For our goal it is important that the convergence of the series $\sum a_n$ in the ring $\mathbb{Z}_p$ is equivalent to the convergence of a_n to zero in $\mathbb{Z}_p$. In the ring $\mathbb{Z}_p[[x]]$ of formal power series with integer p-adic coefficients, we introduce the topology of coefficient convergence (cf. [17, Chapter 1]). The ring $\mathbb{Z}[[x]]$ of formal power series with integer coefficients will be regarded as a subring of $\mathbb{Z}_p[[x]]$.

Theorem 3.1. *We consider the formal power series $\varphi(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ with integer coefficients and $g \in \mathbb{Z}[[x]]$. Then the series $\sum\limits_{n=0}^{\infty} a_n g^{(n)}(z)$ converges in the ring $\mathbb{Z}_p[[x]]$. Thus, the differential operator of infinite order $\varphi\Big(\dfrac{d}{dz}\Big)$ is well defined by a $\mathbb{Z}$-linear mapping from $\mathbb{Z}[[z]]$ to $\mathbb{Z}_p[[z]]$.*

Proof. Let Φ be the Laplace transform of a series φ, and let the Laurent series Φ_- be given by 2.1. Then

$$\Phi_-(z) = \sum_{n=0}^{\infty} \frac{(-1)^n n! a_n}{z^{n+1}}.$$

Since $\lim_{n\to\infty} n! = 0$ in the ring $\mathbb{Z}_p$, we have $\lim_{n\to\infty} (-1)^n n! a_n = 0$ in $\mathbb{Z}_p$. By [20, Lemma 5.4], the series $\sum_{n=0}^{\infty} a_n g^{(n)}(z)$ converges in the topology of coefficient convergence and $\varphi\Big(\frac{d}{dz}\Big) g = \Phi_- * g$, where the convolution of the formal Laurent series with integer coefficients $f(z) = \sum_{n=0}^{\infty} \frac{f_n}{z^{n+1}}$ and the formal power series with integer coefficients g is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} \frac{(-1)^n f_n}{n!} g^{(n)}(z)$$

(cf. [20]). The theorem is proved. □

Example 3.1. Assume that $\varphi(z) = \sum_{n=0}^{\infty} z^n$ and $g \in \mathbb{Z}[[x]]$. Then

$$\Big(\varphi\Big(\frac{d}{dz}\Big) g\Big)(z) = \sum_{n=0}^{\infty} g^{(n)}(z).$$

If $(z) = \sum_{n=0}^{\infty} z^n$, then $\Big(\varphi\Big(\frac{d}{dz}\Big)g\Big)(0) = 0! + 1! + 2! + 3! + 4! + \ldots$. Since the sum of the series $0! + 1! + 2! + 3! + 4! + \ldots$ in all rings $\mathbb{Z}_p$ is not integer (cf., for example, [7, Example 2.1]), we have $\varphi\left(\frac{d}{dz}\right) g \notin \mathbb{Z}[[x]]$, although $\varphi\left(\frac{d}{dz}\right) g \in \mathbb{Z}_p[[z]]$ for all prime p.

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