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ROBUST CONTROLLABILITY OF NONSTATIONARY DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH UNSTRUCTURED UNCERTAINTY

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We consider a nonstationary system of differential-algebraic equations, i.e., the first order ordinary differential equations with variable coefficients and identically singular matrix at the derivative of the unknown vector-valued function. We construct the structural form for the perturbed system and obtain sufficient conditions for the robust (complete, differential, R-) controllability of such systems of index 1 *and* 2. *Bibliography*: 17 *titles.*

1 Introduction

We consider the system of differential equations, called *differential-algebraic equations*

$$A(t)x'(t) + B(t)x(t) + U(t)u(t) = 0, \quad t \in I = [0, +\infty), \tag{1.1}$$

where $A(t)$, $B(t)$, and $U(t)$ are given matrices of size $n \times n$, $n \times n$, and $n \times l$ respectively, $\det A(t) \equiv 0$, $x(t)$ is the unknown n-dimensional function describing the system state, and $u(t)$ is the l-dimensional control function. An important characteristic of differential-algebraic equations is the *unsolvability index* $r : 0 \leqslant r \leqslant n$ (cf. [1, 2]) measured the complexity of the inner structure of the system.

The main difficulty in the study of the robust controllability of differential-algebraic equations is caused by the fact that the inner structure of the system can vary under perturbations of the input data. The known results on robust controllability are obtained only for differential-algebraic equations with constant coefficients and regular matrix pencil (cf. [3]–[7]).

In this paper, we study the robust controllability of nonstationary differential-algebraic equations with unstructured perturbations (in the form of matrix norms) in the matrices at the unknown function and the control function. The proof of our results is based on reducing differential-algebraic equations to the structural form with separated "differential" and "algebraic" parts. Owing to this approach, we could analyze a large class of systems (cf., for example, [8]–[12]).

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2 Definition and Notation

This section contains auxiliaries concerning construction and properties of the equivalent form of differential-algebraic equations, which will be used in the proof of the main results of this paper. With matrix coefficients of the differential-algebraic equations (1.1) we associate the matrices

$$\mathrm{D}_{r,z}(t) = \begin{pmatrix} C_1^1 A(t) & O & \dots & O \\ C_2^1 A'(t) + C_2^2 B(t) & C_2^2 A(t) & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_r^1 A^{(r-1)}(t) + C_r^2 B^{(r-2)}(t) & C_r^2 A^{(r-2)}(t) + C_r^3 B^{(r-3)}(t) & \dots & C_r^r A(t) \end{pmatrix},$$

$$\mathrm{D}_{r,y}(t) = \begin{pmatrix} C_0^0 A(t) & O \\ \begin{pmatrix} C_1^0 A'(t) + C_1^1 B(t) \\ \vdots \\ C_r^0 A^{(r)}(t) + C_r^1 B^{(r-1)}(t) \end{pmatrix} & \mathrm{D}_{r,z}(t) \end{pmatrix},$$

$$\mathrm{D}_{r,x}(t) = \begin{pmatrix} \overline{B}(t) & \mathrm{D}_{r,y}(t) \end{pmatrix},$$

of size $nr \times nr$, $n(r+1) \times n(r+1)$, $n(r+1) \times n(r+2)$ respectively. Hereinafter, $C_i^j = i!/(j!(i-j)!)$ are binomial coefficients and $\overline{B}(t) = \mathrm{column}\,(B(t), B'(t), \dots, B^{(r)}(t))$.

We assume that the entries of the matrices $A(t), B(t), U(t)$ are sufficiently many times continuously differentiable functions on I. Furthermore, we assume that for some $r : 0 \leqslant r \leqslant n$ the condition $\operatorname{rank} \mathrm{D}_{r,z}(t) = \rho = \mathrm{const}$ holds for any $t \in I$ and for all t the matrix $\mathrm{D}_{r,x}(t)$ contains a nonsingular minor of order $n(r+1)$ consisting of ρ columns of the matrix $\mathrm{D}_{r,z}(t)$ and the first n columns of the matrix $\mathrm{D}_{r,y}(t)$. This minor is said to be *resolving*.

We assume that we know which columns of the matrix $\mathrm{D}_{r,x}(t)$ enter the resolving minor. We remove the $n-d$, $d = nr - \rho$, columns of the matrix $\overline{B}(t)$ that do not enter this minor. Permuting columns of $\mathrm{D}_{r,x}(t)$, we obtain the matrix

$$\Lambda_r(t) = \mathrm{D}_{r,x}(t)\ \mathrm{diag}\left(Q^{-1} \begin{pmatrix} O \\ E_d \end{pmatrix},\ Q^{-1}, \dots, Q^{-1} \right), \tag{2.1}$$

where E_d is the identity matrix of order d and Q is the permutation $(n \times n)$-matrix.

The matrix Q^{-1} is constructed by the following rule. Let $i_1, i_2, \dots, i_d$ and $i_{d+1}, i_{d+2}, \dots, i_n$ be the numbers of columns of the matrix $\overline{B}(t)$ which enter or not the resolving minor respectively. The matrix Q^{-1}, multiplied from the left by $\overline{B}(t)$, permutes each (i_{d+k})th column $(k = \overline{1, n-d})$ of the matrix $\overline{B}(t)$ to the kth position and each (i_j)th column $(j = \overline{1, d})$ to the $n-d+j$th position. The matrix Q^{-1} consists of zeros and n units; moreover, all the entries indexed by (i_{d+k}, k) and $(i_j, n-d+j)$ are units.

Lemma 2.1. *We assume that*

(1) $A(t), B(t), U(t), u(t) \in \mathbf{C}^{2r+1}(I)$,

(2) $\operatorname{rank} \mathrm{D}_{r,z}(t) = \rho = \mathrm{const}$ *for any* $t \in I$,

(3) *the matrix* $\mathrm{D}_{r,x}(t)$ *contains a resolving minor,*

(4) $\operatorname{rank} \mathrm{D}_{r+1,y}(t) = \operatorname{rank} \mathrm{D}_{r,y}(t) + n$ *for any* $t \in I$.

Then there exists an operator on I

$$\mathscr{R} = R_0(t) + R_1(t)\frac{d}{dt} + \ldots + R_r(t)\left(\frac{d}{dt}\right)^r \tag{2.2}$$

that reduces (1.1) *to the structural form*

$$x_1'(t) + J_1(t)x_1(t) + \mathscr{H}(t)\overline{u}(t) = 0, \tag{2.3}$$

$$x_2(t) + J_2(t)x_1(t) + \mathscr{G}(t)\overline{u}(t) = 0, \tag{2.4}$$

where

$$\operatorname{column}(x_1(t), x_2(t)) = Qx(t), \quad \overline{u}(t) = \operatorname{column}(u(t), u'(t), \ldots, u^{(r)}(t)),$$

$$\begin{pmatrix} \mathscr{G}(t) \\ \mathscr{H}(t) \end{pmatrix} = \begin{pmatrix} G_0(t) & G_1(t) & \ldots & G_r(t) \\ H_0(t) & H_1(t) & \ldots & H_r(t) \end{pmatrix} = (R_0(t)\ R_1(t)\ \ldots\ R_r(t))\mathscr{P}_r[U(t)],$$

$$\mathscr{P}_r[U(t)] = \begin{pmatrix} C_0^0 U(t) & O & \ldots & O \\ C_1^0 U'(t) & C_1^1 U(t) & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_r^0 U^{(r)}(t) & C_r^1 U^{(r-1)}(t) & \ldots & C_r^r U(t) \end{pmatrix}, \tag{2.5}$$

$$\begin{pmatrix} J_2(t) & E_d \\ J_1(t) & O \end{pmatrix} = (R_0(t)\ R_1(t)\ \ldots\ R_r(t))\overline{B}(t)Q^{-1}.$$

Moreover, the operator (2.2) *possesses the left inverse, its coefficients* $R_j(t)$ $(j = \overline{0,r})$ *are continuous and uniquely found by*

$$(R_0(t)\ R_1(t)\ \ldots\ R_r(t)) = (E_n\ O\ \ldots\ O)\Lambda_r^\top(t)(\Lambda_r(t)\Lambda_r^\top(t))^{-1}, \tag{2.6}$$

whereas all solutions to the system (1.1) *are solutions to the system* (2.3), (2.4) *and the converse assertion is also true.*

Definition 2.1. An n-dimensional vector-valued function $x(t) \in \mathbf{C}^1(I)$ is called a *solution* to the system (1.1) if (1.1) becomes an identity on I under substitution of x.

Definition 2.2. The system (2.3), (2.4) is called the *equivalent form* of the system (1.1).

Using Lemma 2.1, we can obtain an existence and uniqueness criterion for the initial problem for the system (1.1). For this purpose we introduce the initial conditions

$$x(t_0) = x_0, \tag{2.7}$$

where $t_0 \in I$ and $x_0 \in \mathbf{R}^n$ is a given vector.

Corollary 2.1. *Let all the assumptions of Lemma* 2.1 *hold. Then the problem* (1.1), (2.7) *has a solution if and only if*

$$x_{2,0} + J_2(t_0)x_{1,0} + \mathscr{G}(t_0)\overline{u}(t_0) = \vec{0}, \tag{2.8}$$

where $\operatorname{column}(x_{1,0}, x_{2,0}) = Qx_0$. *Moreover, if a solution to the problem* (1.1), (2.7) *exists, then it is unique.*

Definition 2.3. The initial data (2.7) satisfying the condition (2.8) are called *consistent* with the system (1.1).

The proof of Lemma 2.1 and Corollary 2.1 can be found in [13].

Lemma 2.2. *Let* $W(t) \in \mathbf{R}^{n\times n}$, $t \in T$. *If* $||W(t)|| < 1$ *for any* $t \in T$, *then* $\det(E_n \pm W(t)) \neq 0$ *for any* $t \in T$.

The proof of Lemma 2.2 can be found in [14].

Here, by the norm we mean an arbitrary matrix norm preserving the unit, i.e., $||E|| = 1$, where E is the identity matrix of a suitable size.

Definition 2.4. Let $W(t) \in \mathbf{R}^{m\times n}$. The matrix $W^+(t) \in \mathbf{R}^{n\times m}$ is called the *right inverse* of the matrix $W(t)$ on T if $W(t)W^+(t) = E$ for any $t \in T$.

3 Controllability Conditions for Differential-Algebraic Equations

We introduce the notion of controllability for differential-algebraic equations.

Definition 3.1 (cf. [15])**.** The system (1.1) is *completely controllable* on a segment $T = [t_0, t_1]$ if for any $x_0, x_1 \in \mathbf{R}^n$ there is a control $u(t)$ such that the corresponding solution to the system (1.1) satisfies the conditions $x(t_0) = x_0$ and $x(t_1) = x_1$.

Definition 3.2 (cf. [12])**.** The system (1.1) is called *differential-controllable* on a segment T if it is completely controllable on any set $[\tau_0, \tau_1] \subset T$ $(\tau_0 < \tau_1)$.

A vector $x_1 \in \mathbf{R}^n$ is called *reachable* from the initial data vector $x_0 \in \mathbf{R}^n$ at time t_1 if there exists a sufficiently smooth control $u(t)$ such that the corresponding solution to the system (1.1) satisfies the conditions $x(t_0) = x_0$ and $x(t_1) = x_1$.

A set $M(x_0) \subseteq \mathbf{R}^n$ is called the *reachable set* from the initial data vector $x_0 \in \mathbf{R}^n$ if it consists of vectors x_1 reachable from the point x_0 at time t_1. The reachable set M is the union of all reachable sets from all possible consistent initial data vectors [16, 15].

Definition 3.3 (cf. [15])**.** The system (1.1) is *R-controllable* (controllable in the reachable set) on a segment $T = [t_0, t_1]$ if for any consistent initial data vector x_0 and any point x_1 of the reachable set M there is a control $u(t)$ such that the corresponding solution to the system (1.1) satisfies the conditions $x(t_0) = x_0$ and $x(t_1) = x_1$.

According to Definition 3.3, any system of the form (1.1) is R-controllable if its equivalent form does not contain the nondegenerate subsystem (2.3). Otherwise, by the R-controllability of the differential-algebraic equations (1.1) one can mean the complete controllability of the system (2.3). Then, under the assumptions of Lemma 1.1, we can introduce an alternative definition of R-controllability.

Definition 3.4. The system (1.1) is called *R-controllable* on a segment T for $d < n$ if the system (2.3) is completely controllable on T.

Lemma 3.1 (cf. [8, 12, 13])**.** *Let all the assumptions of Lemma* 2.1 *hold on some segment* $T \subset I$. *Assume that*

(1) $\operatorname{rank} \mathscr{G}(t) = d$ *for any* $t \in T$,

(2) *there exists* $\sigma \in T$ *such that* $\operatorname{rank} \mathscr{Q}(\sigma) = n - d$,

(3) $\operatorname{rank} \mathscr{Q}(t) = n - d$ *for almost all* $t \in T$, *i.e., for all points* $t \in T$, *except for points in a set of zero Lebesgue measure.*

Then the system (1.1) *is* R*-controllable on* T *if* (2) *holds, completely controllable on* T *if* (1) *and* (2) *hold, and differentially controllable on* T *if* (1) *and* (3) *hold.*

Here, $\mathscr{Q}(t)$ is the controllability matrix of the system (2.3) defined by

$$\mathscr{Q}(t) = (Q_0(t)\ Q_1(t)\ \dots\ Q_{n-d-1}(t)),$$

$$Q_0(t) = -\mathscr{H}(t),\ Q_i(t) = -J_1(t)Q_{i-1}(t) + Q'_{i-1}(t), \quad i = \overline{1, n-d-1}.$$

4 Robust Controllability Conditions for Differential-Algebraic Equations

Let the system (1.1) be completely (differentially, R-) controllable on some segment $T \subset I$. The robust controllability problem is to find conditions under which the perturbed differential-algebraic equations

$$A(t)x'(t) + (B(t) + \Delta_B(t))x(t) + (U(t) + \Delta_U(t))u(t) = 0 \tag{4.1}$$

remain completely (differentially, R-) controllable on T. Here, $\Delta_B(t)$, $\Delta_U(t)$, $t \in I$, are unknown real (perturbation) matrices of suitable size that satisfy certain smallness conditions on the segment T.

We consider the matrices

$$R_0(t)\Delta_B(t)Q, \quad R_1(t)\Delta_B(t)Q, \quad R_1(t)\Delta'_B(t)Q, \quad \left(R_0(t)\Delta_U(t) + R_1(t)\Delta'_U(t) \quad R_1(t)\Delta_U(t)\right),$$

where $R_0(t)$, $R_1(t)$ are the first coefficients of the operator (2.2) reducing (1.1) to the form (2.3), (2.4) and Q is the permutation matrix in (2.1). Hereinafter, the dependence on the variable t is assumed, but is not reflected in the notation.

We represent these matrices as follows:

$$R_0\Delta_B Q = \begin{pmatrix} \Delta_{0,3} & \Delta_{0,4} \\ \Delta_{0,1} & \Delta_{0,2} \end{pmatrix}, \quad R_1\Delta_B Q = \begin{pmatrix} \Delta_{1,3} & \Delta_{1,4} \\ \Delta_{1,1} & \Delta_{1,2} \end{pmatrix},$$

$$R_1\Delta'_B Q = \begin{pmatrix} \overline{\Delta}_{1,3} & \overline{\Delta}_{1,4} \\ \overline{\Delta}_{1,1} & \overline{\Delta}_{1,2} \end{pmatrix}, \quad \left(R_0\Delta_U + R_1\Delta'_U \quad R_1\Delta_U\right) = \begin{pmatrix} \mathscr{G}_1 \\ \mathscr{H}_1 \end{pmatrix},$$

where the blocks $\Delta_{0,1}$, $\Delta_{1,1}$, $\overline{\Delta}_{1,1}$ have size $(n-d) \times (n-d)$, the blocks $\Delta_{0,2}$, $\Delta_{1,2}$, $\overline{\Delta}_{1,2}$ have size $(n-d) \times d$, the blocks $\Delta_{0,3}$, $\Delta_{1,3}$, $\overline{\Delta}_{1,3}$ have size $d \times (n-d)$, the blocks $\Delta_{0,4}$, $\Delta_{1,4}$, $\overline{\Delta}_{1,4}$ have size $d \times d$, the block $\mathscr{G}_1$ has size $d \times rl$, and the block $\mathscr{H}_1$ has size $(n-d) \times rl$.

Assume that for all $t \in T$ the following estimates hold:

$$\begin{aligned}
&||\Delta_{0,4} + \overline{\Delta}_{1,4}|| < 1, \\
&||\Delta_{1,1} - (\Delta_{0,2} + \overline{\Delta}_{1,2})(E + \Delta_{0,4} + \overline{\Delta}_{1,4})^{-1}\Delta_{1,3}|| < 1, \\
&||\Delta_{1,1}|| < 1, \\
&||\Delta_{0,4} + \overline{\Delta}_{1,4} - \Delta_{1,3}(E + \Delta_{1,1})^{-1}(\Delta_{0,2} + \overline{\Delta}_{1,2})|| < 1.
\end{aligned} \tag{4.2}$$

Then from Lemma 2.2 it follows that the matrices

$$
\begin{aligned}
P_0 &= (E + \Delta_{0,4} + \overline{\Delta}_{1,4})^{-1}, \\
P_1 &= (E + \Delta_{1,1})^{-1}, \\
S_0 &= (P_1^{-1} - (\Delta_{0,2} + \overline{\Delta}_{1,2})P_0\Delta_{1,3})^{-1}, \\
S_1 &= (P_0^{-1} - \Delta_{1,3}P_1(\Delta_{0,2} + \overline{\Delta}_{1,2}))^{-1}
\end{aligned}
$$

are invertible on T and the estimate (4.2) can be written as

$$
||P_0^{-1} - E|| < 1, \quad ||P_1^{-1} - E|| < 1, \quad ||S_0^{-1} - E|| < 1, \quad ||S_1^{-1} - E|| < 1. \tag{4.3}
$$

We consider the perturbed system of differential-algebraic equations (4.1) of index $r = 1$. Let all the assumptions of Lemma 2.1 hold for $r = 1$. Thus, (2.2) is the first order operator

$$
\mathscr{R} = R_0(t) + R_1(t)\frac{d}{dt}. \tag{4.4}
$$

Then the equivalent form of the system (4.1) can be written as

$$
x_1' + J_1x_1 + \Delta_{0,1}x_1 + \Delta_{0,2}x_2 + \Delta_{1,1}x_1' + \Delta_{1,2}x_2' + \overline{\Delta}_{1,1}x_1 + \overline{\Delta}_{1,2}x_2 + (\mathscr{H} + \mathscr{H}_1)\overline{u} = 0, \tag{4.5}
$$

$$
x_2 + J_2x_1 + \Delta_{0,3}x_1 + \Delta_{0,4}x_2 + \Delta_{1,3}x_1' + \Delta_{1,4}x_2' + \overline{\Delta}_{1,3}x_1 + \overline{\Delta}_{1,4}x_2 + (\mathscr{G} + \mathscr{G}_1)\overline{u} = 0. \tag{4.6}
$$

We assume that the matrix blocks $\Delta_{1,2}$ and $\Delta_{1,4}$ vanish on the entire segment T. Then we pass from (4.5), (4.6) to the equations

$$
\begin{aligned}
x_1' &= -P_1[(J_1 + \Delta_{0,1} + \overline{\Delta}_{1,1})x_1 + (\Delta_{0,2} + \overline{\Delta}_{1,2})x_2 + (\mathscr{H} + \mathscr{H}_1)\overline{u}], \\
x_2 &= -P_0[\Delta_{1,3}x_1' + (J_2 + \Delta_{0,3} + \overline{\Delta}_{1,3})x_1 + (\mathscr{G} + \mathscr{G}_1)\overline{u}],
\end{aligned}
$$

which imply

$$
x_1' + \widetilde{J}_1x_1 + \widetilde{\mathscr{H}}\overline{u} = 0, \tag{4.7}
$$

$$
x_2 + \widetilde{J}_2x_1 + \widetilde{\mathscr{G}}\overline{u} = 0, \tag{4.8}
$$

where

$$
\begin{aligned}
\widetilde{J}_1 &= S_0(J_1 + \Delta_{0,1} + \overline{\Delta}_{1,1} - (\Delta_{0,2} + \overline{\Delta}_{1,2})P_0(J_2 + \Delta_{0,3} + \overline{\Delta}_{1,3})), \\
\widetilde{J}_2 &= S_1(J_2 + \Delta_{0,3} + \overline{\Delta}_{1,3} - \Delta_{1,3}P_1(J_1 + \Delta_{0,1} + \overline{\Delta}_{1,1})), \\
\widetilde{\mathscr{H}} &= S_0(\mathscr{H} + \mathscr{H}_1 - \Delta_{0,2}P_0(\mathscr{G} + \mathscr{G}_1)), \\
\widetilde{\mathscr{G}} &= S_1(\mathscr{G} + \mathscr{G}_1 - \Delta_{1,3}P_1(\mathscr{H} + \mathscr{H}_1)).
\end{aligned} \tag{4.9}
$$

Moreover, $\overline{u} = \text{column}\,(u(t), u'(t))$.

We define the right inverses of the matrices $\mathscr{G}$ and $\mathscr{Q}$ respectively by

$$
\mathscr{G}^+ = \mathscr{G}^\top \left(\mathscr{G}\mathscr{G}^\top\right)^{-1}, \quad \mathscr{Q}^+ = \text{column}\,(Q_0^\top, Q_1^\top, \ldots, Q_{n-d-1}^\top) \left(\sum_{i=0}^{n-d-1} Q_iQ_i^\top\right)^{-1},
$$

and the controllability matrix of the system (4.7):

$$\widetilde{\mathscr{Q}} = (\widetilde{Q}_0, \widetilde{Q}_1, \dots \widetilde{Q}_{n-d-1}), \tag{4.10}$$

where

$$\widetilde{Q}_0 = -\widetilde{\mathscr{H}}, \quad \widetilde{Q}_i = -\widetilde{J}_1\widetilde{Q}_i + \widetilde{Q}_i', \quad i = \overline{1, n-d-1}.$$

We set

$$\Delta_{\mathscr{Q}} = \widetilde{\mathscr{Q}} - \mathscr{Q}, \quad \Delta_{\mathscr{G}} = \widetilde{\mathscr{G}} - \mathscr{G}. \tag{4.11}$$

Then we can formulate the following result.

Theorem 4.1. *We assume that*

(1) *all the assumptions of Lemma* 2.1 *hold for* $r = 1$*,*

(2) $\Delta_{1,2}(t) = \Delta_{1,4}(t) \equiv 0$ *on* T*,*

(3) *the estimates* (4.3) *hold for any* $t \in T$*,*

(4) $\operatorname{rank}\mathscr{G}(t) = d$ *and* $\operatorname{rank}\mathscr{Q}(t) = n - d$ *for any* $t \in T$*.*

Then the system of differential-algebraic equations (1.1) *is robustly* (*completely, differentially*) *controllable on a segment* $T \subset I$ *if*

(a) $||\Delta_{\mathscr{G}}(t)\mathscr{G}^+(t)|| < 1$ *for any* $t \in T$*,*

(b) $||\Delta_{\mathscr{Q}}(t)\mathscr{Q}^+(t)|| < 1$ *for any* $t \in T$*.*

Proof. By the assumptions of Lemma 2.1, the operator transforming (1.1) to the equivalent form is of the first order. Taking into account assumptions (2) and (3), we see that the equivalent form of the perturbed equations (4.1) takes the form (4.7), (4.8).

By assumption (4), the matrices $\mathscr{G}$ and $\mathscr{Q}$ have complete ranks on T. Then it is obvious that there exist matrices $\mathscr{G}^+$ and $\mathscr{Q}^+$, which are the right inverses of the matrices $\mathscr{G}$ and $\mathscr{Q}$ respectively on the entire segment T.

The matrices $\widetilde{\mathscr{G}}$ and $\widetilde{\mathscr{Q}}$ are perturbed counterparts of the matrices $\mathscr{G}$ and $\mathscr{Q}$ determined by (4.9) and (4.10). We multiply the matrix $\widetilde{\mathscr{G}}$ by $\mathscr{G}^+$:

$$\widetilde{\mathscr{G}}\mathscr{G}^+ = E + \Delta_{\mathscr{G}}\mathscr{G}^\top(\mathscr{G}\mathscr{G}^\top)^{-1} = E + \Delta_{\mathscr{G}}\mathscr{G}^+.$$

Taking into account Lemma 2.2 and assumption (a), we find

$$\det \widetilde{\mathscr{G}}\mathscr{G}^+ = \det\,(E + \Delta_{\mathscr{G}}\mathscr{G}^+) \neq 0. \tag{4.12}$$

Similarly, we multiply the matrix $\widetilde{\mathscr{Q}}$ by $\mathscr{Q}^+$:

$$\widetilde{\mathscr{Q}}\mathscr{Q}^+ = \left(\sum_{i=0}^{n-d-1} Q_iQ_i^\top + \sum_{i=0}^{n-d-1} \Delta_{Q_i}Q_i^\top\right)\left(\sum_{i=0}^{n-d-1} Q_iQ_i^\top\right)^{-1}$$

$$= E + \sum_{i=0}^{n-d-1} \Delta_{Q_i}Q_i^\top(Q_iQ_i^\top)^{-1} = E + \Delta_{\mathscr{Q}}\mathscr{Q}^+,$$

where $\Delta_{Q_i} = \widetilde{Q}_i - Q_i$. Taking into account Lemma 2.2 and assumption (b), we get

$$\det \widetilde{\mathscr{Q}}\mathscr{Q}^{+} = \det (E + \Delta_{\mathscr{Q}}\mathscr{Q}^{+}) \neq 0. \tag{4.13}$$

The relations (4.12) and (4.13) imply the completeness of row ranks of the matrices $\widetilde{\mathscr{G}}(t)$ and $\widetilde{\mathscr{Q}}(t)$ for all $t \in T$. This means that assumptions (1) and (3) of Lemma 3.1 are satisfied. In this case, the system (4.7), (4.8) and, consequently, the system (4.1) is completely and differentially controllable on T. This implies the robust (complete, differential) controllability of the differential-algebraic equations (1.1) on the segment $T \subset I$. □

Corollary 4.1. *We assume that*

1) *assumptions* (1)–(3) *of Theorem* 4.1 *holds,*

2) $\operatorname{rank} \mathscr{Q}(t) = n - d$ *for all* $t \in T$.

Then the system (1.1) *is robustly* R*-controllable on a segment* $T \subset I$ *if assumption* (b) *of Theorem* 4.1 *holds.*

Proof. By the assumptions of Lemma 2.1 for $r = 1$ (assumption (1) of Theorem 4.1), the systems (1.1) and (2.3), (2.4) are equivalent in the sense of solutions on T. According to Definition 3.4, by the R-controllability of differential-algebraic equations (1.1) we mean the complete controllability of the system (2.3). The same is related to the perturbed equations (4.1) and the equivalent form (4.7), (4.8). Moreover, assumption (2) provides the complete controllability of the system (2.3), whereas assumptions (2) and (3) of Theorem 4.1 guarantee that the perturbed equations (4.1) admit the equivalent form (4.7), (4.8). Thus, if assumption (b) of Theorem 4.1 holds, then the matrix $\widetilde{\mathscr{Q}}$ has complete rank everywhere on T, and, consequently, the system (4.7) is completely controllable on this segment (cf., for example, [15]). This means the robust R-controllability of the differential-algebraic equations (1.1). □

We can obtain the robust controllability conditions for differential-algebraic equations of index $r = 2$. Let the matrix

$$\Theta_{r-1}(t) = (E_{nr}\ O)\Lambda_r(t) \begin{pmatrix} O \\ E_{nr+d} \end{pmatrix}$$

be obtained from $\Lambda_r(t)$ (cf. (2.1)) by eliminating the last n rows and first n columns.

Theorem 4.2. *We assume that*

(1) *assumptions* (1)–(3) *of Lemma* 2.1 *hold for* $r = 2$,

(2) $\operatorname{rank} \Theta_1(t) = n$ *for any* $t \in T$,

(3) *assumptions* (2)–(4) *of Theorem* 4.1 *hold.*

Then the system of differential-algebraic equations (1.1) *is robustly* (*completely, differentially*) *controllable on a segment* $T \subset I$ *provided that assumptions* (a) *and* (b) *of Theorem* 4.1 *hold.*

Proof. As is shown in [17], in assumptions (1) and (2) for a system of differential-algebraic equations of index $r = 2$ the operator transforming (1.1) to the equivalent form (2.3), (2.4) is of the first order, i.e., this operator has the form (4.4); moreover, these systems have the same set of solutions. The further arguments repeat the proof of Theorem 4.1. □

Corollary 4.2. *Let*

(1) *assumptions* (1) *and* (2) *of Theorem* 4.2 *hold,*

(2) *assumptions* (2) *and* (3) *of Theorem* 4.1 *hold,*

(3) *assumption* (2) *of Corollary* 4.1 *holds.*

Then the system of differential-algebraic equations (1.1) *is robustly R-controllable on a segment* $T \subset I$ *provided that assumption* (b) *of Theorem* 4.1 *holds.*

The proof is the same as that of Theorem 4.2 and Corollary 4.1.

5 Example

We consider the system of differential-algebraic equations

$$\begin{pmatrix} 1 & 0 & 0 \\ -\cos t & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x'(t) + \begin{pmatrix} -\cos t & 2\cos t & 0 \\ -\sin^2 t & \cos t & 0 \\ 0 & -2\sin t & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} u(t) = 0, \tag{5.1}$$

where $t \in I = [0, +\infty)$ and $x(t) : I \to \mathbf{R}^3$ is the unknown function. Assume that perturbations at $x(t)$ and $u(t)$ are given by

$$\Delta_B(t) = \begin{pmatrix} 0 & 0 & 0 \\ b_1 & b_2 & b_3 \\ b_4 & b_5 & 0 \end{pmatrix}, \quad \Delta_U(t) = \begin{pmatrix} 0 & 0 \\ y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}, \tag{5.2}$$

where $b_i(t)$ $(i = \overline{1,5})$, $y_j(t)$ $(j = \overline{1,4})$: $I \to \mathbf{R}$. We clarify whether the differential-algebraic equations (5.1), (5.2) are robustly completely controllable. For this purpose we verify the assumptions of Theorem 4.1.

It is obvious that assumption (1) is satisfied. To verify assumptions (2)–(4), we construct the matrices

$$\mathrm{D}_{1,x} = \left(\begin{array}{cc|c|ccc|cc|c} -\cos t & 2\cos t & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\sin^2 t & \cos t & 0 & -\cos t & 1 & 0 & 0 & 0 & 0 \\ 0 & -2\sin t & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \sin t & -2\sin t & 0 & -\cos t & 2\cos t & 0 & 1 & 0 & 0 \\ -\sin 2t & -\sin t & 0 & -\sin^2 t + \sin t & \cos t & 0 & -\cos t & 1 & 0 \\ 0 & -2\cos t & 0 & 0 & -2\sin t & -1 & 0 & 0 & 0 \end{array}\right),$$

$$
\mathrm{D}_{2,y} = \left(\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\cos t & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-\cos t & 2\cos t & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-\sin^2 t + \sin t & \cos t & 0 & -\cos t & 1 & 0 & 0 & 0 & 0 \\
0 & -2\sin t & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\sin t & -2\sin t & 0 & -\cos t & 2\cos t & 0 & 1 & 0 & 0 \\
\cos t - \sin 2t & -\sin t & 0 & 2\sin t - \sin^2 t & \cos t & 0 & -\cos t & 1 & 0 \\
0 & -2\cos t & 0 & 0 & -2\sin t & -1 & 0 & 0 & 0
\end{array}\right)
$$

It is easy to verify that $\operatorname{rank} \mathrm{D}_{1,z}(t) = \rho = 2$ for any $t \in I$. In the matrix $\mathrm{D}_{1,x}(t)$, the columns entering the resolving minor are framed. Thus, the resolving minor includes $\rho = 2$ columns of the matrix $\mathrm{D}_{1,z}(t)$, the first $n = 3$ columns of the matrix $\mathrm{D}_{1,y}(t)$ and the third column of the matrix $\mathrm{D}_{1,x}(t)$. The condition $\operatorname{rank} \mathrm{D}_{2,y} = \operatorname{rank} \mathrm{D}_{1,y} + 3$ is also valid for all $t \in I$. Thus, all the assumptions of Lemma 3.1 are satisfied in the case $r = 1$. From formula (2.6) we define the operator (2.2)

$$
\mathscr{R} = R_0(t) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ \cos t & 1 & 0 \end{pmatrix}.
$$

Then we can write (5.1) in the equivalent form

$$
\begin{aligned}
& x_1'(t) + \begin{pmatrix} -\cos t & 2\cos t \\ -1 & 2\cos^2 t + \cos t \end{pmatrix} x_1(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} u(t) = 0, \\
& (1\ 1)x_2(t) + (0\ 2\sin t)x_1(t) + (0\ 1)u(t) = 0.
\end{aligned}
$$

We construct the matrices $R_0\Delta_B Q$, $R_1\Delta_B Q$, and $R_1\Delta' BQ$ as follows:

$$
R_0\Delta_B Q = \begin{pmatrix} \Delta_{0,3} & \Delta_{0,4} \\ \Delta_{0,1} & \Delta_{0,2} \end{pmatrix} = \left(\begin{array}{cc|c} -b_4 & -b_5 & 0 \\ \hline 0 & 0 & 0 \\ b_1 & b_2 & b_3 \end{array}\right),
$$

$$
R_1\Delta_B Q = \begin{pmatrix} \Delta_{1,3} & \Delta_{1,4} \\ \Delta_{1,1} & \Delta_{1,2} \end{pmatrix} = O,
$$

$$
R_1\Delta_B' Q = \begin{pmatrix} \overline{\Delta}_{1,3} & \overline{\Delta}_{1,4} \\ \overline{\Delta}_{1,1} & \overline{\Delta}_{1,2} \end{pmatrix} = O.
$$

It is obvious that assumption (2) of Theorem 4.1 holds since $\Delta_{1,2} = \Delta_{1,4} \equiv 0$. The estimates (4.3) also hold because $||O|| < 1$.

To verify assumption (4) of Theorem 4.1, we construct the matrices

$$
\begin{pmatrix} \mathscr{G} \\ \mathscr{H} \end{pmatrix} = \left(\begin{array}{cccc} 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right), \quad \mathscr{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 & 2\cos t & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \cos t(2\cos t + 1) & 0 & 0 & 0 \end{pmatrix}.
$$

It is easy to see that $\operatorname{rank}\mathscr{G}(t) = d = 1$ for all $t \in I$ and $\operatorname{rank} Q(t) = n - d = 2$ for all $t \in I$, except for the points $t = \pi/2 + \pi k$, $k = 1, 2, 3, \ldots$.

To verify assumptions (a) and (b), we construct the matrices

$$\Delta_{\mathscr{G}}(t) = (-y_3 \;\; -y_4 \;\; 0 \;\; 0), \quad \mathscr{G}^{+} = (0,\ 1,\ 0,\ 0)^{\top},$$

$$\Delta_{\mathscr{Q}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 2j_1h_1 & 2j_1h_2 & 0 & 0 \\ -h_1 & -h_2 & 0 & 0 & -j_2h_1 - h_1' & -j_2h_2 - h_2' & 0 & 0 \end{pmatrix},$$

$$\mathscr{Q}^{+} = \begin{pmatrix} -q_2 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{\top},$$

where $h_1 = y_1 + y_3b_3 + 1$, $h_2 = b_3(y_4 - 1)$, $j_1 = -\cos t$, $j_2 = \cos t(2\cos t + 1) + b_2 + b_3(b_5 - 2\sin t)$, $q_1 = -\cos t(2\cos t + 1)(\cos t + 3/2)$, $q_2 = -\cos t - 1/2$. Thus, the system (5.1), (5.2) is robustly completely controllable on any segment $T \subset I$ that does not contain the points $t = \pi/2 + \pi k$, $k = 1, 2, 3, \ldots$ provided that

$$\left\| \begin{pmatrix} 2j_1h_1q_1 & 0 \\ h_1q_2 - q_1(j_2h_1 + h_1') & h_1 \end{pmatrix} \right\| < 1, \quad |y_4| < 1.$$

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