DOI 10.1007/s10958-019-04291-0
Journal of Mathematical Sciences, *Vol. 239, No. 1, May, 2019*

On monogenic functions defined in different commutative algebras

Vitalii S. Shpakivskyi

Presented by V. Ya. Gutlyanskii

Abstract. The correspondence between a monogenic function in an arbitrary finite-dimensional commutative associative algebra and a finite collection of monogenic functions in a special commutative associative algebra is established.

Keywords. Commutative associative algebra, monogenic function, characteristic equation, integral representation.

1. Introduction

Probably, P. W. Ketchum [1] was the first who used analytic functions that take their values in a commutative algebra for the construction of solutions of the three-dimensional Laplace equation. He showed that every analytic function $\Phi(\zeta)$ of the variable $\zeta = xe_1 + ye_2 + ze_3$ satisfies the three-dimensional Laplace equation, if the linearly independent elements e_1, e_2, e_3 of a commutative algebra satisfy the condition

$$e_1^2 + e_2^2 + e_3^2 = 0\,, \tag{1.1}$$

since

$$\Delta_3\Phi := \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \equiv \Phi''(\zeta)\ (e_1^2 + e_2^2 + e_3^2) = 0\,, \tag{1.2}$$

where $\Phi'' := (\Phi')'$, and $\Phi'(\zeta)$ is defined by the equality $d\Phi = \Phi'(\zeta)d\zeta$.

Generalizing the work by P. W. Ketchum, M. N. Roşculeţ [2,3] used analytic functions with values in commutative algebras for the study of equations of the form

$$\mathcal{L}_N U(x,y,z) := \sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma}\, \frac{\partial^N U}{\partial x^\alpha\, \partial y^\beta\, \partial z^\gamma} = 0, \quad C_{\alpha,\beta,\gamma} \in \mathbb{R}. \tag{1.3}$$

Considering the variable $\zeta = xe_1 + ye_2 + ze_3$ and an analytic function $\Phi(\zeta)$, we get the following equality for a mixed derivative:

$$\frac{\partial^{\alpha+\beta+\gamma}\Phi}{\partial x^\alpha\, \partial y^\beta\, \partial z^\gamma} = e_1^\alpha\, e_2^\beta\, e_3^\gamma\, \Phi^{(\alpha+\beta+\gamma)}(\zeta) = e_1^\alpha\, e_2^\beta\, e_3^\gamma\, \Phi^{(N)}(\zeta). \tag{1.4}$$

Substituting (1.4) in Eq. (1.3), we obtain the equality

$$\mathcal{L}_N \Phi(\zeta) = \Phi^{(N)}(\zeta) \sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma}\, e_1^\alpha\, e_2^\beta\, e_3^\gamma\,.$$

Translated from Ukrains'kiĭ Matematychnyĭ Visnyk, Vol. 15, No. 2, pp. 272–294, January–March, 2018.
Original article submitted February 06, 2018.

We may conclude that the equality $\mathcal{L}_N\Phi(\zeta) = 0$ holds, if the elements of the algebra $e_1 = 1, e_2, e_3$ satisfy the *characteristic equation*

$$\mathcal{X}(1, e_2, e_3) := \sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma}\, e_2^{\beta}\, e_3^{\gamma} = 0\,. \tag{1.5}$$

If the left-hand side of Eq. (1.5) is expanded in the basis of the algebra, the characteristic equation (1.5) is equivalent to the *characteristic system of equations* generated by Eq. (1.5).

Thus, if condition (1.5) is satisfied, every analytic function Φ with values in any commutative associative algebra satisfies Eq. (1.3), and, respectively, all real-valued components of the function Φ are solutions of Eq. (1.3).

In work [4], some partial differential equations with several variables are considered, and a number of examples of the application of the above-described method are presented.

I. Mel'nichenko [5] proposed to consider the functions Φ twice differentiable by Gâteaux in equalities (1.2) and (1.4). In this case, he described all bases $\{e_1, e_2, e_3\}$ of three-dimensional commutative algebras with 1 over the field $\mathbb{C}$ which satisfy equality (1.1), see [6].

For those three-dimensional commutative algebras associated with the three-dimensional Laplace equation, the constructive description of all monogenic (i.e., continuous and differentiable by Gâteaux) functions with the help of three corresponding holomorphic functions of a complex variable was given in works [7–9].

Works [10, 11] include the constructive description of monogenic functions (related to the equation $\Delta_3\Phi = 0$) with values in some n-dimensional commutative algebras with the help of n corresponding holomorphic functions of a complex variable. Moreover, based on the obtained representations of monogenic functions, some analogs of a number of classical results of complex analysis were proved.

Eventually, work [14] gave analysis of some monogenic functions (related to Eq. (1.3)) with values in an arbitrary commutative associative algebra over the field $\mathbb{C}$ with the help of holomorphic functions of a complex variable.

In the present work, we will show that, for the construction of solutions of Eq. (1.3) in the form of components of monogenic functions with values in finite-dimensional commutative associative algebras, it is sufficient to restrict ourselves by the study of monogenic functions in algebras of a special form.

2. Algebra $\mathbb{A}_n^m$

Let $\mathbb{N}$ be the set of natural numbers, and let $m, n \in \mathbb{N}$ be such that $m \le n$. Let $\mathbb{A}_n^m$ be any commutative associative algebra with 1 over the field of complex numbers $\mathbb{C}$. E. Cartan [12, p. 33] proved that there exists a basis $\{I_k\}_{k=1}^n$ in the algebra $\mathbb{A}_n^m$ that satisfies the following multiplication rules:

1. $\forall\, r, s \in [1, m] \cap \mathbb{N}:$ $\qquad I_r I_s = \begin{cases} 0 & \text{for } \; r \ne s, \\ I_r & \text{for } \; r = s; \end{cases}$

2. $\forall\, r, s \in [m+1, n] \cap \mathbb{N}:$ $\qquad I_r I_s = \sum\limits_{k=\max\{r,s\}+1}^{n} \Upsilon_{r,k}^{s} I_k\,;$

3. $\forall\, s \in [m+1, n] \cap \mathbb{N}\ \exists!\, u_s \in [1, m] \cap \mathbb{N}\ \ \forall\, r \in [1, m] \cap \mathbb{N}:$

$$I_r I_s = \begin{cases} 0 \ \text{ for } \ r \ne u_s\,, \\ I_s \ \text{ for } \ r = u_s\,. \end{cases}$$

In addition, the structural constants $\Upsilon_{r,k}^{s} \in \mathbb{C}$ satisfy the conditions of associativity:

(A 1). $(I_r I_s)I_p = I_r(I_s I_p) \quad \forall\, r,s,p \in [m+1,n] \cap \mathbb{N}$;

(A 2). $(I_u I_s)I_p = I_u(I_s I_p) \quad \forall\, u \in [1,m] \cap \mathbb{N} \;\; \forall\, s,p \in [m+1,n] \cap \mathbb{N}$.

It is obvious that m first basis vectors $\{I_u\}_{u=1}^{m}$ are idempotents and generate a semisimple subalgebra S of the algebra $\mathbb{A}_n^m$, and the vectors $\{I_r\}_{r=m+1}^{n}$ generate a nilpotent subalgebra N of this algebra. The multiplication rules of the algebra $\mathbb{A}_n^m$ imply that $\mathbb{A}_n^m$ is the semidirect sum of an m-dimensional semisimple subalgebra S and an $(n-m)$-dimensional nilpotent subalgebra N, i.e.,

$$\mathbb{A}_n^m = S \oplus_s N.$$

The unit of the algebra $\mathbb{A}_n^m$ is the element $1 = \sum_{u=1}^{m} I_u$.

The algebra $\mathbb{A}_n^m$ contains m maximum ideals

$$\mathcal{I}_u := \Bigg\{ \sum_{k=1,\, k\neq u}^{n} \lambda_k I_k : \lambda_k \in \mathbb{C} \Bigg\}, \qquad u = 1,2,\ldots,m,$$

whose intersection is the radical

$$\mathcal{R} := \Bigg\{ \sum_{k=m+1}^{n} \lambda_k I_k : \lambda_k \in \mathbb{C} \Bigg\}. \tag{2.1}$$

Define now m linear functionals $f_u : \mathbb{A}_n^m \to \mathbb{C}$ by the equalities

$$f_u(I_u) = 1, \quad f_u(\omega) = 0 \;\; \forall\, \omega \in \mathcal{I}_u, \quad u = 1,2,\ldots,m. \tag{2.2}$$

The kernels of the functionals f_u are, respectively, the maximum ideals $\mathcal{I}_u$. Therefore, these functionals are also continuous and multiplicative (see [13]).

3. Monogenic functions

Let

$$e_1 = 1, \quad e_2 = \sum_{r=1}^{n} a_r I_r, \quad e_3 = \sum_{r=1}^{n} b_r I_r \tag{3.1}$$

for $a_r, b_r \in \mathbb{C}$ be a triple of vectors in the algebra $\mathbb{A}_n^m$ which are linearly independent over the field $\mathbb{R}$. This means that the equality

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R},$$

holds iff $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Let $\zeta := x e_1 + y e_2 + z e_3$, where $x,y,z \in \mathbb{R}$. It is obvious that $\xi_u := f_u(\zeta) = x + y a_u + z b_u$, $u = 1,2,\ldots,m$. In the algebra $\mathbb{A}_n^m$, we separate a linear span $E_3 := \{\zeta = x e_1 + y e_2 + z e_3 :\; x,y,z \in \mathbb{R}\}$ generated by the vectors e_1, e_2, e_3.

The following assumption is significant: $f_u(E_3) = \mathbb{C}$ for all $u = 1,2,\ldots,m$, where $f_u(E_3)$ is an image of the set E_3 under the mapping f_u. It is obvious that this takes place iff at least one of the numbers a_u or b_u belongs to $\mathbb{C} \setminus \mathbb{R}$ for each fixed $u = 1,2,\ldots,m$. Theorem 7.1 in [14] established a subclass of equations of the form (1.3) for which the condition $f_u(E_3) = \mathbb{C}$ holds for all $u = 1,2,\ldots,m$.

We put the domain Ω of the three-dimensional space $\mathbb{R}^3$ in correspondence to the domain $\Omega_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x,y,z) \in \Omega\}$ in E_3.

The continuous function $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$ is called *monogenic* in the domain $\Omega_\zeta \subset E_3$, if Φ is differentiable by Gâteaux at every point of this domain. In other words, if, for every $\zeta \in \Omega_\zeta$, there exists an element $\Phi'(\zeta)$ of the algebra $\mathbb{A}_n^m$ such that the equality

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall\, h \in E_3$$

holds, $\Phi'(\zeta)$ is called the *Gâteaux derivative* of a function Φ at the point ζ.

Consider the expansion of the function $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$ in the basis $\{I_k\}_{k=1}^n$:

$$\Phi(\zeta) = \sum_{k=1}^{n} U_k(x,y,z)\, I_k\,. \tag{3.2}$$

Let the functions $U_k : \Omega \to \mathbb{C}$ be $\mathbb{R}$-differentiable in the domain Ω. In other words, for any $(x,y,z) \in \Omega$,

$$U_k(x+\Delta x, y+\Delta y, z+\Delta z) - U_k(x,y,z) = \frac{\partial U_k}{\partial x}\,\Delta x + \frac{\partial U_k}{\partial y}\,\Delta y + \frac{\partial U_k}{\partial z}\,\Delta z +$$

$$+\, o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}\right), \qquad (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \to 0\,.$$

The function Φ is monogenic in the domain Ω_ζ iff, at every point of the domain Ω_ζ, the conditions

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial x}\, e_2\,, \quad \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial x}\, e_3 \tag{3.3}$$

hold.

We note that the expansion of the resolvent takes the form

$$(te_1 - \zeta)^{-1} = \sum_{u=1}^{m} \frac{1}{t - \xi_u}\, I_u + \sum_{s=m+1}^{n} \sum_{k=2}^{s-m+1} \frac{Q_{k,s}}{\left(t - \xi_{u_s}\right)^k}\, I_s \tag{3.4}$$

$$\forall\, t \in \mathbb{C} : \, t \neq \xi_u, \quad u = 1, 2, \ldots, m,$$

where $Q_{k,s}$ are defined by the recurrence relations

$$Q_{2,s} := T_s\,, \quad Q_{k,s} = \sum_{r=k+m-2}^{s-1} Q_{k-1,r}\, B_{r,\,s}\,, \quad k = 3, 4, \ldots, s-m+1,$$

for

$$T_s := ya_s + zb_s\,, \quad B_{r,s} := \sum_{k=m+1}^{s-1} T_k \Upsilon_{r,s}^k\,, \quad s = m+2, \ldots, n,$$

and the natural numbers u_s are defined by rule 3 of the multiplication table of the algebra $\mathbb{A}_n^m$.

Relations (3.4) imply that the points $(x,y,z) \in \mathbb{R}^3$ corresponding to irreversible elements $\zeta \in \mathbb{A}_n^m$ lie on the straight lines

$$L_u : \quad \begin{cases} x + y \operatorname{Re} a_u + z \operatorname{Re} b_u = 0, \\ y \operatorname{Im} a_u + z \operatorname{Im} b_u = 0 \end{cases} \tag{3.5}$$

in the three-dimensional space $\mathbb{R}^3$.

Let the domain $\Omega \subset \mathbb{R}^3$ be convex in the direction of the straight lines $L_u,\ u = 1, 2, \ldots, m$. By D_u, we denote the domain of the complex plane $\mathbb{C}$ onto which the domain Ω_ζ is mapped by the functional f_u.

Theorem A [14]. *Let the domain $\Omega \subset \mathbb{R}^3$ be convex in the direction of the straight lines L_u and let $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Then every monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$ can be presented in the form*

$$\Phi(\zeta) = \sum_{u=1}^{m} I_u \frac{1}{2\pi i} \int\limits_{\Gamma_u} F_u(t)(t-\zeta)^{-1}\, dt + \sum_{s=m+1}^{n} I_s \frac{1}{2\pi i} \int\limits_{\Gamma_{u_s}} G_s(t)(t-\zeta)^{-1}\, dt, \tag{3.6}$$

where F_u is some holomorphic function in the domain D_u, G_s is some holomorphic function in the domain D_{u_s}, and Γ_q is the closed Jordan rectifiable curve lying in the domain D_q, encloses the pointξ_q, and contains no points ξ_ℓ , $\ell = 1, 2, \ldots, m, \ell \neq q$.

By the conditions of Theorem **A**, every monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_n^m$ can be continued to a function monogenic in the domain

$$\Pi_\zeta := \{\zeta \in E_3 : f_u(\zeta) = D_u\, ,\ u = 1, 2, \ldots, m\}. \tag{3.7}$$

Therefore, we consider monogenic functions Φ defined in domains of the form Π_ζ .

4. Characteristic equation in different commutative algebras

We say that the system of equations polynomial over the field $\mathbb{C}$ Q_1 is *reduced* to a system of polynomial equations Q_2, if the system Q_2 follows from the system Q_1 by means of the removal of some number of equations. In turn, the system Q_2 *is a reduction of the system* Q_1. We note that the reduced system Q_2 is not unique the given system of polynomial equations Q_1. The following proposition is obvious.

Proposition 4.1. *Let the system of polynomial equations Q_1 with complex-valued unknowns $t_1, t_2, \ldots, t_n$ have solutions, and let Q_2 be its any reduced system with unknowns $t_{i_1}, t_{i_2}, \ldots, t_{i_k}$, where $i_1, i_2, \ldots, i_k$, $k \leq n$, are pairwise different elements of the set $\{1, 2, \ldots, n\}$. Then all $t_{i_1}, t_{i_2}, \ldots, t_{i_k}$ satisfying the system Q_1 are solutions of the system Q_2.*

For example, the system of equations

$$\begin{aligned} 1 + a_1^2 + b_1^2 &= 0, \\ 1 + a_2^2 + b_2^2 &= 0, \\ a_2 a_3 + b_2 b_3 &= 0 \end{aligned} \tag{4.1}$$

is reduced to the system of equations

$$\begin{aligned} 1 + a_2^2 + b_2^2 &= 0, \\ a_2 a_3 + b_2 b_3 &= 0. \end{aligned} \tag{4.2}$$

Proposition 4.1 means that all values of a_2, b_2, a_3, b_3, satisfying system (4.1) are solutions of system (4.2).

We now prove some auxiliary propositions.

Lemma 4.1. *Let there exist a triple of vectors* $1, e_2, e_3$ *linearly independent over* $\mathbb{R}$ *in the algebra* $\mathbb{A}_n^m$*. Let they satisfy the characteristic equation* (1.5)*. Then, for each* $u \in \{1, 2, \ldots, m\}$*, the characteristic system generated by the equation* $\mathcal{X}(I_u, e_2 I_u, e_3 I_u) = 0$ *is a reduction of the characteristic system generated by Eq.* (1.5)*.*

Proof. Let the left-hand side of Eq. (1.5) in the basis of the algebra have the form

$$\mathcal{X}(1, e_2, e_3) = \sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma}\, e_2^{\beta}\, e_3^{\gamma} = \sum_{k=1}^{n} V_k\, I_k = 0.$$

Respectively, the characteristic system generated by Eq. (1.5) takes the form

$$\begin{aligned} &V_1 = 0,\\ &\ldots\ldots\\ &V_n = 0. \end{aligned} \tag{4.3}$$

Consider now the characteristic system generated by the equation $\mathcal{X}(I_u, e_2 I_u, e_3 I_u) = 0$. We have

$$\mathcal{X}(I_u, e_2 I_u, e_3 I_u) = \sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma}\, I_u\, (e_2 I_u)^{\beta}\, (e_3 I_u)^{\gamma} =$$

$$= I_u \sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma}\, e_2^{\beta}\, e_3^{\gamma} = I_u \sum_{k=1}^{n} V_k\, I_k = V_u + I_u \sum_{k=m+1}^{n} V_k I_k = 0. \tag{4.4}$$

According to rule 3 of the multiplication table of the algebra $\mathbb{A}_n^m$, the product $I_u \sum\limits_{k=m+1}^{n} V_k I_k$ belongs to the radical $\mathcal{R}$. Thus, Eq. (4.4) is equivalent to such characteristic system:

$$\begin{aligned} &V_u = 0,\\ &\ldots\ldots\\ &V_k = 0 \quad \forall\, k \in \{m+1, \ldots, n\} \,:\, I_u I_k = I_k\,. \end{aligned} \tag{4.5}$$

It is obvious that system (4.5) is a reduction of system (4.3). □

By $\operatorname{Rad} e_2$, we denote a part of the vector e_2 from expansion (3.1) that is contained in its radical, i.e., $\operatorname{Rad} e_2 := \sum\limits_{r=m+1}^{n} a_r I_r$. Analogously, $\operatorname{Rad} e_3 := \sum\limits_{r=m+1}^{n} b_r I_r$.

Lemma 4.2. *Let there exist a triple of vectors* $1, e_2, e_3$ *linearly independent over* $\mathbb{R}$ *in the algebra* $\mathbb{A}_n^m = S \oplus_s N$*. Let they satisfy the characteristic equation* (1.5)*. Then, in the algebra* $\mathbb{A}_{n-m+1}^1 = 1 \oplus_s N$ *(where the nilpotent subalgebra* N *is the same as in the algebra* $\mathbb{A}_n^m$*) for each* $u \in \{1, 2, \ldots, m\}$*, there exists a triple of vectors*

$$\begin{aligned} &\widetilde{e}_1(u) = 1,\\ &\widetilde{e}_2(u) := a_u + I_u \operatorname{Rad} e_2\,,\\ &\widetilde{e}_3(u) := b_u + I_u \operatorname{Rad} e_3. \end{aligned} \tag{4.6}$$

The triple is such that the characteristic system generated by the equation $\mathcal{X}(1, \widetilde{e}_2(u), \widetilde{e}_3(u)) = 0$ *is a reduction of the characteristic system generated by the equation* (1.5)*.*

Proof. A consequence of equalities (4.6) is the equalities

$$\widetilde{e}_2^{\beta}(u) = a_u^{\beta} + I_u \sum_{k=1}^{\beta} \mathcal{C}_{\beta}^{k}\, a_u^{\beta-k}\, (\operatorname{Rad} e_2)^k\,,$$
$$\widetilde{e}_3^{\gamma}(u) = b_u^{\gamma} + I_u \sum_{k=1}^{\gamma} \mathcal{C}_{\gamma}^{k}\, b_u^{\gamma-k}\, (\operatorname{Rad} e_3)^k\,. \tag{4.7}$$

In view of formulas (4.7), the characteristic polynomial $\mathcal{X}(1, \widetilde{e}_2(u), \widetilde{e}_3(u)) = 0$ takes the form

$$\sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma}\, \widetilde{e}_2^{\beta}(u)\, \widetilde{e}_3^{\gamma}(u) = \sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma} \Bigg(a_u^{\beta}\, b_u^{\gamma}$$
$$+ I_u\, b_u^{\gamma} \sum_{k=1}^{\beta} \mathcal{C}_{\beta}^{k}\, a_u^{\beta-k}\, (\operatorname{Rad} e_2)^k + I_u\, a_u^{\beta} \sum_{k=1}^{\gamma} \mathcal{C}_{\gamma}^{k}\, b_u^{\gamma-k}\, (\operatorname{Rad} e_3)^k$$
$$+ I_u \sum_{k=1}^{\beta} \mathcal{C}_{\beta}^{k}\, a_u^{\beta-k}\, (\operatorname{Rad} e_2)^k \sum_{p=1}^{\gamma} \mathcal{C}_{\gamma}^{p}\, b_u^{\gamma-p}\, (\operatorname{Rad} e_3)^p \Bigg) = 0. \tag{4.8}$$

We now show that the characteristic systems generated by the equations $\mathcal{X}(1, \widetilde{e}_2(u), \widetilde{e}_3(u)) = 0$ and $\mathcal{X}(I_u, e_2 I_u, e_3 I_u) = 0$ coincide.

For this purpose, we note that a consequence of expansions (3.1) is the representations

$$e_2 = a_1 I_1 + \cdots + a_m I_m + \operatorname{Rad} e_2\,, \quad e_3 = b_1 I_1 + \cdots + b_m I_m + \operatorname{Rad} e_3$$

which yield the relations

$$e_2 I_u = a_u I_u + I_u \operatorname{Rad} e_2\,, \quad e_3 I_u = b_u I_u + I_u \operatorname{Rad} e_3\,. \tag{4.9}$$

From (4.9), we get the equalities

$$e_2^{\beta} I_u = a_u^{\beta} I_u + I_u \sum_{k=1}^{\beta} \mathcal{C}_{\beta}^{k}\, a_u^{\beta-k}\, (\operatorname{Rad} e_2)^k\,,$$
$$e_3^{\gamma} I_u = b_u^{\gamma} I_u + I_u \sum_{k=1}^{\gamma} \mathcal{C}_{\gamma}^{k}\, b_u^{\gamma-k}\, (\operatorname{Rad} e_3)^k\,. \tag{4.10}$$

With regard for formulas (4.10), the characteristic equation $\mathcal{X}(I_u, e_2 I_u, e_3 I_u) = 0$ takes the form

$$I_u \sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma}\, e_2^{\beta}\, e_3^{\gamma} = \sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma} \Bigg(a_u^{\beta}\, b_u^{\gamma}\, I_u$$
$$+ I_u\, b_u^{\gamma} \sum_{k=1}^{\beta} \mathcal{C}_{\beta}^{k}\, a_u^{\beta-k}\, (\operatorname{Rad} e_2)^k + I_u\, a_u^{\beta} \sum_{k=1}^{\gamma} \mathcal{C}_{\gamma}^{k}\, b_u^{\gamma-k}\, (\operatorname{Rad} e_3)^k$$
$$+ I_u \sum_{k=1}^{\beta} \mathcal{C}_{\beta}^{k}\, a_u^{\beta-k}\, (\operatorname{Rad} e_2)^k \sum_{p=1}^{\gamma} \mathcal{C}_{\gamma}^{p}\, b_u^{\gamma-p}\, (\operatorname{Rad} e_3)^p \Bigg) = 0. \tag{4.11}$$

Equalities (4.8) and (4.11) imply obviously that the characteristic systems generated by the equations $\mathcal{X}(1, \widetilde{e}_2(u), \widetilde{e}_3(u)) = 0$, and $\mathcal{X}(I_u, e_2 I_u, e_3 I_u) = 0$ coincide. Now, the proof of the lemma follows from Lemma 4.1. □

Remark 4.1. We note that the algebra $\mathbb{A}^1_{n-m+1} = 1 \oplus_s N$ with basis $\{1, I_{m+1}, \ldots, I_n\}$ is a subalgebra of the algebra $\mathbb{A}^m_n = S \oplus_s N$. Indeed, any element a of the algebra $\mathbb{A}^m_n = S \oplus_s N$ of the form

$$a = a_0 I_1 + a_0 I_2 + \cdots + a_0 I_m + a_{m+1} I_{m+1} + \cdots + a_n I_n$$

$$= a_0(I_1 + \cdots + I_m) + a_{m+1} I_{m+1} + \cdots + a_n I_n = a_0 + a_{m+1} I_{m+1} + \cdots + a_n I_n$$

is a representation of any element of the algebra $\mathbb{A}^1_{n-m+1} = 1 \oplus_s N$.

Proposition 4.1 and Lemma 4.2 yield the following proposition.

Theorem 4.1. *Let, in the algebra* $\mathbb{A}^m_n = S \oplus_s N$, *there exist a triple of vectors* $1, e_2, e_3$ *that are linearly independent over* $\mathbb{R}$ *and satisfy the characteristic equation* (1.5). *Then the triple of vectors* (4.6) *satisfies the the characteristic equation* $\mathcal{X}(1, \widetilde{e}_2(u), \widetilde{e}_3(u)) = 0$ *in the algebra* $\mathbb{A}^1_{n-m+1} = 1 \oplus_s N$ *(where the nilpotent subalgebra* N *is the same as in the algebra* $\mathbb{A}^m_n$*) for each* $u \in \{1, 2, \ldots, m\}$.

Example 4.1. Over the field $\mathbb{C}$, consider the algebra $\mathbb{A}^2_3$ with the multiplication table (see, e.g., [6, p. 32], [8])

$\cdot$	I_1	I_2	I_3
I_1	I_1	0	0
I_2	0	I_2	I_3
I_3	0	I_3	0

. (4.12)

It is obvious that the subalgebra generated by the idempotents I_1, I_2 is a semisimple subalgebra S, and the subalgebra $\{\alpha I_3 : \alpha \in \mathbb{C}\}$ is a nilpotent subalgebra N. Then the algebra $\mathbb{A}^1_2 := 1 \oplus_s N$ coincides with the known biharmonic algebra $\mathbb{B}$ (see, e.g., [15]) and has the multiplication table

$\cdot$	1	I_3
1	1	I_3
I_3	I_3	0

. (4.13)

Let the characteristic equation (1.1) be given in the algebra $\mathbb{A}^2_3$. As is known (see Theorem 1.8 in [6]), the condition of *harmonicity* (1.1) of the vectors $e_1 = 1$, $e_2 = a_1 I_1 + a_2 I_2 + a_3 I_3$, $e_3 = b_1 I_1 + b_2 I_2 + b_3 I_3$ of the algebra $\mathbb{A}^2_3$ is equivalent to the system of equations (4.1).

For the algebra $\mathbb{A}^2_3$ $m = 2$. Therefore, we construct two triples of vectors of the form (4.6) in the algebra $\mathbb{B}$:

$$\widetilde{e}_1(1) = 1,\ \widetilde{e}_2(1) = a_1 + I_1(a_3 I_3) = a_1\,,\ \widetilde{e}_3(1) = b_1 + I_1(b_3 I_3) = b_1 \tag{4.14}$$

and

$$\begin{aligned} &\widetilde{e}_1(2) = 1, \\ &\widetilde{e}_2(2) = a_2 + I_2(a_3 I_3) = a_2 + a_3 I_3\,, \\ &\widetilde{e}_3(2) = b_2 + I_2(b_3 I_3) = b_2 + b_3 I_3\,. \end{aligned} \tag{4.15}$$

By Theorem 4.1, triples (4.14) and (4.15) are harmonic in the algebra $\mathbb{B}$ (i.e., they satisfy condition (1.1)). Indeed, the harmonicity of triple (4.14) is equivalent to the first equation of system (4.1), and the harmonicity of triple (4.15) is equivalent to system (4.2).

Example 4.2. Over the field $\mathbb{C}$, we now consider the algebra $\mathbb{A}_5^3$ with the multiplication table

$\cdot$	I_1	I_2	I_3	I_4	I_5
I_1	I_1	0	0	0	I_5
I_2	0	I_2	0	0	0
I_3	0	0	I_3	I_4	0
I_4	0	0	I_4	0	0
I_5	I_5	0	0	0	0

. (4.16)

We note that the subalgebra generated by the idempotents I_1, I_2, I_3 is a semisimple subalgebra S, and the subalgebra with the basis $\{I_4, I_5\}$ is a nilpotent subalgebra N. Then the algebra $\mathbb{A}_3^1 := 1 \oplus_s N$ coincides with the known algebra $\mathbb{A}_4$ (see, e.g., [6, p. 26]) and has the multiplication table

$\cdot$	1	I_4	I_5
1	1	I_4	I_5
I_4	I_4	0	0
I_5	I_5	0	0

. (4.17)

Let the characteristic equation (1.1) be given in the algebra $\mathbb{A}_5^3$. The condition of harmonicity (1.1) of vectors of the form (3.1) of the algebra $\mathbb{A}_5^3$ is equivalent to the system of equations

$$
\begin{aligned}
&1 + a_u^2 + b_u^2 = 0, \quad u = 1, 2, 3,\\
&a_3 a_4 + b_3 b_4 = 0,\\
&a_1 a_5 + b_1 b_5 = 0.
\end{aligned}
\tag{4.18}
$$

For the algebra $\mathbb{A}_5^3$ $m = 3$. Therefore, we construct three triples of vectors of the form (4.6) in the algebra $\mathbb{A}_4$:

$$
\begin{aligned}
&\widetilde{e}_1(1) = 1,\\
&\widetilde{e}_2(1) = a_1 + I_1(a_4 I_4 + a_5 I_5) = a_1 + a_5 I_5\,,\\
&\widetilde{e}_3(1) = b_1 + I_1(b_4 I_4 + b_5 I_5) = b_1 + b_5 I_5\,,
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
&\widetilde{e}_1(2) = 1,\\
&\widetilde{e}_2(2) = a_2 + I_2(a_4 I_4 + a_5 I_5) = a_2\,,\\
&\widetilde{e}_3(2) = b_2 + I_2(b_4 I_4 + b_5 I_5) = b_2\,,
\end{aligned}
\tag{4.20}
$$

and

$$
\begin{aligned}
&\widetilde{e}_1(3) = 1,\\
&\widetilde{e}_2(3) = a_3 + I_3(a_4 I_4 + a_5 I_5) = a_3 + a_4 I_4\,,\\
&\widetilde{e}_3(3) = b_3 + I_3(b_4 I_4 + b_5 I_5) = b_3 + b_4 I_4\,.
\end{aligned}
\tag{4.21}
$$

By Theorem 4.1, triples (4.19), (4.20), and (4.21) are harmonic in the algebra $\mathbb{A}_4$ (i.e., they satisfy condition (1.1)). Indeed, the harmonicity of triple (4.19) is equivalent to the system of the first and the fifth equations of system (4.15); the harmonicity of triple (4.20) is equivalent to the second equation of system (4.15), and the harmonicity of triple (4.21) is equivalent to the system of the third and fourth equations of system (4.15).

4.1. Linear independence of vectors $1, \widetilde{e}_2(u), \widetilde{e}_3(u)$

It is seen from the presented examples that the vectors $1, \widetilde{e}_2(u), \widetilde{e}_3(u)$ for some $u \in \{1, 2, \ldots, m\}$ can be linearly dependent over the field $\mathbb{R}$. For example, triples (4.14) and (4.20) are always linearly dependent over the field $\mathbb{R}$.

We now establish the necessary and sufficient conditions of linear independence of the vectors $1, \widetilde{e}_2(u), \widetilde{e}_3(u)$ of the algebra $\mathbb{A}^1_{n-m+1} = 1 \oplus_s N$ over the field $\mathbb{R}$.

Lemma 4.3. *Let the vectors* (3.1) *of the algebra* $\mathbb{A}^m_n = S \oplus_s N$ *be linearly independent over the field* $\mathbb{R}$, *and Let* $u \in \{1, 2, \ldots, m\}$ *be fixed. Then*

1. if the vectors $I_u \operatorname{Rad} e_2$, $I_u \operatorname{Rad} e_3 \in \mathbb{A}^m_n$ *are linearly independent over the field* $\mathbb{R}$, *then the vectors* $1, \widetilde{e}_2(u), \widetilde{e}_3(u)$ *of the algebra* $\mathbb{A}^1_{n-m+1} = 1 \oplus_s N$ *are also linearly independent over the field* $\mathbb{R}$;

2. but if the vectors $I_u \operatorname{Rad} e_2$, $I_u \operatorname{Rad} e_3 \in \mathbb{A}^m_n$ *are linearly dependent over the field* $\mathbb{R}$, *then the vectors* $1, \widetilde{e}_2(u), \widetilde{e}_3(u)$ *of the algebra* $\mathbb{A}^1_{n-m+1} = 1 \oplus_s N$ *are linearly independent over the field* $\mathbb{R}$ *iff there exists* $r \in \{m+1, \ldots, n\}$ *such that* $I_u I_r = I_r$ *and at least one of the relations*

$$\operatorname{Im} a_u \operatorname{Re} b_r \neq \operatorname{Im} b_u \operatorname{Re} a_r \quad \textit{or} \quad \operatorname{Im} a_u \operatorname{Im} b_r \neq \operatorname{Im} b_u \operatorname{Im} a_r \tag{4.22}$$

is satisfied.

Proof. We now prove the first proposition of the lemma. By condition, the equality

$$\beta_2 I_u \operatorname{Rad} e_2 + \beta_3 I_u \operatorname{Rad} e_3 = 0, \quad \beta_2, \beta_3 \in \mathbb{R} \tag{4.23}$$

is satisfied iff $\beta_2 = \beta_3 = 0$.

Consider the linear combination

$$\alpha_1 + \alpha_2 \widetilde{e}_2(u) + \alpha_3 \widetilde{e}_3(u) = (\alpha_1 + \alpha_2 a_u + \alpha_3 b_u) +$$

$$+ (\alpha_2 I_u \operatorname{Rad} e_2 + \alpha_3 I_u \operatorname{Rad} e_3) = 0, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}. \tag{4.24}$$

We note that the expression in the second bracket in equality (4.24) takes values in the radical $\mathcal{R}$ of the algebra, and the first bracket is complex-valued. Therefore, condition (4.24) is equivalent to the system of equations

$$\begin{aligned} &\alpha_1 + \alpha_2 a_u + \alpha_3 b_u = 0, \\ &\alpha_2 I_u \operatorname{Rad} e_2 + \alpha_3 I_u \operatorname{Rad} e_3 = 0. \end{aligned} \tag{4.25}$$

The second equation of system (4.25) and condition (4.23) yield $\alpha_2 = \alpha_3 = 0$. Then, from the first equation of system (4.25), we get $\alpha_1 = 0$. Hence, the vectors $1, \widetilde{e}_2(u), \widetilde{e}_3(u)$ are linearly independent over $\mathbb{R}$.

Let u prove the second proposition of the lemma. Consider the equality

$$\beta_1 + \beta_2 e_2 + \beta_3 e_3 = \sum_{s=1}^{m} I_s(\beta_1 + \beta_2 a_s + \beta_3 b_s) + \sum_{k=m+1}^{n} I_k(\beta_2 a_k + \beta_3 b_k) = 0,$$

which is equivalent to the system of equations

$$\begin{aligned} \beta_1 + \beta_2 \operatorname{Re} a_s + \beta_3 \operatorname{Re} b_s &= 0, \\ \beta_2 \operatorname{Im} a_s + \beta_3 \operatorname{Im} b_s &= 0, \qquad s = 1, 2, \ldots, m, \\ \beta_2 \operatorname{Re} a_k + \beta_3 \operatorname{Re} b_k &= 0, \\ \beta_2 \operatorname{Im} a_k + \beta_3 \operatorname{Im} b_k &= 0, \qquad k = m+1, \ldots, n. \end{aligned} \tag{4.26}$$

The linear independence of the vectors $1, e_2, e_3$ over $\mathbb{R}$ means that there exist at least two equations of system (4.26), except for the first one, which are not proportional to each other.

Let us write the condition of linear independence of the vectors $1, \widetilde{e}_2(u), \widetilde{e}_3(u)$ over $\mathbb{R}$. To thi end, we present system (4.25) in the expanded form:

$$\begin{aligned} \alpha_1 + \alpha_2 \operatorname{Re} a_u + \alpha_3 \operatorname{Re} b_u &= 0, \\ \alpha_2 \operatorname{Im} a_u + \alpha_3 \operatorname{Im} b_u &= 0, \\ \alpha_2 \operatorname{Re} a_r + \alpha_3 \operatorname{Re} b_r &= 0, \\ \alpha_2 \operatorname{Im} a_r + \alpha_3 \operatorname{Im} b_r &= 0 \\ \forall\, r \in \{m+1, \dots, n\} &: \quad I_u I_r = I_r\,. \end{aligned} \tag{4.27}$$

By the condition of item *2* of the lemma, the vectors $I_u \operatorname{Rad} e_2$, $I_u \operatorname{Rad} e_3$ are linearly dependent over $\mathbb{R}$. This means that all equalities in system (4.27), except for two first ones, are proportional to one another. It is obvious that, for the linear independence of the vectors $1, \widetilde{e}_2(u), \widetilde{e}_3(u)$ over $\mathbb{R}$ to hold, it is necessary and sufficient that the second equation of system (4.27) be no proportional to at least one other equation (except for the first one) of system (4.27). This is equivalent to conditions (4.22). □

5. Monogenic functions defined in different commutative algebras

In the algebra $\mathbb{A}_n^m = S \oplus_s N$, we consider monogenic functions Φ defined in some domain $\Pi_\zeta \subset E_3$ of the form (3.7). Geometrically, the domain $\Pi \subset \mathbb{R}^3$ which is congruent to the domain $\Pi_\zeta \subset E_3$ is the intersection of m infinite cylinders each of them is parallel to some straight line of m ones L_u, $u = 1, 2, \dots, m$, of the form (3.5). In other words, $\Pi = \cap_{u=1}^m \Pi(u)$, where $\mathbb{R}^3 \supset \Pi(u)$ is an infinite cylinder parallel to the straight line L_u. We have the same for congruent domains in E_3:

$$\Pi_\zeta = \bigcap_{u=1}^{m} \Pi_\zeta(u). \tag{5.1}$$

Analytically, the cylinder $\Pi_\zeta(u)$ is determined by the equality

$$\Pi_\zeta(u) = \{\zeta_u := I_u\, \zeta \,:\, \zeta \in \Pi_\zeta\}.$$

Consider now a function $\Phi : \Pi_\zeta \to \mathbb{A}_n^m$ monogenic in the domain Π_ζ. Denote

$$\Phi_u(\zeta) := I_u\, \Phi(\zeta), \quad u = 1, 2, \dots, m. \tag{5.2}$$

Then the validity of the equality

$$\Phi = (I_1 + \cdots + I_m)\Phi = \sum_{u=1}^{m} \Phi_u \tag{5.3}$$

becomes obvious. In addition, equality (3.6) and the multiplication table of the algebra $\mathbb{A}_n^m$ imply that, for each $u \in \{1, 2, \dots, m\}$, the function Φ_u is monogenic in the whole infinite cylinder $\Pi_\zeta(u)$.

Thus, every function $\Phi : \Pi_\zeta \to \mathbb{A}_n^m$ monogenic in the domain (5.1) can be presented in the form of sum (5.3), where the function Φ_u is monogenic in the whole cylinder $\Pi_\zeta(u)$.

We now consider the monogenic functions $\widetilde{\Phi}$ in the algebra $\mathbb{A}_{n-m+1}^1 = 1 \oplus_s N$. According to Remark 4.1, the algebra $\mathbb{A}_{n-m+1}^1$ is a subalgebra of the algebra $\mathbb{A}_n^m$. Therefore, all cylinders $\Pi_\zeta(u)$ related to equality (5.1) in the algebra $\mathbb{A}^1_{n-m+1}$ coincide with one another. In other words, every monogenic function in algebras of the form $\mathbb{A}^1_{n-m+1}$ is monogenic in a certain single infinite cylinder.

In the following theorem, we will find the connection between monogenic functions in the algebras $\mathbb{A}^m_n = S \oplus_s N$ and $\mathbb{A}^1_{n-m+1} = 1 \oplus_s N$. Prior to the formulation of the result, we introduce some notations.

On vectors of the form (4.6) of the algebra $\mathbb{A}^1_{n-m+1}$, we span a linear space $\widetilde{E}_3(u) := \{\widetilde{\zeta}(u) = x + y\widetilde{e}_2(u) + z\widetilde{e}_3(u) : \ x, y, z \in \mathbb{R}\}$. The triple of vectors (4.6) define one straight line $\widetilde{L}(u)$ of the form (3.5) which corresponds to the set of irreversible elements $\widetilde{\zeta}(u)$ of the space $\widetilde{E}_3(u)$. Let $\widetilde{\Pi}_{\widetilde{\zeta}(u)}$ be some infinite cylinder in $\widetilde{E}_3(u)$ which is parallel to the straight line $\widetilde{L}(u)$.

Theorem 5.1. *Let there exist a triple of linearly independent vectors $1, e_2, e_3$ in the algebra $\mathbb{A}^m_n = S \oplus_s N$ over $\mathbb{R}$. Let they satisfy the characteristic equation (1.5), and let $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. In addition, let a function $\Phi : \Pi_\zeta \to \mathbb{A}^m_n$ of the variable $\zeta = x + ye_2 + ze_3$ be monogenic in the domain $\Pi_\zeta \subset E_3$ of the form (5.1). Then, for each $u \in \{1, 2, \ldots, m\}$, there exists a triple of of vectors (4.6) in the algebra $\mathbb{A}^1_{n-m+1} = 1 \oplus_s N$ (where the nilpotent subalgebra N is the same as in the algebra $\mathbb{A}^m_n$) satisfying the characteristic equation $\mathcal{X}(1, \widetilde{e}_2(u), \widetilde{e}_3(u)) = 0$. Moreover, there exists a function $\widetilde{\Phi}_u : \widetilde{\Pi}_{\widetilde{\zeta}(u)} \to \mathbb{A}^1_{n-m+1}$ of the variable $\widetilde{\zeta}(u)$ which is monogenic in the cylinder*

$$\widetilde{\Pi}_{\widetilde{\zeta}(u)} = \Big\{\widetilde{\zeta}(u) \in \widetilde{E}_3(u) \ : \ f_u\big(\widetilde{\zeta}(u)\big) = f_u(\zeta) \, , \, \zeta \in \Pi_\zeta(u)\Big\}$$

and is such that

$$\Phi_u(\zeta) = I_u \, \widetilde{\Phi}_u\big(\widetilde{\zeta}(u)\big). \tag{5.4}$$

Proof. The existence of triple (4.6) with the property $\mathcal{X}(1, \widetilde{e}_2(u), \widetilde{e}_3(u)) = 0$ was proved in Theorem 4.1. In what follows, let $u \in \{1, 2, \ldots, m\}$ be fixed. We now prove the existence and monogenicity of the function $\widetilde{\Phi}_u$ satisfying equality (5.4) in the domain $\widetilde{\Pi}_{\widetilde{\zeta}(u)}$. With this purpose, we prove firstly the equality

$$I_u \, \zeta^{-1} = I_u \, \widetilde{\zeta}^{-1}(u) \tag{5.5}$$

$$\forall \, \zeta = x + ye_2 + ze_3 \ \ \forall \, \widetilde{\zeta}(u) = x + y\widetilde{e}_2(u) + z\widetilde{e}_3(u), \ x \in \mathbb{C}, y, z \in \mathbb{R}.$$

Equalities (4.10) and (4.6) yield the relations

$$I_u \, e_2 = I_u \, \widetilde{e}_2(u) \, , \quad I_u \, e_3 = I_u \, \widetilde{e}_3(u) \, .$$

In turn, they result in the equality

$$I_u \, \zeta = I_u \, \widetilde{\zeta}(u). \tag{5.6}$$

Consider the difference $I_u \, \zeta^{-1} - I_u \, \widetilde{\zeta}^{-1}(u)$. By the Hilbert formula (see, e.g., Theorem 4.8.2 in [13]), we have

$$I_u \, \zeta^{-1} - I_u \, \widetilde{\zeta}^{-1}(u) = \big(I_u \, \zeta - I_u \, \widetilde{\zeta}(u)\big)\Big(\zeta \, \widetilde{\zeta}(u)\Big)^{-1} = 0$$

due to equality (5.6). Hence, equality (5.5) is proved. Now, formula (5.5) yields the relation

$$I_u(t - \zeta)^{-1} = I_u\big(t - \widetilde{\zeta}(u)\big)^{-1} \tag{5.7}$$

$$\forall \, t \in \mathbb{C} : \ t \neq \xi_u = f_u(\zeta) \quad \forall \, \zeta \in \Pi_\zeta(u) \quad \forall \, \widetilde{\zeta}(u) \in \widetilde{\Pi}_{\widetilde{\zeta}(u)} \, .$$

For a function $\Phi_u(\zeta)$ monogenic in the domain $\Pi_\zeta(u)$, the multiplication table of the algebra $\mathbb{A}_n^m$ and formula (3.6) yield the representation

$$\Phi_u(\zeta) = I_u \frac{1}{2\pi i} \int\limits_{\Gamma_u} \Big(F_u(t) + \sum_{s=m+1}^{n} I_s\, G_s(t) \Big) (t-\zeta)^{-1}\, dt, \tag{5.8}$$

where the functions $F_u\,, G_s$ are defined in Theorem **A**.

In view of relation (5.7), we rewrite representation (5.8) in the form

$$\Phi_u(\zeta) = I_u \frac{1}{2\pi i} \int\limits_{\Gamma_u} \Big(F_u(t) + \sum_{s=m+1}^{n} I_s\, G_s(t) \Big) \big(t-\widetilde{\zeta}(u)\big)^{-1}\, dt. \tag{5.9}$$

We note that the algebra $\mathbb{A}_{n-m+1}^1$ contains a single maximum ideal $\mathcal{I}$ coinciding with radical (2.1) of this algebra $\mathcal{R}$. Therefore, the unique linear continuous multiplicative functional $f : \mathbb{A}_{n-m+1}^1 \to \mathbb{C}$ whose kernel is the radical $\mathcal{R}$ is defined on this algebra. This means that $f\big(\widetilde{\zeta}(u)\big) = x + a_u\, y + b_u\, z$ for every $\widetilde{\zeta}(u) \in \widetilde{E}_3(u)$. With regard for the equality $f_u(\zeta) = x + a_u\, y + b_u\, z$ for any $\zeta \in E_3$, we have the equality

$$f\big(\widetilde{\zeta}(u)\big) = f_u(\zeta). \tag{5.10}$$

Equality (5.10) and the condition $f_u(E_3) = \mathbb{C}$ of the theorem yield the relation $f\big(\widetilde{\zeta}(u)\big) = \mathbb{C}$ for any $\widetilde{\zeta}(u) \in \widetilde{E}_3(u)$.

Thus, we have shown that the conditiona of Theorem **A** hold for monogenic functions in the algebra $\mathbb{A}_{n-m+1}^1$. Then formula (3.6) for the function $\widetilde{\Phi}_u\big(\widetilde{\zeta}(u)\big)$ monogenic in the domain $\widetilde{\Pi}_{\widetilde{\zeta}(u)}$ in the algebra $\mathbb{A}_{n-m+1}^1$ takes the form

$$\widetilde{\Phi}_u\big(\widetilde{\zeta}(u)\big) = \frac{1}{2\pi i} \int\limits_{\gamma} \Big(\widetilde{F}(t) + \sum_{s=m+1}^{n} I_s\, \widetilde{G}_s(t) \Big) \big(t-\widetilde{\zeta}(u)\big)^{-1}\, dt. \tag{5.11}$$

The required formula (5.4) will be a direct consequence of equalities (5.9) and (5.11), if we show that it is possible to set $\gamma \equiv \Gamma_u$, $F_u \equiv \widetilde{F}$ and $G_s \equiv \widetilde{G}_s$ for s such that $I_u\, I_s = I_s$. We now show this.

Equality (5.10) implies that the cylinders $\Pi_\zeta(u) \subset E_3$ and $\widetilde{\Pi}_{\widetilde{\zeta}(u)} \subset \widetilde{E}_3(u)$ are mapped by the corresponding functionals f_u and f into the same domain D of the complex plane $\mathbb{C}$. Thi means that the functions F_u, $\widetilde{F}$, G_s, and $\widetilde{G}_s$ are holomorphic in the same domain D. Hence, we can set $F_u \equiv \widetilde{F}$ and $G_s \equiv \widetilde{G}_s$ in D.

Since the integration curves γ and Γ_u lie in the domain D, we can take $\gamma \equiv \Gamma_u$. Moreover, the curve Γ_u in equality (5.8) encloses the point $f_u(\zeta) = x + a_u\, y + b_u\, z$ by Theorem **A**. Hence, the curve $\gamma \equiv \Gamma_u$ encloses the spectrum of the point $\widetilde{\zeta}(u)$, the point $f\big(\widetilde{\zeta}(u)\big) = x + a_u\, y + b_u\, z$, due to equality (5.10). This is what we require. The theorem is proved. ☐

Remark 5.1. Equalities (5.3) and (5.4) yield the representation

$$\Phi(\zeta) = I_1\, \widetilde{\Phi}_1\big(\widetilde{\zeta}(1)\big) + \cdots + I_m\, \widetilde{\Phi}_m\big(\widetilde{\zeta}(m)\big). \tag{5.12}$$

Remark 5.2. Theorem 4.1 means that the functions Φ and $I_u\widetilde{\Phi}_u$ for all $u = 1, 2, \ldots, m$ satisfy the same differential equation of the form (1.3).

Remark 5.3. Theorem 5.1 asserts that, for the construction of solutions of the differential equation (1.3) in the form of components of monogenic functions with values in commutative algebras, it is sufficient to restrict ourselves to the study of monogenic functions in algebras with the basis $\{1, \eta_1, \eta_2, \ldots, \eta_n\}$, where $\eta_1, \eta_2, \ldots, \eta_n$ are nilpotents. In other words, the number of such n-dimensional commutative associative algebras with 1 over the field $\mathbb{C}$ in which we should study monogenic functions is equal to the number of $(n-1)$-dimensional commutative associative complex nilpotent algebras.

In particular, among two-dimensional commutative associative algebras with 1 over the field $\mathbb{C}$ (only two such algebras exist), it is sufficient to restrict ourselves to the study of monogenic functions in the biharmonic algebra $\mathbb{B}$. Among three-dimensional commutative associative algebras with 1 over the field $\mathbb{C}$ (only four such algebras exist), it is sufficient to restrict ourselves to the study of monogenic functions in two of them (algebras $\mathbb{A}_3$ and $\mathbb{A}_4$ in terms of work [6]). Among four-dimensional commutative associative algebras with 1 over the field $\mathbb{C}$ (only 9 such algebras exist, see [16]), it is sufficient to restrict ourselves to the study of monogenic functions in four of them (algebras $\widetilde{A}_{3,1}$, $\widetilde{A}_{3,2}$, $\widetilde{A}_{3,3}$, and $\widetilde{A}_{3,4}$ from Table 9 in [17], see also Theorem 5.1 in [18]). Among all five-dimensional commutative associative algebras with 1 over the field $\mathbb{C}$ (only 25 such algebras exist, see [16]), it is sufficient to restrict ourselves to the study of monogenic functions in nine algebras (the multiplication tables of all those 9 nilpotent four-dimensional algebras are given in Theorem 6.1 in [18]). and Finally, among all six-dimensional commutative associative algebras with 1 over the field $\mathbb{C}$, it is sufficient to study the monogenic functions in 25 algebras (these 25 five-dimensional algebras are presented in Table 1 in [19]). It is also known (see [20]) that, by starting from a dimension equal to 6, the set of all pairwise nonisomorphic nilpotent commutative algebras over $\mathbb{C}$ is infinite.

Remark 5.4. Theorem 5.1 remains valid in the case where we consider the functions $\Phi : \Pi_\zeta \to \mathbb{A}_n^m$ of the variable $\zeta := \sum\limits_{r=1}^{k} x_r e_r$, $2 \le k \le 2n$, which is monogenic in the domain $\Pi_\zeta \subset E_k$. In this case, we should use Theorem 1 from [21] instead of Theorem **A**.

We now illustrate Theorem 5.1 on the algebras considered in Examples 4.1 and 4.2.

Example 5.1. Let us consider the algebra $\mathbb{A}_3^2$ with the multiplication table (4.12). For the algebra $\mathbb{A}_3^2$, the biharmonic algebra $\mathbb{B}$ with the multiplication table (4.13) is an algebra of the form $1 \oplus_s N$.

According to representation (3.6), every monogenic function Φ with values in the algebra $\mathbb{A}_3^2$ can be presented in the form

$$\Phi(\zeta) = F_1(\xi_1)I_1 + F_2(\xi_2)I_2 + \Big((a_3 y + b_3 z)F_2'(\xi_2) + G_3(\xi_2)\Big)I_3 \tag{5.13}$$

$$\forall\, \zeta \in \Pi_\zeta\,, \quad \xi_u = x + a_u y + b_u z,\ u = 1, 2,$$

where F_1 is some holomorphic function in the domain D_1 , and F_2 and G_3 are some holomorphic functions in the domain D_2 . Since $m = 2$ for $\mathbb{A}_3^2$, the domain Π_ζ is geometrically the intersection of two infinite cylinders: $\Pi_\zeta = \Pi_\zeta(1) \cap \Pi_\zeta(2)$.

We note that representation (5.13) was earlier obtained in [8]. In addition, function (5.13) satisfies some differential equation of the form (1.3).

We now present function (5.13) in the form (5.3):

$$\Phi(\zeta) = \Phi(\zeta)I_1 + \Phi(\zeta)I_2 =: \Phi_1(\zeta) + \Phi_2(\zeta), \tag{5.14}$$

where $\Phi_1(\zeta) = F_1(\xi_1)I_1$ is a monogenic function in the cylinder $\Pi_\zeta(1)$, and the function

$$\Phi_2(\zeta) = F_2(\xi_2)I_2 + \Big((a_3 y + b_3 z)F_2'(\xi_2) + G_3(\xi_2)\Big)I_3$$

is monogenic in the cylinder $\Pi_\zeta(2)$.

Consider the monogenic functions in the algebra $\mathbb{B}$. Representation (3.6) implies that every monogenic function $\widetilde{\Phi}$ with values in the algebra $\mathbb{B}$ can be presented in the form

$$\widetilde{\Phi}(\widetilde{\zeta}) = \widetilde{F}(\widetilde{\xi}) + \Big((a_3y + b_3z)\widetilde{F}'(\widetilde{\xi}) + \widetilde{G}(\widetilde{\xi})\Big)I_3 \quad \forall\, \widetilde{\zeta} \in \widetilde{\Pi}_{\widetilde{\zeta}},\ \widetilde{\xi} = f(\widetilde{\zeta}), \tag{5.15}$$

where $\widetilde{F}, \widetilde{G}$ are some holomorphic functions in the domain D. The domain $\widetilde{\Pi}_{\widetilde{\zeta}}$ is an infinite cylinder. Equality (5.15) was established in a special case in [15].

Theorem 5.1 asserts the following:

1) in the algebra $\mathbb{B}$, there exists a triple of the vectors $1, \widetilde{e}_2(1)\,, \widetilde{e}_3(1)$ which satisfies the same characteristic equation as the triple $1, e_2, e_3 \in \mathbb{A}_3^2$. In this case, the relations $\xi_1 \equiv \widetilde{\xi}$ and $D_1 \equiv D$ hold. In addition, there exists a function $\widetilde{\Phi}$ monogenic in $\mathbb{B}$ which is such that

$$I_1\, \widetilde{\Phi}_1\big(\widetilde{\zeta}(1)\big) = \Phi_1(\zeta)\,. \tag{5.16}$$

2) in the algebra $\mathbb{B}$, there exists a triple of the vectors $1, \widetilde{e}_2(2)\,, \widetilde{e}_3(2)$ which satisfies the same characteristic equation as the triple $1, e_2, e_3 \in \mathbb{A}_3^2$. In this case, the relations $\xi_2 \equiv \widetilde{\xi}$ and $D_2 \equiv D$ hold. In addition, there exists a monogenic function $\widetilde{\Phi}$ such that

$$I_2\, \widetilde{\Phi}_2\big(\widetilde{\zeta}(2)\big) = \Phi_2(\zeta)\,. \tag{5.17}$$

The required triples of vectors $1, \widetilde{e}_2(1)\,, \widetilde{e}_3(1)$, and $1, \widetilde{e}_2(2)\,, \widetilde{e}_3(2)$ were determined in Example 4.1. Consider case 1). Indeed, for triple (4.14), we have $\widetilde{\zeta}(1) = x + a_1y + b_1z \equiv \xi_1 \equiv \widetilde{\xi}$, $D_1 \equiv D$. Set $\widetilde{F} \equiv F_1$ and $\widetilde{G} \equiv G_3$ in D. Then equality (5.15) can be written in the form

$$\widetilde{\Phi}_1(\widetilde{\zeta}(1)) = F_1(\xi_1) + \Big((a_3y + b_3z)F_1'(\xi_1) + G_3(\xi_1)\Big)I_3\,. \tag{5.18}$$

Multiplying equality (5.18) by I_1, we verify the validity of equality (5.16).

Consider case 2). Indeed, for triple (4.15), we have

$$\widetilde{\zeta}(2) = x + y\widetilde{e}_2(2) + z\widetilde{e}_3(2) = x + a_2y + b_2z + a_3xI_3 + b_3yI_3\,.$$

It is obvious that $f(\widetilde{\zeta}(2)) = x + a_2y + b_2z = \xi_2 \equiv \widetilde{\xi}$, $D_2 \equiv D$. Set $\widetilde{F} \equiv F_2$ and $\widetilde{G} \equiv G_3$ in D. Then equality (5.15) can be written in the form

$$\widetilde{\Phi}_2(\widetilde{\zeta}(2)) = F_2(\xi_2) + \Big((a_3y + b_3z)F_1'(\xi_2) + G_3(\xi_2)\Big)I_3\,. \tag{5.19}$$

Multiplying equality (5.19) by I_2, we verify the validity of equality (5.17).

Thus, equality (5.12) is true:

$$\Phi(\zeta) = I_1\, \widetilde{\Phi}_1\big(\widetilde{\zeta}(1)\big) + I_2\, \widetilde{\Phi}_2\big(\widetilde{\zeta}(2)\big),$$

where Φ takes values in the algebra $\mathbb{A}_3^2$, and $\widetilde{\Phi}_1\big(\widetilde{\zeta}(1)\big) and \widetilde{\Phi}_2\big(\widetilde{\zeta}(2)\big)$ take values in $\mathbb{B}$.

Example 5.2. Consider the algebra $\mathbb{A}_5^3$ with the multiplication table (4.16). For the algebra $\mathbb{A}_5^3$, the algebra $\mathbb{A}_4$ with the multiplication table (4.17) is an algebra of the form $1 \oplus_s N$.

According to representation (3.6), every monogenic function Φ with values in the algebra $\mathbb{A}_5^3$ can be given in the form

$$\Phi(\zeta) = F_1(\xi_1)I_1 + F_2(\xi_2)I_2 + F_3(\xi_3)I_3 + \Big((a_4y + b_4z)F_3'(\xi_3) + G_3(\xi_3)\Big)I_4 +$$

$$+\Big((a_5y+b_5z)F_1'(\xi_1)+G_5(\xi_1)\Big)I_5 \tag{5.20}$$

$$\forall\,\zeta\in\Pi_\zeta\,,\quad \xi_u=x+a_uy+b_uz,\ u=1,2,3,$$

where F_1 and G_5 are some holomorphic functions in the domain D_1, F_2 is some holomorphic function in the domain D_2, and F_3 and G_3 are some holomorphic functions in the domain D_3. For $\mathbb{A}_5^3$ $m=3$. Therefore, the domain Π_ζ is geomerically the intersection of three infinite cylinders: $\Pi_\zeta=\Pi_\zeta(1)\cap\Pi_\zeta(2)\cap\Pi_\zeta(3)$.

Function (5.20) can be presented in the form (5.3):

$$\Phi(\zeta)=\Phi(\zeta)I_1+\Phi(\zeta)I_2+\Phi(\zeta)I_3=:\Phi_1(\zeta)+\Phi_2(\zeta)+\Phi_3(\zeta), \tag{5.21}$$

where

$$\Phi_1(\zeta)=F_1(\xi_1)I_1+\Big((a_5y+b_5z)F_1'(\xi_1)+G_5(\xi_1)\Big)I_5$$

is a monogenic function in the cylinder $\Pi_\zeta(1)$, the function $\Phi_2(\zeta)=F_2(\xi_2)I_2$ is monogenic in the cylinder $\Pi_\zeta(2)$, and the function

$$\Phi_3(\zeta)=F_3(\xi_3)I_3+\Big((a_4y+b_4z)F_3'(\xi_3)+G_3(\xi_3)\Big)I_4$$

is monogenic in the cylinder $\Pi_\zeta(3)$.

Consider the monogenic functions in the algebra $\mathbb{A}_4$. Representation (3.6) implies that every monogenic function $\widetilde{\Phi}$ with values in the algebra $\mathbb{A}_4$ can be presented in the form

$$\widetilde{\Phi}(\widetilde{\zeta})=\widetilde{F}(\widetilde{\xi})+\Big((a_4y+b_4z)\widetilde{F}'(\widetilde{\xi})+\widetilde{G}_3(\widetilde{\xi})\Big)I_4$$

$$+\Big((a_5y+b_5z)\widetilde{F}'(\widetilde{\xi})+\widetilde{G}_5(\widetilde{\xi})\Big)I_5\quad\forall\,\widetilde{\zeta}\in\widetilde{\Pi}_{\widetilde{\zeta}},\ \widetilde{\xi}=f(\widetilde{\zeta}), \tag{5.22}$$

where $\widetilde{F},\widetilde{G}_3,\widetilde{G}_5$ are some holomorphic functions in the domain $D\subset\mathbb{C}$. The domain $\widetilde{\Pi}_{\widetilde{\zeta}}$ is an infinite cylinder.

For triple (4.19) of the algebra $\mathbb{A}_4$, we have

$$\widetilde{\zeta}(1)=x+y\widetilde{e}_2(1)+z\widetilde{e}_3(1)=x+a_1y+b_1z+a_5xI_5+b_5yI_5\,.$$

It is obvious that $f(\widetilde{\zeta}(1))=x+a_1y+b_1z=\xi_1\equiv\widetilde{\xi}$, $D_1\equiv D$. We set $\widetilde{F}\equiv F_1$, $\widetilde{G}_3\equiv G_3$, $\widetilde{G}_5\equiv G_5$ in D. Then equality (5.22) can be written in the form

$$\widetilde{\Phi}_1(\widetilde{\zeta}(1))=F_1(\xi_1)+\Big((a_4y+b_4z)F_1'(\xi_1)+G_3(\xi_1)\Big)I_4$$

$$+\Big((a_5y+b_5z)F_1'(\xi_1)+G_5(\xi_1)\Big)I_5\,. \tag{5.23}$$

Multiplying equality (5.23) by I_1, we verify the validity of the equality

$$I_1\,\widetilde{\Phi}_1\big(\widetilde{\zeta}(1)\big)=\Phi_1(\zeta)\,.$$

For triple (4.20) of the algebra $\mathbb{A}_4$, we have

$$\widetilde{\zeta}(2)=x+y\widetilde{e}_2(2)+z\widetilde{e}_3(2)=x+a_2y+b_2z.$$

It is obvious that $f(\widetilde{\zeta}(2)) = \widetilde{\zeta}(2) = x + a_2 y + b_2 z = \xi_2 \equiv \widetilde{\xi}$, $D_2 \equiv D$. Set $\widetilde{F} \equiv F_2$, $\widetilde{G}_3 \equiv G_3$, and $\widetilde{G}_5 \equiv G_5$ in D. Then equality (5.22) takes the form

$$\widetilde{\Phi}_2(\widetilde{\zeta}(2)) = F_2(\xi_2) + \Big((a_4 y + b_4 z)F_2'(\xi_2) + G_3(\xi_2)\Big)I_4 +$$

$$+\Big((a_5 y + b_5 z)F_2'(\xi_2) + G_5(\xi_2)\Big)I_5\,. \tag{5.24}$$

Multiplying equality (5.24) by I_2, we verify the validity of the equality

$$I_2\,\widetilde{\Phi}_2\big(\widetilde{\zeta}(2)\big) = \Phi_2(\zeta)\,.$$

Eventually, for triple (4.21), we get

$$\widetilde{\zeta}(3) = x + y\widetilde{e}_2(3) + z\widetilde{e}_3(3) = x + a_3 y + b_3 z + a_4 x I_4 + b_4 y I_4\,.$$

It is obvious that $f(\widetilde{\zeta}(3)) = x + a_3 y + b_3 z = \xi_3 \equiv \widetilde{\xi}$, $D_3 \equiv D$. Set $\widetilde{F} \equiv F_3$, $\widetilde{G}_3 \equiv G_3$, and $\widetilde{G}_5 \equiv G_5$ in D. Then equality (5.22) can be rewritten in the form

$$\widetilde{\Phi}_3(\widetilde{\zeta}(3)) = F_3(\xi_3) + \Big((a_4 y + b_4 z)F_3'(\xi_3) + G_3(\xi_3)\Big)I_4$$

$$+\Big((a_5 y + b_5 z)F_3'(\xi_3) + G_5(\xi_3)\Big)I_5\,. \tag{5.25}$$

Multiplying equality (5.25) by I_3, we verify the validity of the equality

$$I_3\,\widetilde{\Phi}_3\big(\widetilde{\zeta}(3)\big) = \Phi_3(\zeta)\,.$$

Thus, equality (5.12) is true:

$$\Phi(\zeta) = I_1\,\widetilde{\Phi}_1\big(\widetilde{\zeta}(1)\big) + I_2\,\widetilde{\Phi}_2\big(\widetilde{\zeta}(2)\big) + I_3\,\widetilde{\Phi}_3\big(\widetilde{\zeta}(3)\big),$$

where Φ takes values in the algebra $\mathbb{A}_5^3$, and $\widetilde{\Phi}_1\big(\widetilde{\zeta}(1)\big)$, $\widetilde{\Phi}_2\big(\widetilde{\zeta}(2)\big)$, and $\widetilde{\Phi}_3\big(\widetilde{\zeta}(3)\big)$ take values in $\mathbb{A}_4$.

REFERENCES

1. P. W. Ketchum, "Analytic functions of hypercomplex variables," *Trans. Amer. Math. Soc.*, **30**(4), 641–667 (1928).
2. M. N. Roşculeţ, "Algebre infinite asociate la ecuaţii cu derivate parţiale, omogene, cu coeficienţi constanţi de ordin oarecare," *Studii şi Cercetări Matem.*, **6**(3–4), 567–643 (1955).
3. M. N. Roşculeţ, "Algebre infinite, comutative, asociate la sisteme de ecuaţii cu derivate parţiale," *Studii şi Cercetări Matem.*, **7**(3–4), 321–371 (1956).
4. A. Pogorui, R. M. Rodriguez-Dagnino, and M. Shapiro, "Solutions for PDEs with constant coefficients and derivability of functions ranged in commutative algebras," *Math. Meth. Appl. Sci.*, **37**(17), 2799–2810 (2014).
5. I. P. Mel'nichenko, "The representation of harmonic mappings by monogenic functions," *Ukr. Math. J.*, **27**(5), 499–505 (1975).
6. I. P. Mel'nichenko and S. A. Plaksa, *Commutative Algebras and Spatial Potential Fields* [in Russian], Institute of Mathematics of the NAS of Ukraine, 2008.

7. S. A. Plaksa and V. S. Shpakovskii, “Constructive description of monogenic functions in a harmonic algebra of the third rank,” *Ukr. Math. J.*, **62**(8), 1251–1266 (2011).
8. S. A. Plaksa, R. P. Pukhtaevich, “Constructive description of monogenic functions in a three-dimensional harmonic algebra with one-dimensional radical,” *Ukr. Math. J.*, **65**(5) (2013), 740–751.
9. R. P. Pukhtaievych, “Monogenic functions in a three-dimensional harmonic semisimple algebra,” *Zb. Pr. Inst. Mat. NAN Ukr.*, **10**(4–5), 352–361 (2013).
10. S. A. Plaksa and V. S. Shpakivskyi, “Monogenic functions in a finite-dimensional algebra with unit and radical of maximal dimensionality,” *J. Alger. Math. Soc.*, **1**, 1–13 (2014).
11. S. A. Plaksa and R. P. Pukhtaievych, “Constructive description of monogenic functions in n-dimensional semisimple algebra,” *An. Şt. Univ. Ovidius Constanţa*, **22**(1), 221–235 (2014).
12. E. Cartan, “Les groupes bilinéares et les systèmes de nombres complexes,” *Ann. de la Faculté des Sci. de Toulouse*, **12**(1), 1–64 (1898).
13. E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Mat. Soc., Providence, RI, 1957.
14. V. S. Shpakivskyi, “Constructive description of monogenic functions in a finite-dimensional commutative associative algebra,” *Adv. Pure Appl. Math.*, **7**(1), 63–75 (2016).
15. S. V. Grishchuk and S. A. Plaksa, “Monogenic functions in a biharmonic algebra,” *Ukr. Math. J.*, **61**(12), 1865–1876 (2009).
16. G. Mazzola, “Generic finite schemes and Hochschiid cocycles,” *Comment. Math. Helvetici*, **55**, 267–293 (1980).
17. D. Burde and W. de Graaf, “Classification of Novicov algebras,” *Applic. Alg. Engineer., Commun. Comput.*, **24**(1), 1–15 (2013)
18. A. S. Hegazim and H. Abdelwahab, “Classification of five-dimensional nilpotent Jordan algebras,” *Linear Algebra Its Applic.*, **494**, 165–218 (2016).
19. B. Poonen, “Isomorphism types of commutative algebras of finite rank over an algebraically closed field,” *Contemp. Math.*, **463**, 111–120 (2008).
20. D. A. Suprunenko, “On maximum commutative subalgebras of the full linear algebra,” *Uspekhi Mat. Nauk*, **11**(3) (1956), 181–184 (1956).
21. V. S. Shpakivskyi, “Monogenic functions in finite-dimensional commutative associative algebras,” *Zb. Pr. Inst. Mat. NAN Ukr.*, **12**(3), 251–268 (2015).

Translated from Ukrainian by V.V. Kukhtin

Vitalii Stanislavovych Shpakivskyi
Institute of Mathematics of the NAS of Ukraine, Kyiv, Ukraine
E-Mail: shpakivskyi86@gmail.com