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LEONTOVICH–FOCK PARABOLIC EQUATION METHOD IN THE NEUMANN DIFFRACTION PROBLEM ON A PROLATE BODY OF REVOLUTION

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This paper continues a series of publications on the shortwave diffraction of the plane wave on prolate bodies of revolution with axial symmetry in the Neumann problem. The approach, which is based on the Leontovich–Fock parabolic equation method for the two parameter asymptotic expansion of the solution, is briefly described. Two correction terms are found for the Fock's main integral term of the solution expansion in the boundary layer. This solution can be continuously transformed into the ray solution in the illuminated zone and decays exponentially in the shadow zone. If the observation point is in the shadow zone near the scatterer, then the wave field can be obtained with the help of residue theory for the integrals of the reflected field, because the incident field does not reach the shadow zone. The obtained residues are necessary for the unique construction of the creeping waves in the boundary layer of the scatterer in the shadow zone. Bibliography: 16 *titles.*

We consider a shortwave diffraction of a plane incident wave on the strictly convex, prolate body of revolution. The geometric characteristics of the scatterer (i.e., radii of curvatures of the surface of body of revolution) are assumed to be much larger than the incident wavelength. The incident wave propagates along the axis of revolution. The total wave field U is the sum of the incident $U^{\mathtt{inc}}$ and reflected $U^{\mathtt{ref}}$ waves, $U = U^{\mathtt{inc}} + U^{\mathtt{ref}}$. The field is constructed in the vicinity of the light-shadow border (i.e., in the penumbra of Fock's region, [1]), which is the "seed" zone for fields both in the vicinity of the limit rays and in the shadowed part of the body. The shortwave field in the illuminated area near the scatterer is described by means of the ray method. The field U satisfies the Helmholtz equation with Neumann or Dirichlet boundary conditions. Fock's boundary layer $\mathcal{O}(sk^{\frac{1}{3}}) = \mathcal{O}(1)$, $\mathcal{O}(nk^{\frac{2}{3}}) = \mathcal{O}(1)$ is introduced in a neighborhood of point $s = 0$, which belongs to the geometric border (Equator) of the shadow; here k is the wavenumber, n is the distance along the outer normal on the scatterer, and s is the arclength of the geodesic. The ray method does not work in the vicinity of the light-shadow border, i.e., in the Fock's boundary layer. The total wave field in the Fock's zone can be represented as $U = e^{iks}(W^{\mathtt{inc}} + W^{\mathtt{ref}})$, where e^{iks} is the oscillating factor of the wave field along the geodesic; the function W is called the attenuation function. Introducing dimensionless coordinates σ, ν instead of s and n, and rewriting e^{ikz} in the new coordinates σ, ν, we obtain the first three terms of the expansion $W^{\mathtt{inc}}$ in the form

$$W(\sigma,\nu) = W_0^{\mathtt{inc}} + \frac{W_1^{\mathtt{inc}}}{k^{\frac{1}{3}}} + \frac{W_2^{\mathtt{inc}}}{k^{\frac{2}{3}}} + \mathcal{O}(k^{-1}), \;\; k \gg 1,$$

here $W_0^{\mathtt{inc}}$ is the main, $W_1^{\mathtt{inc}}$ is the first, and $W_2^{\mathtt{inc}}$ is the second terms of the asymptotic expansion. The functions $W_i^{\mathtt{inc}}$, $i = 0, 1, 2$, have the form of integrals of linear combinations of the Airy function $v(t)$ and its derivative $v'(t)$ with polynomials in the dimensionless normal coordinate ν in Fock's region. We apply the Leontovich–Fock parabolic equation method [1,4]

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to the function under investigation

$$W^{\mathtt{ref}}(\sigma,\nu) = W_0^{\mathtt{ref}} + \frac{W_1^{\mathtt{ref}}}{k^{\frac{1}{3}}} + \frac{W_2^{\mathtt{ref}}}{k^{\frac{2}{3}}} + \mathcal{O}(k^{-1}), \ \ k \gg 1.$$

The functions $W_i^{\mathtt{ref}}, i = 0,1,2$ satisfy the system of recurrence equations (1)

$$\mathcal{L}_0 W_0^{\mathtt{ref}} = 0, \quad \mathcal{L}_0 W_1^{\mathtt{ref}} + \mathcal{L}_1 W_0^{\mathtt{ref}} = 0, \tag{1}$$
$$\mathcal{L}_0 W_2^{\mathtt{ref}} + \mathcal{L}_1 W_1^{\mathtt{ref}} + \mathcal{L}_2 W_0^{\mathtt{ref}} = 0, \quad \text{and so on.}$$

The attenuation functions $W_i^{\mathtt{ref}}$, $i = 0,1,2$, are also integrals of the Airy functions $w_1(t)$ and $w_1'(t)$ (in Fock's definition) and polynomials of ν.

These formulas give a continuous transition from the field in the illuminated area (ray expansion) to the full shadow, where the solution decays exponentially as $\sigma > 0$. Moreover, the integrals $W_i^{\mathtt{ref}}$, $i = 0,1,2$, can be represented as a sum of residues of the roots of the denominators of the integrand in the shadow. The latter allows one to generate creeping waves from the initial data taken from the asymptotic expansion. Owing to the prolate character of the body, the constructed two-parameters asymptotic expansion in the Fock's region allows one to obtain an approximate expressions for the wave field, depending on the large parameters of the problem. Thus the system of recurrence differential equations (1) keeps its asymptotic nature, provided that the prolongation parameter is equal to

$$\mathbf{\Lambda}_0 = 2\boldsymbol{M}_0^{2-\varepsilon}, \ \ 0 < \varepsilon < 2, \ \ \mathbf{\Lambda}_0 = \frac{\rho_0}{f(0)}, \ \ \boldsymbol{M}_0 = \left(\frac{k\rho_0}{2}\right)^{\frac{1}{3}}. \tag{2}$$

Here the prolongation parameter $\mathbf{\Lambda}_0$ equals the ratio of curvature radii along the geodesics and along the Equator, and $\boldsymbol{M}_0$ is the Fock's parameter.

1. Statement of the problem and geometry

We follow the statement of the problem from [2, 3], however the solution was constructed based on a set of completely different methods [1, 4, 5], which allowed one to observe the influence of both curvature radii of the body of revolution on the wave field.

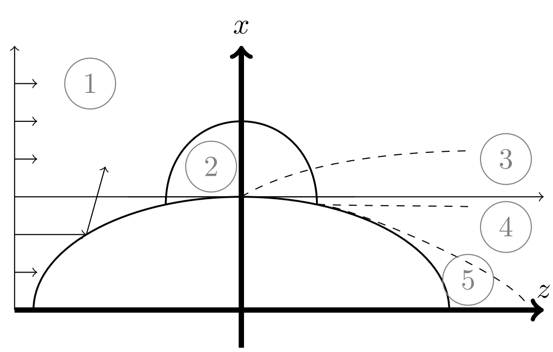

Fig. 1. The figure presents five main areas of irradiation in the plane wave diffraction on the body of revolution Ω: illuminated region 1, region 2 is a neighborhood of the point of contact of the limit ray (Fock's region), region 3 is a penumbra zone in the vicinity of the limit ray, region 4 is the shadow region, region 5 is the surface layer in the shadow region, where the creeping waves are formed.

Assume that the surface $\partial\Omega$ of the scatterer Ω is generated by rotating a flat strictly convex curve $x = f(z)$ about the z axis. The cross-section of $\partial\Omega$ by the plane $z = 0$ is called the

Equator and coincides with the light-shadow border, which appears as a result of the incidence of the plane wave $U^{\mathtt{inc}} = \exp(ikz)$ onto Ω. Here $k \gg 1$ is a wavenumber, and $k = \frac{2\pi}{\lambda}$, where λ is a wave length (see Fig. 1).

The wave field U satisfies the Helmholtz equation with the Neumann boundary condition on the surface $\partial\Omega$, i.e.,

$$(\Delta + k^2)U = 0, \quad \frac{\partial}{\partial n}U|_{n=0} = 0, \tag{3}$$

also U satisfies the limiting absorption principle, which suggests that for small $\operatorname{Im} k > 0, U \to 0$ for large distances from the penumbra region. The Dirichlet boundary condition has been considered in papers [4,5], and we present those results in this paper for convenience of comparing the Dirichlet and the Neuman wave field.

Together with the Cartesian coordinates $\{x, y, z\}$ and the polar coordinates $r = \sqrt{x^2 + y^2}$, $x = r\cos\varphi$, $y = r\sin\varphi$, $0 \le \varphi \le 2\pi$, we introduce new curvilinear coordinates $\{s, n, \varphi\}$ in the vicinity of the Equator, where s is the arclength along the geodesics (meridians), measured from the Equator (i.e., $s = 0$), so that, $s < 0$ corresponds to the illuminated area of $\partial\Omega$, while $s > 0$ corresponds to the shadowed part of the surface. The relationship between s and z is clear from the formula

$$s = \int_0^z \sqrt{1 + (f'(z))^2}\, dz. \tag{4}$$

The radius vector of a point M outside Ω in the polar coordinates (r, φ, z) has the form

$$\mathbf{R}(M) = r[\cos\varphi \mathbf{e}_x + \sin\varphi \mathbf{e}_y] + z\mathbf{e}_z.$$

If a point belongs to $\partial\Omega$, then $r = f(z)$, and the unit outward normal to $\partial\Omega$ is

$$\mathbf{n} = \frac{\nabla[r - f(z) = 0]}{|\nabla[r - f(z) = 0]|} = \frac{(\cos\varphi \mathbf{e}_x + \sin\varphi \mathbf{e}_y) - f'_z(z)\mathbf{e}_z}{\sqrt{1 + (f'_z)^2}}.$$

The radius vector of a point on the surface $\partial\Omega$ is equal to

$$\mathbf{R}(s) = f(z(s))[\cos\varphi\, \mathbf{e}_x + \sin\varphi\, \mathbf{e}_y] + z(s)\mathbf{e}_z = f(z(s))\mathbf{e}_r + z(s)\mathbf{e}_z,$$

where the unit normal vector is $\mathbf{e}_r = (\cos\varphi\, \mathbf{e}_x + \sin\varphi\, \mathbf{e}_y)$. Morevover, the tangent vectors along the meridians are orthogonal to tangent vectors along parallels on the surface of the scatterer. In $\{s, n, \varphi\}$ coordinate system, the radius vector takes the form

$$\mathbf{R}(M) = \mathbf{R}(s) + n\mathbf{n} = \mathbf{R}(s) + n\frac{\mathbf{e}_r - f'_z\mathbf{e}_z}{\sqrt{1 + (f'_z)^2}}.$$

In order to find the Lamé coefficients, we use the total differential $d\mathbf{R}(M)$:

$$d\mathbf{R}(M) = d\mathbf{R}(s) + d(n\mathbf{n}) = \frac{\partial \mathbf{R}(s)}{\partial z}dz + \frac{\partial \mathbf{R}(s)}{\partial \varphi}d\varphi + dn\mathbf{n} + nd\mathbf{n}.$$

Thus,

$$(d\mathbf{R}(M), d\mathbf{R}(M)) = \left(1 + \frac{n}{\rho(s)}\right)^2 ds^2 + dn^2 + \left(f + \frac{n}{\sqrt{1 + (f'_z)^2}}\right)^2 d\varphi^2.$$

The function $\rho(s)$ is the curvature radius of a meridian at a point s. The square of a linear distance element, which is equal to

$$dS^2 = h_s^2 ds^2 + dn^2 + h_\varphi^2 d\varphi^2,$$

defines the Lamé coefficients

$$h_s = 1 - n\frac{f''(z(s))}{[1+(f'(z(s)))^2]^{3/2}} = 1 + \frac{n}{\rho(s)}, \quad h_n = 1,$$
$$h_\varphi = f(z(s)) + \frac{n}{\sqrt{1+(f'(z(s)))^2}}, \quad \frac{1}{\rho(s)} = \frac{-f''(z(s))}{[1+(f'(z(s)))^2]^{3/2}}. \tag{5}$$

We note that z in formula (4) is taken to be a function of s, i.e., $z = z(s)$ can be obtained as an inverse of (4). Hence, finding the inverse of (4) we get

$$z(s) = s - \frac{s^3}{3!\rho_0^2} + \frac{3\rho_0'}{4!\rho_0^3}s^4 + \left[\frac{-11(\rho_0')^2 + 4\rho_0\rho_0'' + 1}{5!\rho_0^4}\right]s^5 + \mathcal{O}(s^6).$$

Here $\rho_0 = \rho(s)|_{s=0}$ is the radius of curvature of the geodesics (meridians) on the light-shadow border. Multiplying the radius vector in the (s, n, φ) coordinate system by $\mathbf{e}_z$, we obtain the expansion of z with respect to s and n :

$$z = z(s) + n(\mathbf{n}, \mathbf{e}_z) = z(s) + n\frac{-f'(z(s))}{\sqrt{1+[f'(z(s))]^2}}.$$

Taking into account the expansion of $f(z(s))$ with respect to s, we have

$$z = s + \left(-\frac{s^3}{3!\rho_0^2} + \frac{ns}{\rho_0}\right) + \left(\frac{3\rho_0' s^4}{4!\rho_0^3} - \frac{ns^2\rho_0'}{2!\rho_0^2}\right) + \left(\frac{\alpha s^5}{5!\rho_0^4} - \frac{\beta n s^3}{3!\rho_0^3}\right) + \mathcal{O}(s^6, ns^4). \tag{6}$$

Next from

$$\frac{1}{\rho(s)} = \frac{-f''(z(s))}{[1+(f'(z(s)))^2]^{3/2}} \quad \text{and} \quad \frac{dz}{ds} = \frac{1}{\sqrt{1+(f'(z(s)))^2}},$$

we get

$$f_z'(0) = 0, \quad f_z^{(2)} = f_{zz}''(0) = -\frac{1}{\rho_0}, \quad f_z^{(3)} = f_{zzz}'''(0) = -\frac{d}{ds}\frac{1}{\rho}\Big|_{s=0}, \ \dots .$$

Throughout this text all the expressions that describe the Fock's region, contain functions of s, evaluated at $s = 0$. The coefficients α, β in (6) have the form

$$\alpha = 4\rho_0\rho_0'' - 11(\rho_0')^2 + 1, \quad \beta = \rho_0\rho_0'' - 2(\rho_0')^2 + 1. \tag{7}$$

We note that we consider an axially symmetric case for the sake of simplicity, similarly to papers [2, 3], i.e., we are interested in a solution that does not depend on the angle φ, which means that $\frac{\partial U}{\partial \varphi} = 0$. Therefore, the solution in the three-dimensional case can be constructed in any cross-section over φ, say, $\varphi = 0$.

In the vicinity of the points on $\partial\Omega$ that belong to the light-shadow border, i.e., to the region 2 (see Fig. 1), we introduce scaling factors s and n as in the Leontovich–Fock parabolic-equation method: $sk^{\frac{1}{3}} = \mathcal{O}(1)$, $nk^{\frac{2}{3}} = \mathcal{O}(1)$, $k \gg 1$, where $\mathcal{O}(1)$ means that the right-hand sides of the equalities are proportional to constants. These constants are parts of the formulas for the dimensionless variables $\sigma = \mathcal{O}(1)$ and $\nu = \mathcal{O}(1)$,

$$\sigma = \frac{k^{1/3}s}{2^{1/3}\rho_0^{2/3}} = \frac{\boldsymbol{M}_0}{\rho_0}s, \quad \nu = \frac{2^{1/3}k^{2/3}n}{\rho_0^{1/3}} = \frac{2\boldsymbol{M}_0^2}{\rho_0}n. \tag{8}$$

The width of the penumbra region $|s| \le \frac{\rho_0}{\boldsymbol{M}_0}|\sigma|$ is of order $\mathcal{O}(\frac{1}{\boldsymbol{M}_0})$, here $\boldsymbol{M}_0$ is Fock's dimensionless parameter (2), $\boldsymbol{M}_0 \gg 1$.

In the shortwave diffraction of a plane incident wave from a strongly prolongated scatterer, the field expansion inherits not only the properties of the large Fock's parameter $\boldsymbol{M}_0$, but also the properties of the second parameter of body enlongation $\boldsymbol{\Lambda}_0$ (2), see [4, 5].

The solution in the surface layer 2 was constructed by the two-scale expansion; the parameter $\mathbf{\Lambda}_0$ appeared in the system of recurrence equations for the boundary layer in the ratio $\frac{\mathbf{\Lambda}_0}{2\boldsymbol{M}_0^2}$ and its positive exponents. The asymptotic nature of the expansion is not broken if $\frac{\mathbf{\Lambda}_0}{2\boldsymbol{M}_0^2}$ is of order $\mathcal{O}(\boldsymbol{M}_0^{-\varepsilon})$ for $0 < \varepsilon < 2$. If $\varepsilon = 2$, then $\mathbf{\Lambda}_0 = 2\boldsymbol{M}_0^{2-\varepsilon} = \mathcal{O}(1)$ and both curvature radii do not differ (which is not the case of a prolate body).

When $\varepsilon = 0$, the parameter $\mathbf{\Lambda}_0$ compensates Fock's parameter and the system of recurrence equations in the boundary layer looses its asymptotic nature and all the equations in the system get some singularity in their coefficients, see [4, 6].

In this research we have found the wave field near surface $\partial\Omega$ in the Fock's region 2, see Fig. 1. The solution in the illuminated region 1 was found by the ray method [1, 4, 7]. The region of the limit ray 3 was investigated in the monography by V. A. Fock [1] and in book [8]; Fridlander-Keller diffracted rays are developed in the shadowed region 4 far from the border. The asymptotics of the solution in region 2 generates initial data for creeping waves in the boundary layer 5. Regions 4 and 5 will be considered separately.

2. Incident field in Fock's region

2.1. Field $U^{\texttt{inc}}$ expansion with respect to σ and ν. Consider the incident wave $U^{\texttt{inc}} = \exp(ikz)$, which satisfies (3) with boundary conditions of either Dirichlet, or Neumann type on the surface of the scatterer $\partial\Omega$. We need an expansion of the incident wave in s and n coordinates in Fock's region in order to construct the reflected wave $U^{\texttt{ref}}$. The incident wave expansion (8) has the following form in the dimensionless coordinates (σ, ν):

$$U^{\texttt{inc}} = e^{ikz} = e^{\left[iks + i(\nu\sigma - \frac{\sigma^3}{3})\right]} \left(1 + i\left(\frac{2}{k\rho_0}\right)^{\frac{1}{3}} \left[\frac{\rho_0'}{4}(\sigma^4 - 2\nu\sigma^2)\right] \right.$$
$$\left. + \left(\frac{2}{k\rho_0}\right)^{\frac{2}{3}} \left[\frac{2i\alpha\sigma^5}{5!} - \frac{i\beta\nu\sigma^3}{3!} - \frac{(\rho_0')^2}{2\cdot 4^2}(\sigma^8 - 4\nu\sigma^6 + 4\nu^2\sigma^4)\right] + \mathcal{O}\Big(\frac{1}{k}\Big)\right). \tag{9}$$

Here expression (6) with respect to s, n was rewritten in the new stretched coordinates σ, ν. One should notice that the terms in the square brackets is of the same order with respect to the wavenumber k, and the order of each term is $k^{-1/3}$. The latter implies the following expression for the attenuation function of the incident wave $W^{\texttt{inc}}$:

$$U^{\texttt{inc}} = e^{iks} W^{\texttt{inc}} = e^{iks} \sum_{m=0} W_m^{\texttt{inc}} k^{-\frac{m}{3}} = e^{iks} W_0^{\texttt{inc}} \sum_{m=0} k^{-\frac{m}{3}} P_m^{\texttt{inc}}, \tag{10}$$

where the $P_m^{\texttt{inc}}$ are polynomials with respect to σ and ν, and the attenuation coefficient $W_0^{\texttt{inc}} = e^{i(\nu\sigma - \sigma^3/3)}$ is outside of the sum. This paper presents first three terms of expansion (10) for the incident and reflected waves. We omit cumbersome calculations in order not to overload the paper.

2.2. Three terms of the expansion $W^{\texttt{inc}}$ with respect to σ, ν. We start by finding three terms of expansion (10) of the incident field $e^{ikz} = U^{\texttt{inc}} = e^{iks} W^{\texttt{inc}}$:

$$\begin{aligned} W^{\texttt{inc}} &= W_0^{\texttt{inc}} + k^{-\frac{1}{3}} W_1^{\texttt{inc}} + k^{-\frac{2}{3}} W_2^{\texttt{inc}} + \ldots \\ &= W_0^{\texttt{inc}} \left[P_0^{\texttt{inc}} + k^{-\frac{1}{3}} P_1^{\texttt{inc}} + k^{-\frac{2}{3}} P_2^{\texttt{inc}} + \mathcal{O}(k^{-1})\right], \end{aligned}$$

where $W_0^{\texttt{inc}} = e^{i(\nu\sigma - \frac{\sigma^3}{3})}$ is the attenuation function (9). Comparing the obtained result with (9), it is clear that the amplitude of the incident wave equals one, i.e., $P_0^{\texttt{inc}} = 1$. Hence the main term of expansion (10) has the form

$$W_0^{\texttt{inc}}(\sigma,\nu) = \exp\left\{i\left(\sigma\nu - \frac{\sigma^3}{3}\right)\right\} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{i\sigma\zeta} v(\zeta-\nu)d\zeta, \tag{11}$$

where $v(\zeta-\nu)$ is the real-valued Airy function in the form defined by V. A. Fock [1]. The representation for the first term

$$W_1^{\texttt{inc}}(\sigma,\nu) = e^{i(\nu\sigma - \frac{\sigma^3}{3})}\left[i\left(\frac{2}{\rho_0}\right)^{\frac{1}{3}}\frac{\rho_0'}{4}(\sigma^4 - 2\nu\sigma^2)\right]$$

implies that $P_1^{\texttt{inc}}$ takes the form

$$P_1^{\texttt{inc}}(\sigma,\nu) = i\left(\frac{2}{\rho_0}\right)^{\frac{1}{3}}\frac{\rho_0'}{4}(\sigma^4 - 2\nu\sigma^2).$$

One of the most important technical moments is that in the Fock's region, the scaled coordinates σ and ν turn the region where the reflected wave is found, into the half-plane $\{-\infty < \sigma < \infty, \nu > 0\}$. Thus it is convenient to single out the coordinate σ in the Fourier transform, see (11). Such a transform can be applied owing to the following fundamental equality (its full mathematical justification can be found in [8]):

$$(-i\sigma)^m \exp\left\{i\left(\sigma\nu - \frac{\sigma^3}{3}\right)\right\} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{i\sigma\zeta}\frac{d^m}{d\zeta^m}v(\zeta-\nu)\,d\zeta. \tag{12}$$

Airy functions satisfy the Airy equation $v''(t) = tv(t)$, which can transform relation (12) in such a way that

$$(-i\sigma)^m \exp\left\{i\left(\sigma\nu - \frac{\sigma^3}{3}\right)\right\} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{i\sigma\zeta}[P_m(\zeta,\nu)v(\zeta-\nu) + Q_m(\zeta,\nu)v'(\zeta-\nu)]d\zeta, \tag{13}$$

where P_m and Q_m (polynomials with respect to ν with coefficients dependent on ζ) get more complicated as m grows.

Therefore we obtain a formula for the next term of the attenuation function $W_1^{\texttt{inc}}(\sigma,\nu)$:

$$W_1^{\texttt{inc}}(\sigma,\nu) = \frac{1}{\sqrt{\pi}}i\rho_0'\left(\frac{2}{\rho_0}\right)^{\frac{1}{3}}\int_{-\infty}^{\infty} e^{i\sigma\zeta}\left[\frac{v'(\zeta-\nu)}{2} + \frac{\zeta^2-\nu^2}{4}v(\zeta-\nu)\right]d\zeta, \tag{14}$$

and thus the polynomials take the forms

$$P_1^{\texttt{inc}}(\zeta,\nu) = \frac{\zeta^2-\nu^2}{4},\ Q_1^{\texttt{inc}}(\zeta,\nu) = \frac{1}{2}.$$

The polynomial $P_2^{\texttt{inc}}$ looks more complicated, since it includes σ^8 and ν^4, however it can be explicitely written from (6) in the expansion for $\exp(ikz)$. The representation

$$W_2^{\texttt{inc}}(\sigma,\nu) = e^{i(\nu\sigma - \frac{\sigma^3}{3})}\left[\left(\frac{2}{\rho_0}\right)^{\frac{2}{3}}\left(\frac{2\alpha\sigma^5 i}{5!} - \frac{\beta i\nu\sigma^3}{3!} - \frac{(\rho_0')^2}{2\cdot 4^2}(\sigma^4 - 2\nu\sigma^2)^2\right)\right] + \dots$$

implies that

$$P_2^{\texttt{inc}}(\sigma,\nu) = \left(\frac{2}{\rho_0}\right)^{\frac{2}{3}}\left(\frac{2\alpha\sigma^5 i}{5!} - \frac{\beta i\nu\sigma^3}{3!} - \frac{(\rho_0')^2}{2\cdot 4^2}(\sigma^4 - 2\nu\sigma^2)^2\right).$$

The second term $W_2^{\texttt{inc}}$ (and further terms as well) of the incident field expansion has form (13) in the Fock's region. The term for $m = 2$ (we are interested in it) in expansion (10) has the form

$$W_2^{\texttt{inc}}(\sigma,\nu) = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\rho_0}\right)^{\frac{2}{3}} \int\limits_{-\infty}^{\infty} e^{i\sigma\zeta}\left[P_2^{\texttt{inc}} v(\zeta-\nu) + Q_2^{\texttt{inc}} v'(\zeta-\nu)\right] d\zeta,$$

whereas the polynomials $P_2^{\texttt{inc}}(\zeta,\nu)$ and $Q_2^{\texttt{inc}}(\zeta,\nu)$ are equal to

$$P_2^{\texttt{inc}}(\zeta,\nu) = -\frac{8\alpha}{5!}(\zeta-\nu) - \frac{\beta}{3!}\nu - \frac{{\rho_0'}^2}{8}[4\zeta + 3(\zeta-\nu) + \frac{1}{4}(\zeta-\nu)^4 + \nu(\zeta-\nu)^3 + \nu^2(\zeta-\nu)^2],$$

$$Q_2^{\texttt{inc}}(\zeta,\nu) = -\frac{2\alpha}{5!}(\zeta-\nu)^2 - \frac{\beta}{3!}\nu(\zeta-\nu) - \frac{{\rho_0'}^2}{4}\left[\frac{3}{2}(\zeta-\nu)^2 + 3\nu(\zeta-\nu) + \nu^2\right],$$

where, as before, α, β are defined by (7). The formulas presented above for $W_m^{\texttt{inc}}(\sigma,\nu)$, $m = 0, 1, 2$, give an idea of the analytic structure of the reflected field.

3. The reflected wave in Fock's region

3.1. Axially symmetric solution. Among the solutions of the Helmholtz equation (3), we are interested only in the axially symmetric ones (because of the symmetry of the scatterer), i.e., a solution that satisfies the condition $\frac{\partial U}{\partial \varphi} = 0$. We seek the solution in the form $U = \exp(iks)W(s,n)$, where $\exp(iks)$ describes the main oscillations of the field, and W is an attenuation function (it slowly changes the amplitude, i.e., the relative change in amplitude is small compared to the wavenumber k). Upon separating the factor e^{iks}, the Helmholtz equation can be rewritten for the attenuation functions

$$[(\Delta + k^2)e^{iks}W] = e^{iks}g^{-1}(\mathcal{A}W + \mathcal{B}W) = 0,$$

where $g = (4\boldsymbol{M}_0^4)/\rho_0^2$, and the operators $\mathcal{A}$ and $\mathcal{B}$ are equal to

$$\mathcal{A}W = k^2(1 - h_s^{-2})W + \left(2ik\frac{\partial W}{\partial s} + \frac{\partial^2 W}{\partial s^2}\right)h_s^{-2} + \frac{\partial^2 W}{\partial n^2}$$

$$-\left(ikW + \frac{\partial W}{\partial s}\right)h_s^{-2}\frac{\partial \ln h_s}{\partial s} + \frac{\partial W}{\partial n}\frac{\partial \ln h_s}{\partial n},$$

$$\mathcal{B}W = \frac{1}{h_\varphi}\left[\frac{df}{ds}\frac{1}{h_s}\left(ikW + \frac{\partial W}{\partial s}\right) + \frac{\partial h_\varphi}{\partial n}\frac{\partial W}{\partial n}\right], \tag{15}$$

$$\mathcal{A}W + \mathcal{B}W = g\sum_{m=0} k^{-\frac{m}{3}}\mathcal{L}_m.$$

We note that the operator $\mathcal{A}$ corresponds to the two-dimensional diffraction problem on a convex body in the plane $\varphi = \text{const}$, whereas $\mathcal{B}$ is responsible for the three-dimensional properties of the solution. Therefore, we are mostly interested in the operator $\mathcal{B}$. Regarding the two-dimensional problem, the detailed discussion of the asymptotic expansion construction can be found in [1]; the mathematically rigorous construction scheme can be found in [8–10] and [11].

3.2. Reflected field series expansion. The mathematical justification of the results presented in this section can be found in the previous papers on shortwave diffraction on a strictly convex body (see [10, 12] and [13]). The reflected wave $U^{\mathtt{ref}} = e^{iks}W^{\mathtt{ref}}$ is constructed in the form of an asymptotic series with respect to the powers of $k^{-m/3}$, $m = 0, 1, 2, \dots$, similarly to series (10):

$$W^{\mathtt{ref}} = \sum_{m=0} W_m^{\mathtt{ref}} k^{-\frac{m}{3}}. \tag{16}$$

We introduce a system of recurrence equations for $W_m^{\mathtt{ref}}$, and then write down the coefficients of the expansion.

3.3. System of equations for the reflected field. Expanding the coefficients of the initial Helmholtz equation (3) with respect to the powers of s, n, we pass to the stretched coordinates σ, ν, as in (8), and eventually obtain

$$(\Delta + k^2)e^{iks}W^{\mathtt{ref}} = e^{iks}g^{-1}g \sum_{m=0} k^{-\frac{m}{3}} \mathcal{L}_m W^{\mathtt{ref}} = 0, \tag{17}$$

where $g = (4\boldsymbol{M}_0^4)/(\rho_0^2)$. The differential operators $\mathcal{L}_m$, $m = 0, 1, 2, \dots$, (of the first and second order) with respect to σ and ν, contain polynomials in σ and ν as their factors. When m increases, the explicit formulas for $\mathcal{L}_m$ become cumbersome. We present them here only for the cases of $m = 0, 1, 2$.

Substituting the series for $W^{\mathtt{ref}}$ (16) into (17), and equating the coefficients at equal powers of k to zero, we find a system of recurrence equations (1). In order to solve system (1), we supply it with boundary conditions on the surface of the body $\nu = 0$, hence the solutions are, respectively, $W_m^{\mathtt{ref}}$. In the case of the Neumann problem, the required boundary condition will be

$$\frac{\partial}{\partial \nu}(W_m^{\mathtt{ref}} + W_m^{\mathtt{inc}})|_{\nu=0} = 0, \quad m = 0, 1, 2, \dots. \tag{18}$$

For the Dirichlet problem,

$$(W_m^{\mathtt{ref}} + W_m^{\mathtt{inc}})|_{\nu=0} = 0, \quad m = 0, 1, 2, \dots, \tag{19}$$

here we have also taken into account the conditions on the field as $\nu \to +\infty$, and hence $W^{\mathtt{ref}}$ describes the wave leaving the body of revolution. A discussion on supplying correct boundary conditions can be found in [1, 4, 8, 9].

3.4. The main term of the reflected field. The first equation of system (1) has the form

$$\left(i\frac{\partial}{\partial \sigma} + \frac{\partial^2}{\partial \nu^2} + \nu\right)W_0^{\mathtt{ref}}(\sigma, \nu) = 0.$$

The last equation has its variables σ and ν separated, and thus, under the Fourier transform, one can have the Airy equation in ν. The solution is taken in the form

$$W_0^{\mathtt{ref}}(\sigma, \nu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{i\sigma\zeta} B_0(\zeta) w_1(\zeta - \nu) d\zeta, \tag{20}$$

where $B_0(\zeta)$ is an arbitrary function (for now), $w_1(\zeta - \nu)$ is the Airy function in Fock's definition [1]. The choice of the exact form follows from the limiting absorption principle, namely, we claim that for the small positive imaginary part of the wavenumber k, $\operatorname{Im} k > 0$, the Airy function $w_1(\zeta - \nu)$ goes to zero as $\nu \to +\infty$, see [1]. Equivalently we can say that formula (20) now describes the wave leaving the scatterer as $\nu \to +\infty$.

The function $B_0(\zeta)$ can be found from boundary condition (18) or (19), where we take $m = 0$, for the Neumann and Dirichlet problems, respectively,

$$B_0^{\mathtt{Neu}}(\zeta) = -\frac{v'(\zeta)}{w_1'(\zeta)}, \quad B_0^{\mathtt{Dir}}(\zeta) = -\frac{v(\zeta)}{w_1(\zeta)}. \tag{21}$$

The main term of the expansion of the reflected wave $W_0^{\mathtt{ref}}$ can be uniquely determined.

3.5. Nonhomogeneous equations. The remaining equations in (1) are nonhomogeneous, and we want to construct the general solution for each of them, while the homogeneous equation is already solved and its general solution has form (20) with its own function $B_m(\zeta)$, $m = 1, 2, \dots$. (in addition different for the Dirichlet and Neumann problems). The particular solution can be found by the method used in [4], where instead of $v(\zeta - \nu)$ and its derivative $v'(\zeta - \nu)$ we take $w_1(\zeta - \nu)$ and $w_1'(\zeta - \nu)$.

We mentioned that the variables separate in system (1), and after σ has been singled out, we can use the Fourier transform in (12) implying explicit formulas for the operators $\widetilde{\mathcal{L}}_m$, thus obtained from the operators $\mathcal{L}_m$. We took into account the fact that $v(\zeta - \nu)$ is replaced by $w_1(\zeta - \nu)$, and the contour of integration should be regularized:

$$\widetilde{\mathcal{L}}_0 = \frac{\partial^2}{\partial \nu^2} + (\nu - \zeta),$$

where $\frac{\partial^2}{\partial \nu^2} + (\nu - \zeta)$ is the Airy operator;

$$\widetilde{\mathcal{L}}_1 = -i\left(\frac{2}{\rho_0}\right)^{1/3} \rho_0' \, \nu \, \frac{\partial}{\partial \zeta},$$

$$\widetilde{\mathcal{L}}_2 = \left(\frac{2}{\rho_0}\right)^{\frac{2}{3}} \left[\frac{\rho_0 \rho_0'' - 2\rho_0'^2}{2} \nu \frac{\partial^2}{\partial \zeta^2} - \frac{3\nu^2}{4} + \nu\zeta - \frac{\zeta^2}{4} + \frac{1}{2}\frac{\partial}{\partial \nu} + \frac{\mathbf{\Lambda}_0}{2}\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \nu}\right)\right].$$

3.6. The first term in the reflected field $W_1^{\mathtt{ref}}$. Copying the expression for the first term in the expansion of the incident field $W_1^{\mathtt{inc}}(\sigma, \nu)$ (see (14)), we assume that the first term of the reflected field has the form

$$W_1^{\mathtt{ref}}(\sigma, \nu) = i\left(\frac{\rho_0'}{\sqrt{\pi}}\right)\left(\frac{2}{\rho_0}\right)^{\frac{1}{3}} \int_{-\infty}^{\infty} e^{i\sigma\zeta}\, W_1^{\mathtt{ref}}(\zeta, \nu) d\zeta, \tag{22}$$

where

$$W_1^{\mathtt{ref}}(\zeta, \nu) = B_1(\zeta) w_1(\zeta - \nu) + P_1(\zeta, \nu) w_1(\zeta - \nu) + Q_1(\zeta, \nu) w_1'(\zeta - \nu), \tag{23}$$

the term $B_1(\zeta) w_1(\zeta - \nu)$ satisfies the nonhomogeneous equation:

$$\widetilde{\mathcal{L}}_0 B_1(\zeta) w_1(\zeta - \nu) = 0.$$

The polynomials $P_1(\zeta, \nu)$ and $Q_1(\zeta, \nu)$ are polynomials in ν of the first and second degree, respectively, with coefficients dependent on ζ. These polynomials (and also $W_1^{\mathtt{ref}}$) can be found from the equation

$$\widetilde{\mathcal{L}}_0 W_1 + \widetilde{\mathcal{L}}_1 W_0 = 0.$$

To this end we rewrite the last equation in expanded form

$$\left[\frac{\partial^2}{\partial \nu^2} + (\nu - \zeta)\right]\Big(P_1(\zeta, \nu) w_1(\zeta - \nu) + Q_1(\zeta, \nu) w_1'(\zeta - \nu)\Big)$$
$$= \nu\left[\frac{\partial B_0(\zeta)}{\partial \zeta} w_1(\zeta - \nu) + B_0(\zeta) w_1'(\zeta - \nu)\right].$$

On the other hand,

$$\left[\frac{\partial^2}{\partial\nu^2}+(\nu-\zeta)\right]\Big(P_1(\zeta,\nu)w_1(\zeta-\nu)+Q_1(\zeta,\nu)w_1'(\zeta-\nu)\Big)$$
$$=w_1(\zeta-\nu)\Big(P_1''-2[\zeta-\nu]Q_1'+Q_1\Big)+w_1'(\zeta-\nu)\Big(-2P_1'+Q_1''\Big).$$

We equate both sides in the last two equations (since the left-hand sides are equal), and collect the factors of $w_1(\zeta-\nu)$ and $w_1'(\zeta-\nu)$ in order to get the required polynomials:

$$P_1''-2[\zeta-\nu]Q_1'+Q_1=\nu\,\frac{\partial B_0(\zeta)}{\partial\zeta}\quad\text{and}\quad -2P_1'+Q_1''=\nu\,B_0(\zeta).$$

The choice of degrees of $P_1(\zeta,\nu),Q_1(\zeta,\nu)$ (the second and the first degree, respectively) is made in line with the form of the incident field. Equating the left-hand and right-hand side coefficients with equal powers of ν in the last relations, we get polynomials

$$P_1=\nu^2P_1^{(2)}\quad\text{and}\quad Q_1=Q_1^{(1)}\nu+Q_1^{(0)}.$$

Let us find $B_1(\zeta)$. The coefficients follow from the condition

$$\frac{\partial\nu}{\partial n}\frac{\partial}{\partial\nu}\left(W_1^{\texttt{inc}}+W_1^{\texttt{ref}}\right)\big|_{n=0}=\frac{2\boldsymbol{M}_0^2}{\rho_0}\frac{\partial}{\partial\nu}\left(W_1^{\texttt{inc}}+W_1^{\texttt{ref}}\right)\big|_{\nu=0}=0,$$

here the function $W_1^{\texttt{inc}}(\sigma,\nu)$ is defined by formula (14), and the attenuation function $W_1^{\texttt{ref}}$ is expressed by (22), (23). Omitting cumbersome calculations, we present the final form of the correction term $W_1^{\texttt{ref}}(\sigma,\nu)$ of order $\mathcal{O}(k^{-\frac{1}{3}})$ of the reflected field in Fock's region $W_0^{\texttt{ref}}(\sigma,\nu)$ for the Neumann boundary conditions:

$$W_1^{\texttt{ref}}(\sigma,\nu)=\left(\frac{2}{\rho_0}\right)^{1/3}\frac{i\rho_0'}{\sqrt{\pi}}\int\limits_{-\infty}^{+\infty}e^{i\sigma\zeta}\{B_1(\zeta)w_1(\zeta-\nu)$$
$$+P_1^{(2)}\nu^2w_1(\zeta-\nu)+(Q_1^{(1)}\nu+Q_1^{(0)})w_1'(\zeta-\nu)\}d\zeta. \tag{24}$$

The coefficients B_1, $P_1^{(2)}$, $Q_1^{(1)}$, and $Q_1^{(0)}$ have the form

$$B_1^{\texttt{Neu}}(\zeta)=\left[-\frac{\zeta^2v'(\zeta)}{4w_1'(\zeta)}-\frac{5}{6}\frac{\zeta}{(w_1'(\zeta))^2}+\frac{2}{3}\frac{\zeta^3w_1(\zeta)}{(w_1'(\zeta))^3}\right];\quad P_1^{(2)\texttt{Neu}}(\zeta)=\frac{v'(\zeta)}{4w_1'(\zeta)};$$
$$Q_1^{(1)\texttt{Neu}}(\zeta)=\frac{(-\zeta)}{3(w_1'(\zeta))^2};\quad Q_1^{(0)\texttt{Neu}}(\zeta)=\left(\frac{-2\zeta^2}{3(w_1'(\zeta))^2}-\frac{v'(\zeta)}{2w_1'(\zeta)}\right).$$

In the case of the Dirichlet problem, the same coefficients B_1, $P_1^{(2)}$, $Q_1^{(1)}$, and $Q_1^{(0)}$ have the form

$$B_1^{\texttt{Dir}}(\zeta)=\left[-\frac{\zeta^2v(\zeta)}{4w_1(\zeta)}+\frac{1}{2}\frac{1}{w_1^2(\zeta)}-\frac{2}{3}\frac{\zeta w_1'(\zeta)}{w_1^3(\zeta)}\right];\quad P_1^{(2)\texttt{Dir}}(\zeta)=\frac{v(\zeta)}{4w_1(\zeta)};$$
$$Q_1^{(1)\texttt{Dir}}(\zeta)=\frac{1}{3w_1^2(\zeta)};\quad Q_1^{(0)\texttt{Dir}}(\zeta)=\left(\frac{2\zeta}{3w_1^2(\zeta)}-\frac{v(\zeta)}{2w_1(\zeta)}\right).$$

We note that we used the following identity in the expressions for the coefficients:

$$v'(t)w_1(t)-v(t)w_1'(t)=-1.$$

3.7. The second term $W_2^{\mathtt{ref}}$ of the reflected field. Finally, we obtain the asymptotic expansion for $W_2^{\mathtt{ref}}(\sigma,\nu)$, which consists of the general solution of the homogeneous equation and a particular solution of the nonhomogeneous equation, namely, the third equation in system (1). We present it here again

$$\widetilde{\mathcal{L}}_0 W_2^{\mathtt{ref}} = -\widetilde{\mathcal{L}}_1 W_1^{\mathtt{ref}} - \widetilde{\mathcal{L}}_2 W_0^{\mathtt{ref}} - \dots .$$

We look for $W_2^{\mathtt{ref}}(\sigma,\nu)$ in the form

$$W_2^{\mathtt{ref}}(\sigma,\nu) = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\rho_0}\right)^{\frac{2}{3}} \int\limits_{-\infty}^{\infty} e^{i\sigma\zeta} W_2^{\mathtt{ref}}(\zeta,\nu)d\zeta, \tag{25}$$

where

$$W_2^{\mathtt{ref}}(\zeta,\nu) = B_2(\zeta)w_1(\zeta-\nu) + P_2(\zeta,\nu)w_1(\zeta-\nu) + Q_2(\zeta,\nu)w_1'(\zeta-\nu).$$

As before the first term satisfies the homogeneous equation

$$\widetilde{\mathcal{L}}_0 B_2(\zeta)w_1(\zeta-\nu) = 0.$$

We choose the form of the polynomials P_2 and Q_2 using the form of the polynomials of the incident wave

$$P_2(\zeta,\nu) = P_2^{(4)}\nu^4 + P_2^{(3)}\nu^3 + P_2^{(2)}\nu^2 + P_2^{(1)}\nu;\ Q_2 = Q_2^{(3)}\nu^3 + Q_2^{(2)}\nu^2 + Q_2^{(1)}\nu + Q_2^{(0)}.$$

Similarly to the way in which we found the polynomials P_1 and Q_1 for $W_1^{\mathtt{ref}}(\zeta,\nu)$, after some cumbersome computations, for both Dirichlet and Neumann boundary conditions, we find $P_2(\zeta,\nu)$ and $Q_2(\zeta,\nu)$:

$$\begin{cases} P_2^{(4)} = \dfrac{(\rho_0')^2 v'(\zeta)}{32 w_1'(\zeta)} \\ P_2^{(3)} = -\dfrac{(\rho_0')^2}{18}\left[\dfrac{1}{(w_1'(\zeta))^2} - \dfrac{2\zeta^2 w_1(\zeta)}{[w_1'(\zeta)]^3}\right] \\ P_2^{(2)} = \dfrac{(\rho_0')^2}{2}\cdot\dfrac{\zeta^3 w_1(\zeta)}{[w_1'(\zeta)]^3} - \dfrac{(\rho_0')^2\zeta^2 v'(\zeta)}{16 w_1'(\zeta)} - \dfrac{\zeta}{24[w_1'(\zeta)]^2}\left[7(\rho_0')^2 + 6\rho_0\rho_0''\right] \\ P_2^{(1)} = \dfrac{v'(\zeta)}{40 w_1'(\zeta)}\left[(\rho_0')^2 - 4\rho_0\rho_0'' + 4\right] - \dfrac{(\rho_0')^2\zeta^2}{6[w_1'(\zeta)]^2} \end{cases}$$

$$\begin{cases} Q_2^{(3)} = -\dfrac{(\rho_0')^2\zeta}{12[w_1'(\zeta)]^2} \\ Q_2^{(2)} = \dfrac{v'(\zeta)}{40 w_1'(\zeta)}\left[(\rho_0')^2 - 4\rho_0\rho_0'' - 6\right] - \dfrac{(\rho_0')^2\zeta^2}{6[w_1'(\zeta)]^2} \\ Q_2^{(1)} = \dfrac{2}{3}(\rho_0')^2\dfrac{\zeta^4[w_1(\zeta)]^2}{[w_1'(\zeta)]^4} - \dfrac{5}{36}\dfrac{\zeta^3(\rho_0')^2}{[w_1'(\zeta)]^2} - \dfrac{\zeta^4 w_1(\zeta)}{[w_1'(\zeta)]^3}\cdot\dfrac{7(\rho_0')^2 + 3\rho_0\rho_0''}{9} \\ \qquad + \dfrac{\zeta v'(\zeta)}{w_1'(\zeta)}\dfrac{[(\rho_0')^2 + \rho_0\rho_0'' + 4]}{30} + \dfrac{1}{[w_1'(\zeta)]^2}\cdot\dfrac{[(\rho_0')^2 + 3\rho_0\rho_0'']}{18} \\ Q_2^{(0)} = \dfrac{4(\rho_0')^2}{3}\dfrac{\zeta^5[w_1(\zeta)]^2}{[w_1'(\zeta)]^4} - \dfrac{5}{18}\dfrac{(\rho_0')^2\zeta^4}{[w_1'(\zeta)]^2} - \dfrac{\zeta^3 w_1(\zeta)}{[w_1'(\zeta)]^3}\cdot\dfrac{[23(\rho_0')^2 + 6\rho_0\rho_0'']}{9} \\ \qquad + \dfrac{\zeta^2 v'(\zeta)}{w_1'(\zeta)}\dfrac{[23(\rho_0')^2 + 8\rho_0\rho_0'' + 2]}{120} + \dfrac{\zeta}{[w_1'(\zeta)]^2}\left[\dfrac{\mathbf{\Lambda}_0}{2} + \dfrac{[25(\rho_0')^2 + 30\rho_0\rho_0'']}{36}\right]. \end{cases} \tag{26}$$

Similarly to the way in which we obtained $B_1(\zeta)$, we find $B_2(\zeta)$, which together with formulas (25)–(26), gives the required second term $W_2^{\mathtt{ref}}$ of the reflected field. Finally, the coefficient $B_2(\zeta)$ equals

$$\begin{aligned} B_2(\zeta) &= Q_2^{(1)} + P_2^{(1)} \frac{w_1(\zeta)}{w_1'(\zeta)} - Q_2^{(0)} \frac{\zeta w_1(\zeta)}{w_1'(\zeta)} \\ &+ \frac{v(\zeta)}{w_1'(\zeta)} \left[\frac{8\alpha}{5!} - \frac{\beta}{3!} + \frac{3(\rho_0')^2}{8} + \frac{2\alpha}{5!}\zeta^3 + \frac{3}{8}(\rho_0')^2\zeta^3 \right] \\ &+ \frac{v'(\zeta)}{w_1'(\zeta)} \left[\frac{8\alpha}{5!}\zeta - \frac{\beta}{3!}\zeta + \frac{(\rho_0')^2}{8}\left(7\zeta + \frac{\zeta^4}{4}\right) + \frac{4\alpha}{5!}\zeta \right], \end{aligned} \tag{27}$$

where α and β are given by (7). The prolongation of the body reveals only in the second large parameter $\mathbf{\Lambda}_0$ (2), which appears only in the operator $\mathcal{B}$ (see (15)) and in the coefficient $Q_2^{(0)}(\zeta)$ (26), which occurs in $W_2^{\mathtt{ref}}(\zeta,\nu)$ at $w_1'(\zeta-\nu)$ and at $w_1(\zeta-\nu)(-\frac{\zeta\, w_1(\zeta)}{w_1'(\zeta)})$ in the coefficient $B_2(\zeta)$, see (27).

4. The total field

The first term in the expansion of the total field can be expressed as

$$U_0(s,n) = e^{iks} W_0(\sigma,\nu) = e^{iks}(W_0^{\mathtt{inc}} + W_0^{\mathtt{ref}}).$$

Substitute the representations for the main term of the incident (11) and reflected (20), (21) waves, in order to obtain the total field

$$U_0(s,n) = \frac{e^{iks}}{\sqrt{\pi}} \int_\Gamma e^{i\sigma\zeta} \Big[v(\zeta-\nu) - B_0(\zeta) w_1(\zeta-\nu) \Big] d\zeta, \tag{28}$$

where the function $B_0(\zeta)$ (21) is substituted according to the type of the boundary conditions.

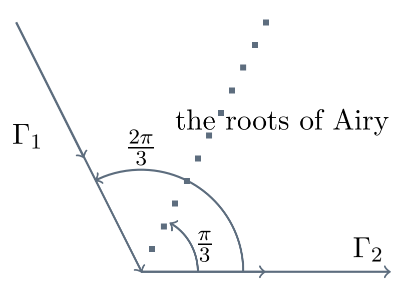

Fig. 2. Integration contour $\Gamma = \Gamma_1 \cup \Gamma_2$ goes along the straight line from $\infty e^{i\frac{2\pi}{3}}$ to 0, and then from 0 to ∞ along the positive real axis in the complex plane of ζ. This contour envelops the roots of the Airy function $w_1(\zeta_q), q = 1,2,\dots$ in the first quadrant, or the roots of its derivative $w_1'(\zeta_p), p = 1,2,\dots$. The roots of the Airy function $w_1(\zeta)$ and its derivative belong to the ray $e^{i\frac{\pi}{3}}$.

It is important to choose the contour of integration Γ, which should go around the first quadrant of the complex plane of ζ in the positive direction. All the zeros of the integrand in (28) belong to the first quadrant of the complex plane of ζ. For example, we may assume that the contour Γ goes from the infinity along the ray $\arg\zeta = \frac{2\pi}{3}$ toward zero, and then from zero to ∞, along the positive real axis (see Fig. 2).

We rewrite integral (28), using the relations between the Airy functions: $v(t)$ can be expressed via the functions $w_1(t), w_2(t)$ according to the formula

$$v(t) = \frac{w_1(t) - w_2(t)}{2i},$$

and then integral (28) can be rewritten as follows:

$$U_0(s,n) = e^{iks}\frac{1}{\sqrt{\pi}}\frac{i}{2}\int_\Gamma e^{i\sigma\zeta}\left[w_2(\zeta-\nu) - B_0(\zeta)w_1(\zeta-\nu)\right]d\zeta,$$

and $B_0(\zeta)$ is chosen according to the type of the boundary conditions: Neumann or Dirichlet:

$$B_0^{\mathtt{Neu}}(\zeta) = -\frac{w_2'(\zeta)}{w_1'(\zeta)}, \quad B_0^{\mathtt{Dir}}(\zeta) = -\frac{w_2(\zeta)}{w_1(\zeta)}.$$

5. Comparison with the result by V. I. Ivanov [14]

We found three terms of the asymptotic expansion of the wave filed using the Leontovich–Fock parabolic equation method

$$U = e^{iks}\left(W_0 + \frac{W_1}{k^{\frac{1}{3}}} + \frac{W_2}{k^{\frac{2}{3}}} + \mathcal{O}(k^{-1})\right), \ k \gg 1. \tag{29}$$

Compare expansion (29) with the solution found by V. I. Ivanov [14]. Paper [14] contains numerical comparison of the exact solution and the asymptotic correction term $e^{iks}W_2k^{-\frac{2}{3}}$ (the second term) to the Fock's formula ($e^{iks}W_0$) near a circle (or a circular cylinder in the orthogonal cross-section) and a sphere. Here $\rho_0 = a$, where a is the radius of the circle or the sphere. Consequently, we should assume $\rho_0' = \rho_0'' = 0$, $f(0) = \infty$, $\mathbf{\Lambda}_0 = 0$ with the operator $\mathcal{B}$ vanishing in formulas (25)–(27). By the axially symmetric nature of the problem, $\mathbf{\Lambda}_0 = 1$ for the sphere, since $\rho_0 = f(0) = a$.

In the case of a diffraction problem for a plane wave on a circle, formulas (25)–(27) coincide precisely with formulas (10), (11) from [14], namely:

$$U^{\mathtt{Neu}} = U_0^{\mathtt{Neu}}$$
$$-\frac{e^{iks}}{60\boldsymbol{M}_0^2\sqrt{\pi}}\int\limits_\Gamma e^{i\sigma\zeta}\left([4\zeta+6\nu]\Psi(\zeta,\nu)+[\zeta-\nu][\zeta+9\nu]\overline{\Psi}(\zeta,\nu) - \frac{\zeta^3-6}{[w_1'(\zeta)]^2}w_1(\zeta-\nu)\right)d\zeta + \mathcal{O}\left(\frac{1}{\boldsymbol{M}_0^4}\right),$$

$$\Psi(\zeta,\nu) = v(\zeta-\nu) - \frac{v'(\zeta)}{w_1'(\zeta)}w_1(\zeta-\nu), \quad \overline{\Psi} = -\frac{\partial\Psi}{\partial\nu}.$$

Here

$$U_0^{\mathtt{Neu}} = \frac{e^{iks}}{\sqrt{\pi}}\int\limits_\Gamma e^{i\sigma\zeta}\Psi(\zeta,\nu)d\zeta$$

is the Fock's asymptotics, and all the other terms in (29) are correction ones. We can choose a broken line (see Fig. 2) as a contour Γ, since the roots of the integrand belong to the first quadrant. In the case of diffraction on a sphere, there is similarity in the results neither for the incident nor for the reflected wave. By the axially symmetric nature of the problem, the incident wave formula should coincide with the two-dimensional formula on a circle. The latter is not the case in the results presented in [14]. This conclusion is true for both Dirichlet and Neumann cases.

6. The boundary current in the Neumann problem $\frac{\partial U}{\partial n}|_{n=0} = 0$

6.1. Correction terms with factor $\frac{\mathbf{\Lambda}_0}{2\boldsymbol{M}_0^2}$. The constructed two-scale asymptotic expansion of the field in the Fock's region allows one to obtain the approximate formulas for determining the wave field depending of the ratio between the large parameters in the problem: the Fock's parameter $\boldsymbol{M}_0$ and the prolongation parameter $\mathbf{\Lambda}_0$ of the body of revolution (2). This parameter appears in the system of recurrence equations (1) only as ratio $\frac{\mathbf{\Lambda}_0}{2\boldsymbol{M}_0^2}$ and its integer powers. The parameter $\mathbf{\Lambda}_0$ firstly appears from the operator $\mathcal{B}$ (15) in the third equation of system (1) after expanding the Lamé coefficient h_φ^{-1} (5) as a series in powers of σ and ν. Therefore, the solution of system (1) will keep its asymptotic nature with respect to $\boldsymbol{M}_0^{-m}$, $m = 0, 1, 2$, provided that $\mathbf{\Lambda}_0 = 2\boldsymbol{M}_0^{2-\varepsilon}$, $0 < \varepsilon < 2$. The magnitude of ε depends on the actual values of the parameters k, ρ_0, $f(0)$ in the region 2 (see Fig. 1), i.e., in the Fock's region.

6.2. Dimensionless currents: the Fock's current and the Fock's current with a correction term. The following value $I^{\mathtt{Dir}}$ is called the current for the attenuation function $W = W^{\mathtt{inc}} + W^{\mathtt{ref}}$ for the Dirichlet boundary condition (see [4], we present these formulas here in order to compare them with a similar result for the Neumann boundary conditions)

$$I^{\mathtt{Dir}} = \frac{\partial}{\partial n}\Big(W^{\mathtt{inc}} + W^{\mathtt{ref}}\Big)\Big|_{n=0} = \frac{2\boldsymbol{M}_0^2}{\rho_0}\frac{\partial}{\partial \nu}\Big(W^{\mathtt{inc}} + W^{\mathtt{ref}}\Big)\Big|_{\nu=0}.$$

For further comparison we present the form of the dimensionless current with the correction term

$$k^{-1} I^{\mathtt{Dir}} = \frac{1}{\boldsymbol{M}_0\sqrt{\pi}}\left[\int_\Gamma \frac{e^{i\sigma\zeta}}{w_1(\zeta)}d\zeta + \frac{\mathbf{\Lambda}_0}{2\boldsymbol{M}_0^2}\int_\Gamma e^{i\sigma\zeta}\left[\frac{\zeta}{w_1(\zeta)} - \frac{[w_1'(\zeta)]^2}{[w_1(\zeta)]^3}\right]d\zeta\right]. \tag{30}$$

We note that the current that corresponds to the Fock's field is

$$I_0^{\mathtt{Dir}} = \frac{2\boldsymbol{M}_0^2}{\rho_0}\frac{1}{\sqrt{\pi}}\int_\Gamma \frac{e^{i\sigma\zeta}d\zeta}{w_1(\zeta)}.$$

The following value $I^{\mathtt{Neu}}$ is called the current for the attenuation function $W = W^{\mathtt{inc}} + W^{\mathtt{ref}}$ for the Neumann boundary conditions

$$I^{\mathtt{Neu}} = \Big(W^{\mathtt{inc}} + W^{\mathtt{ref}}\Big)\Big|_{\nu=0}.$$

The dimensionless current, corresponding to the total field $W_0^{\mathtt{Neu}}$, which is equal to

$$W_0^{\mathtt{Neu}} = \frac{1}{\sqrt{\pi}}\int_\Gamma e^{i\sigma\zeta}[v(\zeta-\nu) - \frac{v'(\zeta)}{w_1'(\zeta)}w_1(\zeta-\nu)]d\zeta,$$

has the form

$$I_0^{\mathtt{Neu}} = \frac{1}{\sqrt{\pi}}\int_\Gamma e^{i\sigma\zeta}\left[\frac{v(\zeta)w_1'(\zeta) - v'(\zeta)w_1(\zeta)}{w_1'(\zeta)}\right]d\zeta = \frac{1}{\sqrt{\pi}}\int_\Gamma e^{i\sigma\zeta}\frac{d\zeta}{w_1'(\zeta)},$$

since the Wronskian $v(\zeta)w_1'(\zeta) - v'(\zeta)w_1(\zeta) = 1$. The current with the correction term is equal to

$$I^{\mathtt{Neu}} = \frac{1}{\sqrt{\pi}}\left[\int_\Gamma \frac{e^{i\sigma\zeta}}{w_1'(\zeta)}d\zeta + \frac{\mathbf{\Lambda}_0}{2\boldsymbol{M}_0^2}\int_\Gamma e^{i\sigma\zeta}\left[\frac{\zeta}{w_1'(\zeta)} - \frac{\zeta^2 w_1^2(\zeta)}{[w_1'(\zeta)]^3}\right]d\zeta\right]. \tag{31}$$

Consider the illuminated area 1 (see Fig. 1). The total wave field $U = U^{\mathtt{inc}} + U^{\mathtt{ref}} = e^{iks}(W^{\mathtt{inc}} + W^{\mathtt{ref}})$ can be represented by formulas for the Dirichlet and Neumann cases of the following type:

$$W^{\mathtt{inc}} = e^{i(\nu\sigma - \frac{\sigma^3}{3})},$$

$$W^{\mathtt{ref}} = \mp\sqrt{\frac{1}{3} - \frac{2\sigma}{3\sqrt{\sigma^2+3\nu}}}\exp\left[i\left(-\frac{5}{27}\sigma^3 - \frac{\nu\sigma}{3} + \frac{4}{27}(\sigma^2+3\nu)^{\frac{3}{2}}\right)\right], \sigma < 0.$$

The minus sign in front of the radical corresponds to the Dirichlet, and the plus sign to the Neumann boundary condition. V. A. Fock showed [1] that as $\sigma \to -\infty$, the reflected field $W^{\mathtt{ref}}$ coincided with the reflected wave in the boundary layer. Constraining the problem by $\nu = \mathcal{O}(1), \sigma \to -\infty$, $W^{\mathtt{ref}}_{-\infty}$ is simplified, and the total wave field can be represented by the relation

$$W_{-\infty} = e^{i(\nu\sigma - \frac{\sigma^3}{3})} \mp \frac{\sqrt{2\sigma^2+\nu}}{\sqrt{2\sigma^2+3\nu}} e^{-i(\nu\sigma + \frac{\sigma^3}{3})}. \tag{32}$$

The dimensionless current, generated by ray formula (32), has the following form for the Neumann conditions

$$I^{\mathtt{Neu}}_{\mathtt{ray}} = 2e^{-i\frac{\sigma^3}{3}}, \quad \sigma \to -\infty. \tag{33}$$

For the Dirichlet condition, the dimensionless current is equal to

$$k^{-1} I^{\mathtt{Dir}}_{\mathtt{ray}} = \frac{2i\sigma}{\boldsymbol{M}_0} e^{-i\frac{\sigma^3}{3}}, \quad \sigma \to -\infty. \tag{34}$$

6.3. Fock's dimensionless currents. Thus the Fock's dimensionless current for the Dirichlet and the Neumann conditions are

$$k^{-1} I^{\mathtt{Dir}}_0 = \frac{1}{\boldsymbol{M}_0\sqrt{\pi}} \int_\Gamma e^{i\sigma\zeta} \frac{d\zeta}{w_1(\zeta)}; \tag{35}$$

$$I^{\mathtt{Neu}}_0 = \frac{1}{\sqrt{\pi}} \int_\Gamma e^{i\sigma\zeta} \frac{d\zeta}{w_1'(\zeta)}; \tag{36}$$

here $\boldsymbol{M}_0 = \left(\frac{k\rho_0}{2}\right)^{\frac{1}{3}}$ is Fock's parameter (2).

7. Penumbra and shadow regions 4,5

Recall that the light-shadow border on the surface of the scatterer $\partial\Omega$ corresponds to $s = 0$, and for small n and s growing, the observation point (s, n) gets into a shadowed zone near the scatterer (Fig. 3). The latter condition corresponds to $\sigma \to +\infty$ and to the boundedness of $\nu = \mathcal{O}(1)$ in the stretched coordinates (σ, ν). The wave field can be obtained using residue theory, since the function $\exp(i\sigma\zeta)$ decays exponentially with $\sigma \gg 1$ in the upper half-plane $\operatorname{Im}\zeta > 0$, in the same manner as the function $\exp(-\sigma\operatorname{Im}\zeta)$. The integration contour in diffraction formulas for $W^{\mathtt{ref}}_0, W^{\mathtt{ref}}_1, W^{\mathtt{ref}}_2$ can be moved to the upper half-plane. Hence the residues at the roots of the integrands will be singled out. The integrals W_0, W_1, W_2 (20), (21), (24), (25)–(27) can be represented as sums of residues, at the roots of the Airy function ($w_1(\zeta_q) = 0$, $q = 1, 2, \dots$) for the Dirichlet and at the roots of the derivative of the Airy function ($w_1'(\zeta_p') = 0$, $p = 1, 2, \dots$) in the case of the Neumann problem on $\partial\Omega$.

Then the full solution in the shadow region

$$U^{\mathtt{Dir,\ Neu}} = e^{iks}\left[W_0(s,n) + \frac{W_1(s,n)}{k^{\frac{1}{3}}} + \frac{W_2(s,n)}{k^{\frac{2}{3}}} + \mathcal{O}(k^{-1})\right]$$

is given by functions (from Fock's formula for W_0)

$$W_0^{\mathtt{Dir}} = -i2\sqrt{\pi}\sum_{q=1}\frac{w_1(\zeta_q-\nu)}{[w_1'(\zeta_q)]^2}e^{i\sigma\zeta_q} \tag{37}$$

and

$$W_0^{\mathtt{Neu}} = i2\sqrt{\pi}\sum_{p=1}\frac{w_1(\zeta_p'-\nu)}{\zeta_p'[w_1(\zeta_p')]^2}e^{i\sigma\zeta_p'}, \tag{38}$$

and so on. Roots ζ_q, $q=1,2,\dots$ belong to ray $\arg\zeta=\frac{\pi}{3}$ and increase in magnitude (Fig. 2). Here ζ_1' and ζ_1 take the values

$$\zeta_1' = 1.01879\cdot e^{i\frac{\pi}{3}},\ \zeta_1 = 2.33811\cdot e^{i\frac{\pi}{3}}.$$

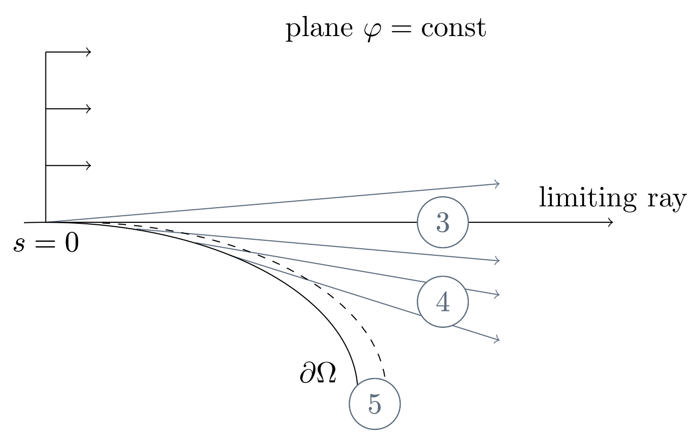

Fig. 3. Penumbra and shadow regions. The field in the shadow region 4 between regions 3 and 5 has ray nature of a leaving wave. Region 5 is the creeping waves zone with $\nu=\mathcal{O}(1), s=\mathcal{O}(1)$ (37), (38).

Using the asymptotic expression for $w_1(\zeta-\nu)$ with ν large compared to the roots

$$w_1(\zeta-\nu) \underset{\nu\to+\infty}{\sim} \frac{1}{\sqrt{(\nu-\zeta)}}\exp(i\frac{2}{3}((\nu-\zeta)^{\frac{3}{2}})+i\frac{\pi}{4}),$$

we get approximate values for $W_0^{\mathtt{Dir}}$ and $W_0^{\mathtt{Neu}}$:

$$e^{iks}W_0^{\mathtt{Dir}} \underset{\nu\to+\infty}{\sim} \frac{-ie^{i\frac{\pi}{4}}2\sqrt{\pi}}{[w_1'(\zeta_1)^2]}\cdot e^{i(\sigma-\sqrt{\nu})\zeta_1}\cdot e^{i\frac{2}{3}\nu^{\frac{3}{2}}}\cdot e^{iks}, \tag{39}$$

$$e^{iks}W_0^{\mathtt{Neu}} \underset{\nu\to+\infty}{\sim} \frac{ie^{i\frac{\pi}{4}}2\sqrt{\pi}}{[w_1(\zeta_1')^2]}\cdot e^{i(\sigma-\sqrt{\nu})\zeta_1'}\cdot e^{\frac{2}{3}\nu^{\frac{3}{2}}}\cdot e^{iks}. \tag{40}$$

Let us analyze formulas (39) and (40), see [1]. We note that the inequality $\sigma-\sqrt{\nu}=0$ gives the geometric border for the shadow. Increasing positive values of $\sigma-\sqrt{\nu}$ correspond to the points positioned further deep in the shadow. In the area, where $\sigma-\sqrt{\nu}$ is not large (and can be both positive or negative), we have penumbra region. Thus, when $\sigma-\sqrt{\nu}$ increases, functions (39) and (40), as well as the field, decay exponentially. Moreover, the highest oscillating factor is $\exp(iks+\frac{2}{3}i\nu^{\frac{3}{2}})$ in formulas (39) and (40) as $\nu\to+\infty$. Moving away from the boundary $\partial\Omega$, the ray field in both the Dirichlet and the Neumann cases is a sum of rays of the leaving wave, which has the surface of the scatterer as its caustic (see Fig. 3). Note that the formal solution corresponding to the creeping waves was found by Friedlander and Keller [15]. Moreover, diffraction rays in region 4 were first introduced by J. B. Keller in paper [16].

Looking again at formulas (37), (38) for residues, one can see that $e^{iks}W_0^{\mathtt{Dir}}$ and $e^{iks}W_0^{\mathtt{Neu}}$ give us exactly the creeping waves. Formally, the creeping wave is taken in the following form in the layer $\nu = \mathcal{O}(1), s = \mathcal{O}(1)$:

$$U = e^{iks} \cdot e^{ik^{\frac{1}{3}}\Phi(s)} \sum_{m=0} V_m(s,\nu) k^{-\frac{1}{3}},$$

where for the Dirichlet case

$$\Phi(s) = \frac{\zeta_q}{2^{\frac{1}{3}}} \int_0^s \rho^{-\frac{2}{3}}(s)ds, \ V_{0,q} = A_0(s) w_1(\zeta_q - \nu),$$

where for the Neumann case

$$\Phi(s) = \frac{\zeta_p'}{2^{\frac{1}{3}}} \int_0^s \rho^{-\frac{2}{3}}(s)ds, \ V_{0,p} = A_0(s) w_1(\zeta_p' - \nu).$$

If we pass from the coordinate σ to s, in the vicinity of $s = 0$, formulas (37), (38) will give us the creeping waves.

8. Conclusions

The shortwave diffraction of a plane incident wave from a strictly convex body of revolution is under investigation. Geometric properties of the body of revolution (curvature radii of the surface of the scatterer) are assumed to be much bigger than the incident wavelength. The problem's solution gives a clear physical picture of shortwave diffraction. The field is described by the formulas of the ray method in the illuminated region of the scatterer. The ray method fails in the vicinity of the light-shadow border, where the ray are tangent to the boundary. In the latter region, Fock's boundary layer method is applied, which gives another asymptotic expansion. The expansion of the field there has a form of integrals with linear combinations of Airy functions and its derivatives and polynomials of the normal coordinate as integrands. Fock's asymptotics is used for the initial data to construct creeping waves in the shadowed part of the boundary, for the diffraction rays (first introduced by Keller) in the shadow far from the surface, and for the construction of the wave field in the limiting ray vicinity, where light turns into shadow.

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