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ON AN EXPONENTIAL FUNCTIONAL FOR GAUSSIAN PROCESSES AND ITS GEOMETRIC FOUNDATIONS

R. A. Vitale*

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After setting geometric notions, we revisit an exponential functional which has arisen in several contexts, with special attention to a set of geometric parameters and associated inequalities. Bibliography: 32 titles.

1. Introduction

It is an honor and pleasure to contribute to this volume. V. N. Sudakov's work has had a great influence on my own interests. In that spirit, what follows is a note on an exponential functional that bears on the structure of bounded Gaussian processes. The content is largely expository and begins with a review of relevant notions from classical convex geometry and their extension to infinite dimensions. We then recall the exponential functional, including a basic inequality, and a set of geometric parameters. The latter are re-examined for an alternate representation and then related inequalities are discussed.

2. Background

In what follows, aspects of geometric convexity not otherwise referenced can be found in the excellent monograph [19]. As stated there, the key feature of the *Brunn–Minkowski theory* is the interaction of volume evaluation and vector addition of convex bodies (nonempty, compact, convex subsets): For convex bodies $K_1, K_2, \dots, K_n$ in $\mathbb{R}^d$ and positive coefficients $\lambda_1, \lambda_2, \dots, \lambda_n$,

$$\mathrm{vol}_d\,(\lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_n K_n) = \sum_{i_1, i_2, \cdots, i_d = 1}^{n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_d} V(K_{i_1}, K_{i_2}, \dots, K_{i_d}), \tag{1}$$

where, without loss of generality, the "mixed volumes" $V(\cdots)$ are taken to be symmetric in their arguments. For the special case of a parallel body $K + \lambda B_d$ (B_d is the unit ball in $\mathbb{R}^d$), (1) is the classical Steiner formula

$$\mathrm{vol}_d(K + \lambda B_d) = \sum_{j=0}^{d} \lambda^j \binom{d}{j} W_j(K), \tag{2}$$

where

$$W_j(K) = V(\underbrace{K, K, \cdots, K}_{k-j}, \underbrace{B_d, B_d, \cdots, B_d}_{j}), \quad 0 \le j \le d,$$

are the *quermassintegrals* or *Minkowski functionals* (one should note that the latter term also refers to a different object in the literature). Unfortunately, they have the inconvenient property of depending on d, the dimension of the specific ambient space. A modified collection is free of this property: The *intrinsic volumes* [2, 16] are given by

$$V_j(K) = \frac{\binom{d}{j}}{\kappa_j} W_{d-j}(K), \quad 0 \le j \le d. \tag{3}$$

*University of Connecticut Storrs, CT, USA, e-mail: r.vitale@uconn.edu.

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Here κ_j is the volume of B_j, and one can extend (3) by taking $V_j(K) = 0$ for $d < j$ (by contrast, infinite–dimensional K will have $V_j(K) > 0$ for all j). We note the equality $V_0(K) = 1$ and three other specific cases: $V_d(K) = \mathrm{vol}_d(K)$, $V_{d-1}(K) = (1/2)S_{d-1}(K)$ (i.e., 1/2 the surface area of K), and $V_1(K)$, which is a mean-width type functional normalized so that if K is a line segment, then $V_1(K)$ is its length.

The corresponding version of the Steiner formula reads

$$\mathrm{vol}_d(K + \lambda B_d) = \sum_{i=0}^{d} \lambda^j \kappa_j V_{d-j}(K). \tag{4}$$

The *Alexandrov–Fenchel inequality* asserts that

$$V^2(K_1, K_2, K_3, \ldots, K_d) \geq V(K_1, K_1, K_3, \ldots, K_d)\, V(K_2, K_2, K_3, \ldots, K_d) \tag{5}$$

for convex bodies $K_1, K_2, \ldots, K_d$ in $I\!R^d$. Specifying to intrinsic volumes and making an appropriate adjustment of constants, (5) can be shown to imply the logconcavity of the sequence $\{j!\, V_j(K)\}_{j=0}^{\infty}$:

$$(j!\, V_j(K))^2 \geq (j-1)! V_{j-1}(K) \cdot (j+1)! V_{j+1}(K), \qquad j = 1, 2, \ldots, \tag{6}$$

and a direct consequence [2, 17]:

$$V_j(K) \leq \frac{V_1^j(K)}{j!} \quad j = 1, 2, \ldots. \tag{7}$$

3. Extension of intrinsic volumes to infinite-dimensional bodies

It was the celebrated insight of Sudakov ([21–23], Theorem 1 below) which connected the geometric structure just described and Gaussian processes. This was subsequently elaborated by Chevet and Tsirelson. We give a brief review.

For a convex body K in a Hilbert space ($\iff \ell_2$), consider a Gaussian process $\{X_t, t \in K\}$[1] that is *isonormal*:

$$t \longmapsto X_t \sim N(0, \sigma_t^2),$$

where $\sigma_t^2 = \mathrm{Var}\, X_t = \|t\|^2$ and $\mathrm{Cov}\left(X_t, X_{\widehat{t}}\right) = \langle t, \widehat{t}\rangle$ (scalar product). An important question is whether there is a version that is a.s. bounded; it was formulated by Dudley [3] as follows: is K a *GB-set*?

On the geometric side, making use of the monotonicity of $V_1(\cdot)$, set

$$V_1(K) = \sup\left\{V_1(\widehat{K}) \,:\, \widehat{K} \subseteq K,\ \ \widehat{K} \text{ finite-dimensional}\right\}. \tag{8}$$

Sudakov established the following fact.

Theorem 1. *K is a GB-set if and only if $V_1(K)$ is finite.*

In what follows, we assume that all the relevant sets K are GB.

Chevet [2] similarly extended by monotonicity the other intrinsic volumes $V_j, j = 2, 3, \ldots$, established (7), and thereby concluded that

$$V_1(K) < \infty \implies V_j(K) < \infty, \quad j = 2, 3, \ldots.$$

Sudakov showed specifically that

$$V_1(K) = \sqrt{2\pi}\ \mathrm{E} \sup_{t \in K} X_t\,. \tag{9}$$

[1] Here, and below, $t \in K$ means by convention that t ranges over a countable dense (respectively, any) subset of K.

In an important step, Tsirelson [25] placed (9) within a family of representations for all of the intrinsic volumes. Accommodating technical issues somewhat differently, a sketch is as follows: For given j, consider

$$X_t^{j*} = \left(X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(j)} \right),$$

where the components are independent copies of X_t, together with the vector process

$$X_K^{j*} = \{ X_t^{j*},\, t \in K \}.$$

The closed convex hull

$$Y_{j,K} = \overline{\mathrm{conv}} \left(X_K^{j*} \right)$$

is a candidate for a random convex body in $I\!R^j$, and, accordingly, its measurability must be established. To do this, we make use of its support function $h_{Y_{j,K}} : S^{j-1} \to I\!R^1$ given by

$$h_{Y_{j,K}}(u) = \sup \{ \langle y, u \rangle \,|\, y \in Y_{j,K} \} = \sup \left\{ \langle x, u \rangle \,|\, x \in X_K^{j*} \right\} = \sup \left\{ \sum_{i=1}^{j} X_t^{(i)} u_i \,|\, t \in K \right\},$$

which is evidently a random variable for each u. Now the measurability of $Y_{j,K}$ coincides with the measurability of the quantity $\delta_H \left(Y_{j,K}, L \right)$ for every convex body L in $I\!R^j$, where δ_H is the Hausdorff metric. This is confirmed by recalling that

$$\delta_H \left(Y_{j,K}, L \right) = \sup \left\{ |h_{Y_{j,K}}(u) - h_L(u)| \,|\, u \in \text{a countable, dense subset of } S^{j-1} \right\}.$$

With the foregoing in place, Tsirelson's representation [25, Theorem 6] is:

$$V_j(K) = \frac{(2\pi)^{j/2}}{j!\,\kappa_j} \,\mathrm{E}\, \mathrm{vol}_j \left(Y_{j,K} \right), \qquad j = 1, 2, \ldots. \tag{10}$$

For what follows, and in view of the standard isonormal map $t \mapsto X_t = \langle t, Z \rangle = \sum_1^\infty t_i Z_i$, where $\{Z_i\}_1^\infty$ is a sequence of standard normal random variables, we introduce the suggestive notation

$$Z_{[j,\infty]} K = Y_{j,K}, \tag{11}$$

where $Z_{[j,\infty]}$ is a $j \times \infty$ matrix of independent, standard normal random variables. Finally, we mention that an alternate proof of the representation was given by the author [31] based on a theorem of Hadwiger characterizing intrinsic volumes ([6], see also [10]).

4. The Wills functional

In various forms, the functional of the title has arisen independently in (i) geometry [7,8,32] (from where we take its name), (ii) maximum likelihood estimation of location [24–26], and (iii) financial mathematics [1], see also [27–29]. For a convex body K in $I\!R^d$, the *Wills functional* is given by [32][2]

$$W(K) = \sum_{j=0}^{d} V_j((1/\sqrt{2\pi})K) = \sum_{j=0}^{d} (1/(2\pi)^{j/2}) V_j(K). \tag{12}$$

A different expression for $W(K)$ takes the form

$$\int_{I\!R^d} e^{-\pi\, \mathrm{dist}^2 (x, (1/\sqrt{2\pi})K)} \mathrm{d}x, \tag{13}$$

[2]We note that the scaling of K by $1/\sqrt{2\pi}$ does not appear in the original formulation of Wills as followed in the geometry literature and also in [27]. The present normalization was adopted by the author in [29] as somewhat better fitted to Gaussian contexts; see also [25].

where $\operatorname{dist}(x, \left(1/\sqrt{2\pi}\right) K) = \inf_{t \in \left(1/\sqrt{2\pi}\right) K} \|x - t\|$. Following [7], the equivalence of the two expressions was shown in [27], and we repeat that here for the reader's convenience. Consider

$$W(K) = \mathrm{E} \operatorname{vol}_d \left((1/\sqrt{2\pi}) K + \Lambda B_d \right), \tag{14}$$

where Λ is a random variable with density $f(\lambda) = 1(\lambda \geq 0) 2\pi \, \lambda \, e^{-\pi \lambda^2} x$. Expanding the volume expression, taking expectations, and making note of the equalities $\mathrm{E} \, \Lambda^j = \frac{1}{\kappa_j}, j = 0, 1, 2, \ldots,$ yields (12). For the second representation, again start with (14), but now set

$$\operatorname{vol}_d((1/\sqrt{2\pi})K + \Lambda B_d) = \int_{\mathbb{R}^d} 1 \left[\operatorname{dist}\left(x, (1/\sqrt{2\pi})K\right) \leq \Lambda \right] \mathrm{d}x.$$

Taking expectations and invoking Fubini gives us (13).

Now we make a change of variables $z = \sqrt{2\pi}\, x$ in (13) to get, equivalently,

$$\left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} e^{-(1/2)\operatorname{dist}^2(z,K)} \mathrm{d}z = \left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} e^{\sup_{t \in K}[\langle t, z\rangle - (1/2)\|t\|^2]} e^{-(1/2)\|z\|^2} \, \mathrm{d}z.$$

For an isonormal Gaussian process $\{X_t, t \in K\}$ given by $X_t = \langle t, Z \rangle$, where Z is d-dimensional standard normal, we have thus shown that

$$W(K) = \mathrm{E} \, e^{\sup_{t \in K}\left[X_t - (1/2)\sigma_t^2\right]}. \tag{15}$$

Extension of the domain of W to infinite-dimensional K is naturally done via finite-dimensional approximation as in (8). Representation (15), and also (12) in the form

$$W(K) = \sum_{j=0}^{\infty} (1/(2\pi)^{j/2}) V_j(K) \tag{16}$$

are maintained. Tsirelson [25] gave a proof of this using specifically polytopal approximants and a result of Chevet [2]. He further established, by inserting the domination (7) into (16), the inequality

$$W(K) \leq e^{(1/\sqrt{2\pi}) V_1(K)}, \tag{17}$$

or, equivalently,

$$\mathrm{E} \, e^{\sup_{t \in K}\left\{X_t - (1/2)\sigma_t^2\right\}} \leq e^{\mathrm{E} \sup_{t \in K} X_t} \tag{18}$$

([25], see also [17, 27, 28] and Remark 1 below). The latter guarantee that (15) and (16) are, in fact, finite for any GB set K and are interesting in their own right as well. In Sec. 6, we discuss variants.

The asymptotic form of $W(rK)$, $r \to \infty$, was studied in [29]. The context there (see also [11]) was a geometric treatment of the Itô–Nisio phenomenon [9] which showed that, in a weak sense, a local neighborhood of a discontinuity of $\{X_t, t \in K\}$ generically resembles a ball of small radius and high dimension. Relevant here is the following: For $t \in K$, let $B(t, \varepsilon)$ be the t-centered ball of radius ε and set

$$\delta(t) = \lim_{\varepsilon \to 0} \left[\sup_{s \in K \cap B(t,\varepsilon)} X_s - \inf_{s \in K \cap B(t,\varepsilon)} X_s \right].$$

Each of these limits is an almost sure constant. Considering them as numbers, set $\Delta(K) = \sup_{t\in K} \delta(t)$ (departing from convention, we regard this as over *all* $t \in K$). Then

$$W(rK) = e^{(\Delta(K)/2)r+o(r)}. \tag{19}$$

An important tool in [29] was a class of geometric parameters $\{m_j(K)\}_1^\infty$ such that

$$\mathrm{E}\sup_{t\in K} X_t = m_1(K) \geq \cdots \geq m_{j-1}(K) \geq m_j(K) \geq \cdots \to \Delta(K)/2. \tag{20}$$

In what follows, we further examine their structure and discuss related inequalities.

5. Quasi-widths

Following [29, 30], we set

$$m_j(K) = \frac{jV_j(K)}{\sqrt{2\pi}V_{j-1}(K)} \quad j = 1, 2, \ldots. \tag{21}$$

For each j, $m_j(rK)$ is homogeneous of degree 1 in r, and accordingly, we call it the *quasi-width of order* j. Note that

$$m_1(K) = (1/\sqrt{2\pi})V_1(K) = \mathrm{E}\sup_{t\in K} X_t \tag{22}$$

and that, as a consequence of (6), the quasi-widths form a decreasing sequence. For a further understanding, we derive an alternate expression to (21). In the numerator, recall that

$$V_j(K) = \frac{(2\pi)^{j/2}}{j!\,\kappa_j}\,\mathrm{E}\,\mathrm{vol}_j\left(Z_{[j,\infty]}K\right). \tag{23}$$

Similarly, in the denominator,

$$V_{j-1}(K) = \frac{(2\pi)^{(j-1)/2}}{(j-1)!\,\kappa_{j-1}}\,\mathrm{E}\,\mathrm{vol}_{j-1}(Z_{[j-1,\infty]}K), \tag{24}$$

which we re-express by noting that in distribution,

$$Z_{[(j-1),\infty]} = \Pi_{j-1} Z_{[j,\infty]},$$

where the independent matrix Π_{j-1} consists of the first $j-1$ rows of a random $j \times j$ orthogonal matrix. Then

$$\begin{aligned}\mathrm{E}\,\mathrm{vol}_{j-1}(Z_{[j-1,\infty]}K) &= \mathrm{E}\,\mathrm{vol}_{j-1}(\Pi_{j-1}Z_{[j,\infty]}K) \\ &= \mathrm{E}\{\mathrm{E}\left[\mathrm{vol}_{j-1}(\Pi_{j-1}Z_{[j,\infty]}K) \mid Z_{[j,\infty]}K\right]\}.\end{aligned} \tag{25}$$

Now *Kubota's integral recursion* [2, 19, 25] in the special case of *Cauchy's surface area formula* implies that

$$\mathrm{E}\,\mathrm{vol}_{j-1}(\Pi_{j-1}K_0) = \frac{2\kappa_{j-1}}{j\,\kappa_j}V_{j-1}(K_0) = \frac{\kappa_{j-1}}{j\kappa_j}S_{j-1}(K_0)$$

for j-dimensional K_0. Applying this to the inner expectation in the final expression in (25), we get the equality

$$\mathrm{E}\left[\mathrm{vol}_{j-1}(\Pi_{j-1}Z_{[j,\infty]}K) \mid Z_{[j,\infty]}K\right] = \frac{\kappa_{j-1}}{j\,\kappa_j}S_{j-1}(Z_{[j,\infty]}K).$$

It follows that

$$\mathrm{E}\,\mathrm{vol}_{j-1}(Z_{[j-1,\infty]}K) = \frac{\kappa_{j-1}}{j\,\kappa_j}\,\mathrm{E}\,S_{j-1}(Z_{[j,\infty]}K).$$

Inserting this into (24) gives us the equality

$$V_{j-1}(K) = \frac{(2\pi)^{(j-1)/2}}{j!\,\kappa_j}\,\mathrm{E}\left[S_{j-1}(Z_{[j,\infty]}K)\right]. \tag{26}$$

Substituting this and (23) into (21), we finally get the equality

$$m_j(K) = \frac{j \cdot \mathrm{E}\,\mathrm{vol}_j(Z_{[j,\infty]}K)}{\mathrm{E}\,S_{j-1}(Z_{[j,\infty]}K)} \ , \tag{27}$$

which was our goal. It expresses $m_j(K)$ in terms of the mean behavior of the single j-dimensional random convex body $Z_{[j,\infty]}K$.

6. A class of inequalities

We turn now to a generalization of (17). Specifically, following [30], we show that a class of bounds in terms of quasi-widths comes about by varying the domination (7); recall from (6) that $a_j = j!\,V_j(K),\ j = 0, 1, 2, \ldots$, is a log-concave sequence:

$$\log a_j \le \log a_i + (\log a_{i+1} - \log a_i)(j - i)$$

for all $i, j = 0, 1, 2, \ldots$. Equivalently, for any fixed $i \in \{0, 1, 2, \ldots\}$, this can be read as

$$V_j(K) \le \frac{i!\,V_i(K)}{j!}\left(\frac{(i+1)V_{i+1}(K)}{V_i(K)}\right)^{j-i} \qquad j = 0, 1, 2, \ldots. \tag{28}$$

It is of interest to re-express this. From (21), it follows that

$$\frac{(i+1)V_{i+1}(K)}{V_i(K)} = (2\pi)^{1/2} m_{i+1}(K), \tag{29}$$

and taking the product of (21) for $j = 1, 2, \ldots, i$ shows that

$$i!V_i(K) = (2\pi)^{i/2} \prod_{j=1}^{i} m_j(K). \tag{30}$$

Substituting (29) and (30) into (28) and re-arranging gives us the estimate

$$V_j(K) \le c_i(K) \cdot \frac{(2\pi)^{j/2}\, m_{i+1}^j(K)}{j!}, \tag{31}$$

where

$$c_i(K) = \frac{\prod_{j=1}^{i} m_j(K)}{m_{i+1}^i(K)} = \prod_{j=1}^{i} \frac{m_j(K)}{m_{i+1}(K)} \tag{32}$$

(taking $c_0(K) = 1$). Finally, substituting the domination (31) into (16) yields the estimate

$$W(K) \le c_i(K) e^{m_{i+1}(K)}, \quad i = 0, 1, 2, \ldots, \tag{33}$$

thus generalizing (17) (i.e., $i = 0$) to the other quasi-widths (we note that there is a minor typo in the corresponding expression in [30]).

A class of deviation bounds can also be deduced. First note that it follows from (33) with $r \ge 0$ that

$$W(rK) \le c_i(K) e^{m_{i+1}(K)r}, \quad i = 0, 1, 2, \ldots, \tag{34}$$

using the fact that $c_i(rK)$ is homogeneous of degree 0 in r. Then, following [27], one can re-express (34) as

$$\mathrm{E}\, e^{\sup_{t\in K}\left\{rX_t - r^2(1/2)\sigma_t^2\right\}} \le c_i(K) e^{m_{i+1}(K)r}. \tag{35}$$

Setting $\sigma^2 = \sup_{t\in K} \sigma_t^2$ and then re-arranging, we show that

$$\mathrm{E}\, e^{r\left[\sup_{t\in K} X_t - m_{i+1}(K)\right]} \le c_i(K) e^{(1/2)\sigma^2 r^2}. \tag{36}$$

Applying Markov's inequality, we see that

$$\mathrm{P}(\sup_{t\in K} X_t - m_{i+1}(K) \geq a) \leq c_i(K)e^{(1/2)\sigma^2 r^2 - ar}$$

for $a > 0$, and minimizing the bound at $r = a/\sigma^2$ finally yields the bound

$$\mathrm{P}(\sup_{t\in K} X_t - m_{i+1}(K) \geq a) \leq c_i(K)e^{-a^2/(2\sigma^2)}, \quad i = 0, 1, 2, \ldots. \tag{37}$$

The case $i = 0$ is well known in the probability literature in the form

$$\mathrm{P}(\sup_{t\in K} X_t - \mathrm{E}\sup_{t\in K} X_t \geq a) \leq e^{-a^2/(2\sigma^2)}$$

(see, e.g., [12, 14]), similarly shown in [27].

In a different vein, one can think of looking for bounds sharper than those in (33). One option is to express (31) for both $i \geq 1$ and $i - 1$. Then, for a given j, choose the domination that is tighter. This amounts to using the first domination for $j \geq i$ and the second for $j \leq i-1$ (the two being the same at $j = i$). That is,

$$V_j(K) \leq \begin{cases} c_{i-1}(K) \cdot \frac{(2\pi)^{j/2}\, m_i^j(K))}{j!}, & j = 0, 1, 2, \ldots, i-1, \\ c_i(K) \cdot \frac{(2\pi)^{j/2}\, m_{i+1}^j(K))}{j!}, & j = i, i+1, i+2, \ldots, \end{cases}$$

and, consequently,

$$W(K) \leq c_{i-1}(K) \sum_{j=0}^{i-1} \frac{m_i^j(K)}{j!} + c_i(K) \sum_{j=i}^{\infty} \frac{m_{i+1}^j(K)}{j!}.$$

Finally, echoing a comment in [30], we note that the natural way in which quasi-widths emerge in the derivation of (33), as well as their appearance in (20), suggests that they bear further examination as functionals of interest for both K and the process $\{X_t, t \in K\}$. In this regard, we mention as well the functionals $c_i(K), i = 1, 2, \ldots$, which, as noted, are homogeneous of degree 0 in r and thus can be regarded as "shape" parameters for K.

7. Final remarks

(1) A significant generalization of (18), including a left-tail probability bound, was shown by Borell [1].

(2) Following the above discussion, it is not possible to let $i \to \infty$ in (37), make use of (20), and produce the analogous statement with $m_i(K)$ replaced by $\Delta/2$; the reason is the absence of established control over the $c_i(K)$. However, a result of this type was shown in [29] using other means, in which the explicit intermediate estimates (35) and (37) are bypassed (note that the statement of Theorem 4 there has a typographical error ("=" should be "$\leq$"), and, in any case, does not always reflect the exact asymptotics as claimed (see, e.g., [13, 14]); the reader is also cautioned that in [29], the definition of "oscillation" carries a factor of 1/2 compared to the conventional definition).

(3) For additional geometric understanding of $m_2(K)$ (via $V_2(K)$), see [2] and [4].

(4) In view of the key role which (6) played in the discussion above, we note that it appeared in [18] as the *ultra-logconcavity of order* ∞ of the sequence $\{V_j(K)\}_1^\infty$. In that study (relating to negative dependence of random variables), the closure of the class of such sequences under convolution was conjectured. This was verified in [15] with a later, geometrically-based, proof in [5] using a theorem of Shephard [20] involving mixed volumes and a special case of (1).

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