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# ASYMPTOTIC BOUNDS FOR THE SOLUTIONS OF A FUNCTIONAL-DIFFERENTIAL EQUATION WITH LINEARLY TRANSFORMED ARGUMENT

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We establish new properties of solutions of a functional-differential equation with linearly transformed argument.

In the present paper, we investigate the equation

$$x'(t) = ax(t) + bx(qt) + cx'(qt) + f(t), \tag{1}$$

where $a \in \mathrm{R}$, $a \neq 0$, $\{b, c\} \subset \mathrm{C}$, $0 < q < 1$. For $c = 0$, this equation was studied in [1]. In what follows, the numbers $M_i$ are nonnegative constants.

We need the following lemma [1]:

**Lemma 1.** *Suppose that*

*(1)* $\mu < 0$, $\gamma > 0$, *and* $l$ *is a complex number such that* $|l| = e^{\gamma\mu}$;

*(2)* $W(s)$ *is a solution of the difference equation*

$$W(s) - lW(s + \mu) = G(s),$$

*where* $G(s)$ *is a continuous function such that*

$$|G(s)| \leq M_1 e^{-\beta s}, \quad s \geq s_0,$$

*for some positive quantities* $\beta$, $s_0$, *and* $|W(s)| \leq M_2$ *for* $s \in [s_0 + \mu, s_0]$.

*Then:*

*(i)* $|W(s)| \leq M_3 e^{-\gamma s}$, $s \geq s_0$ *for* $\gamma < \beta$;

*(ii)* $|W(s)| \leq M_3 s e^{-\gamma s}$, $s \geq s_0$ *for* $\gamma = \beta$;

*(iii)* $|W(s)| \leq M_3 e^{-\beta s}$, $s \geq s_0$ *for* $\gamma > \beta$.

We prove the following theorem:

**Theorem 1.** *Suppose that*

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*(1)* $a < 0$ *and* $bc \neq 0$;

*(2) for the parameter* $j \in \mathrm{N}\bigcup\{0\}$, *the inequality*

$$v_0 \stackrel{\text{df}}{=} \frac{\ln\left(\frac{|b|}{a}\right)}{\ln q^{-1}} \geq v_{\min} \stackrel{\text{df}}{=} \frac{\ln\left(\left|cq^j\right| q^{-1} + \frac{\left|bq^j + acq^j q^{-1}\right|}{a}\right)}{\ln q^{-1}}$$

*is true;*

*(3) the function* $f(t)$ *belongs to* $C^{j+1}[1, +\infty)$ *and* $f^{(m)}(t) = O\left(t^{\alpha - m}\right), t \to +\infty, \alpha \in \mathrm{R}, m = \overline{0, j+1}$.

*Then, for each* $j + 2$ *times continuously differentiable solution* $x(t)$ *of Eq. (1), the following estimates are true:*

*(i) if* $\alpha < v_0$, *then* $x(t) = O\left(t^{v_0}\right), t \to +\infty$;

*(ii) if* $\alpha = v_0$, *then* $x(t) = O\left(t^{v_0} \ln t\right), t \to +\infty$;

*(iii) if* $\alpha > v_0$, *then* $x(t) = O\left(t^{\alpha}\right), t \to +\infty$.

***Proof.*** Differentiating Eq. (1) $j$ times, we get

$$x^{(j+1)}(t) = ax^{(j)}(t) + bq^j x^{(j)}(qt) + cq^j x^{(j+1)}(qt) + f^{(j)}(t).$$

By the change of variables $x^{(j)}(t) = t^v y(t)$, where $v \stackrel{\text{df}}{=} \max\{v_{\min} + \varepsilon, \alpha - j\}$ and $\varepsilon > 0$ is an arbitrary number, we obtain

$$y'(t) = \left(a - \frac{v}{t}\right) y(t) + \left(bq^j q^v + \frac{vcq^j q^{v-1}}{t}\right) y(qt) + cq^j q^v y'(qt) + t^{-v} f^{(j)}(t),$$

$$y(t) = e^{a(t-t_0)}\left\{y(t_0) - cq^j q^{v-1} y(qt_0)\right\} + cq^j q^{v-1} y(qt)$$

$$+ \int_{t_0}^{t} e^{a(t-s)}\left\{\left(bq^j q^v + acq^j q^{v-1}\right) y(qs) - \frac{v}{s}\, y(s) + \frac{vcq^j q^{v-1}}{s}\, y(qs)\right\} ds$$

$$+ \int_{t_0}^{t} e^{a(t-s)} s^{-v} f^{(j)}(s)\, ds, \quad t_0 \geq 1.$$

By using the inequality $v \geq \alpha - j$ and conditions (i) and (iii), we estimate $|y(t)|$ for $t_0 \leq t \leq T$ as follows:

$$|y(t)| \leq e^{a(t-t_0)}\left|y(t_0) - cq^j q^{v-1} y(qt_0)\right| + |c| q^j q^{v-1} |y(qt)|$$

$$+ \int_{t_0}^{t} e^{a(t-s)}\left\{\left|bq^j q^v + acq^j q^{v-1}\right| |y(qs)| + \frac{|v|}{s}\, |y(s)| + \frac{|vc| q^j q^{v-1}}{s}\, |y(qs)|\right\} ds$$

$$+ \int_{t_0}^{t} e^{a(t-s)} \left| s^{-v} f^{(j)}(s) \right| ds \le e^{a(t-t_0)} \left| y(t_0) - cq^j q^{v-1} y(qt_0) \right|$$

$$+ |c| q^j q^{v-1} \sup_{qt_0 \le u \le T} |y(u)| + \int_{t_0}^{t} e^{a(t-s)} \left\{ \left| bq^j q^v + acq^j q^{v-1} \right| \sup_{qt_0 \le u \le T} |y(u)| \right.$$

$$\left. + \frac{|v|}{t_0} \sup_{qt_0 \le s \le T} |y(s)| + \frac{|vc| q^j q^{v-1}}{t_0} \sup_{qt_0 \le u \le T} |y(u)| \right\} ds + \frac{M_6}{|a|}$$

$$\le \left| y(t_0) - cq^j q^{v-1} y(qt_0) \right| + \left( |c| q^j q^{v-1} + \left| \frac{bq^j}{a} q^v + cq^j q^{v-1} \right| + \frac{|v|}{|a| t_0} + \frac{|vc| q^j q^{v-1}}{|a| t_0} \right)$$

$$\times \sup_{qt_0 \le u \le T} |y(u)| + \frac{M_6}{|a|} \le \left| y(t_0) - cq^j q^{v-1} y(qt_0) \right|$$

$$+ \left( |c| q^j q^{v-1} + \left| \frac{bq^j}{a} q^v + cq^j q^{v-1} \right| + \frac{|v|}{|a| t_0} + \frac{|vc| q^j q^{v-1}}{|a| t_0} \right) \sup_{qt_0 \le u \le t_0} |y(u)|$$

$$+ \left( |c| q^j q^{v-1} + \left| \frac{bq^j}{a} q^v + cq^j q^{v-1} \right| + \frac{|v|}{|a| t_0} + \frac{|vc| q^j q^{v-1}}{|a| t_0} \right) \sup_{t_0 \le u \le T} |y(u)| + \frac{M_6}{|a|}.$$

This yields

$$\sup_{t_0 \le t \le T} |y(t)| \le \left| y(t_0) - cq^j q^{v-1} y(qt_0) \right|$$

$$+ \left( |c| q^j q^{v-1} + \left| \frac{bq^j}{a} q^v + cq^j q^{v-1} \right| + \frac{|v|}{|a| t_0} + \frac{|vc| q^j q^{v-1}}{|a| t_0} \right) \sup_{qt_0 \le u \le t_0} |y(u)|$$

$$+ \left( |c| q^j q^{v-1} + \left| \frac{bq^j}{a} q^v + cq^j q^{v-1} \right| + \frac{|v|}{|a| t_0} + \frac{|vc| q^j q^{v-1}}{|a| t_0} \right) \sup_{t_0 \le u \le T} |y(u)| + \frac{M_6}{|a|}.$$

Since $v > v_{\min}$, the inequality

$$|c| q^j q^{v-1} + \left| \frac{bq^j}{a} q^v + cq^j q^{v-1} \right| = q^v \left( |c| q^j q^{-1} + \left| \frac{bq^j}{a} + cq^j q^{-1} \right| \right) < 1$$

is true and, for sufficiently large $t_0$, we get

$$|c| q^j q^{v-1} + \left| \frac{bq^j}{a} q^v + cq^j q^{v-1} \right| + \frac{|v|}{|a| t_0} + \frac{|vc| q^j q^{v-1}}{|a| t_0} < 1.$$

Therefore,

$$\sup_{t_0\le t\le T}|y(t)| \le \left\{1-|c|q^jq^{v-1}-\left|\frac{bq^j}{a}q^v+cq^jq^{v-1}\right|-\frac{|v|}{|a|t_0}-\frac{|vc|q^jq^{v-1}}{|a|t_0}\right\}^{-1}$$

$$\times\left\{\left|y(t_0)-cq^jq^{v-1}y(qt_0)\right|+\left(|c|q^jq^{v-1}+\left|\frac{bq^j}{a}q^v+cq^jq^{v-1}\right|\right.\right.$$

$$\left.\left.+\frac{|v|}{|a|t_0}+\frac{|vc|q^jq^{v-1}}{|a|t_0}\right)\sup_{qt_0\le u\le t_0}|y(u)|+\frac{M_6}{|a|}\right\}.$$

Since $T$ is arbitrary, we conclude that $y(t)=O(1)$, $t\to+\infty$, and $x^{(j)}(t)=t^vy(t)=O\left(t^v\right)$, $t\to+\infty$.

Performing the change of variables

$$x^{(j-1)}(t)=t^{v_*}y(t),$$

where $v_*\ge\max\{v,\alpha-(j-1)\}$ and

$$v_*>\frac{\ln\left(\frac{|bq^{j-1}|}{a}\right)}{\ln q^{-1}}=v_0-(j-1),$$

in the equation

$$x^{(j)}(t)=ax^{(j-1)}(t)+bq^{j-1}x^{(j-1)}(qt)+cq^{j-1}x^{(j)}(qt)+f^{(j-1)}(t),$$

$$x^{(j-1)}(t)=-\frac{b}{a}q^{j-1}x^{(j-1)}(qt)-\frac{c}{a}q^{j-1}x^{(j)}(qt)+\frac{1}{a}x^{(j)}(t)-\frac{1}{a}f^{(j-1)}(t)$$

$$\stackrel{\text{df}}{=}-\frac{b}{a}q^{j-1}x^{(j-1)}(qt)+g(t),$$

we obtain

$$y(t)=-\frac{b}{a}q^{j-1}q^{v_*}y(qt)+t^{-v_*}g(t).$$

We define an auxiliary coefficient

$$c_1\stackrel{\text{df}}{=}-\frac{b}{a}q^{j-1}q^{v_*}$$

and an inhomogeneity $g_1(t)\stackrel{\text{df}}{=}t^{-v_*}g(t)$ and estimate them, in view of the choice of $v_*$, as follows:

$$|g_1(t)|=O\left(t^{\max\{v,\alpha-(j-1)\}-v_*}\right)\le M_7,\quad t\to\infty,$$

$$|c_1| = \exp\left\{\left(\frac{\ln\left|\frac{bq^{j-1}}{a}\right|}{\ln q^{-1}} - v_*\right)\ln q^{-1}\right\} < 1.$$

Then

$$y(t) = c_1 y(qt) + g_1(t)$$

and, for $q^{-1} \le t \le T$, we find

$$|y(t)| \le |c_1|\,|y(qt)| + M_7 \le |c_1| \sup_{q^{-1}\le t\le T} |y(qt)| + M_7$$

$$\le |c_1| \sup_{1\le t\le q^{-1}} |y(t)| + |c_1| \sup_{q^{-1}\le t\le T} |y(t)| + M_7.$$

This yields

$$\sup_{q^{-1}\le t\le T} |y(t)| \le |c_1| \sup_{1\le t\le q^{-1}} |y(t)| + |c_1| \sup_{q^{-1}\le t\le T} |y(t)| + M_7,$$

$$\sup_{q^{-n-1}\le t\le T} |y(t)| \le (1-|c_1|)^{-1}\left(|c_1| \sup_{q^{-n}\le t\le q^{-n-1}} |y(t)| + M_7\right),$$

where $T$ is an arbitrary number. Hence,

$$x^{(j-1)}(t) = t^{v_*} y(t) = O\left(t^{v_*}\right), \quad t\to\infty,$$

where

$$v_* = \max\left\{v_0 - (j-1) + \varepsilon, \max\left\{v_{\min} + \varepsilon, \alpha - j\right\}, \alpha - (j-1)\right\}$$

$$= \max\left\{v_0 - (j-1) + \varepsilon, v_{\min} + \varepsilon, \alpha - (j-1)\right\}.$$

Repeating this process, we conclude that

$$x^{(j-2)}(t) = O\left(t^{\max\{v_0-(j-2)+\varepsilon;\max\{v_0-(j-1)+\varepsilon, v_{\min}+\varepsilon, \alpha-(j-1)\}, \alpha-(j-2)\}}\right)$$

$$= O\left(t^{\max\{v_0-(j-2)+\varepsilon, v_{\min}+\varepsilon, \alpha-(j-2)\}}\right), \quad t\to\infty.$$

After several repetitions, by the condition of the theorem $v_0 \ge v_{\min}$, we get

$$x(t) = O\left(t^{\max\{v_0+\varepsilon, v_{\min}+\varepsilon, \alpha\}}\right) = O\left(t^{\max\{v_0+\varepsilon, \alpha\}}\right) = O\left(t^{\max\{v_0, \alpha\}+\varepsilon}\right), \quad t\to\infty.$$

Similarly, the derivative can be estimated as follows:

$$x'(t) = O\left(t^{\max\{v_0-1+\varepsilon,\alpha-1\}}\right) = O\left(t^{\max\{v_0+\varepsilon,\alpha\}-1}\right) = O\left(t^{\max\{v_0,\alpha\}+\varepsilon-1}\right), \quad t \to \infty.$$

For the sake of brevity, we define $h \stackrel{\text{df}}{=} \max\{v_0,\alpha\}+\varepsilon$, $\mu = \ln q$ and perform the change of variables

$$t = e^s, \quad W(s) = t^{-h}x(t)$$

in Eq. (1). Then $x(t) = e^{hs}W(s)$ and

$$W(s) - \left(\frac{bq^h}{-a}\right)W(s+\mu) = a^{-1}\left(hW(s) + W'(s) - c\left\{he^{h\mu}W(s+\mu) + e^{h\mu}W'(s+\mu)\right\}e^{-\mu}\right)$$

$$\times e^{-s} - a^{-1}e^{-hs}f\left(e^s\right).$$

Since $W(s) = t^{-h}x(t) = e^{-hs}x\left(e^s\right) = O(1)$, $s \to +\infty$, we find

$$W'(s) = -he^{-hs}x\left(e^s\right) + e^{-hs}x'\left(e^s\right)e^s = O(1), \quad s \to +\infty.$$

Let

$$l \stackrel{\text{df}}{=} \frac{bq^h}{-a}$$

and

$$G(s) \stackrel{\text{df}}{=} a^{-1}\left(hW(s) + W'(s) - c\left\{he^{h\mu}W(s+\mu) + e^{h\mu}W'(s+\mu)\right\}e^{-\mu}\right)e^{-s} - a^{-1}e^{-hs}f\left(e^s\right).$$

Thus, we get

$$W(s) - lW(s+\mu) = G(s),$$

where

$$|l| = \left|\frac{bq^h}{-a}\right| = \left|\frac{bq^h}{bq^{v_0}}\right| = q^{h-v_0} = e^{(h-v_0)\mu}, \quad |G(s)| \le M_8e^{-s} + M_9e^{(\alpha-h)s}, \quad s \ge s_0 > 0.$$

The subsequent reasoning in the proof of Theorem 1 coincides with the proof presented in [1].

*Case (i)*: $\alpha < v_0$. If $v_0 < \alpha+1$, then we choose $h$ such that $v_0 < h < \alpha+1$. Then $|G(s)| \le (M_8+M_9)e^{(\alpha-h)s}$, $s \ge s_0$. We define

$$\gamma \stackrel{\text{df}}{=} h - v_0 \quad \text{and} \quad \beta \stackrel{\text{df}}{=} h - \alpha.$$

Thus, $\gamma < \beta$. By Lemma 1, we conclude that $|W(s)| \le M_{10}e^{(v_0-h)s}$, $s \ge s_0$, i.e., $x(t) = e^{hs}W(s) = O\left(t^{v_0}\right)$, $t \to +\infty$.

If $v_0 \geq \alpha + 1$, then we choose

$$h = v_0 + \frac{1}{2}.$$

Hence, $|G(s)| \leq (M_8 + M_9)e^{-s}$, $s \geq s_0$. We define

$$\gamma \stackrel{\text{df}}{=} h - v_0 \quad \text{and} \quad \beta \stackrel{\text{df}}{=} 1.$$

Then $\gamma < \beta$. By Lemma 1, we conclude that $|W(s)| \leq M_{11} e^{(v_0 - h)s}$, $s \geq s_0$, i.e., $x(t) = O\left(t^{v_0}\right), t \to +\infty$.

*Case (ii)*: $\alpha = v_0$. We choose

$$h = v_0 + \frac{1}{2}.$$

Then $|G(s)| \leq (M_8 + M_9)e^{(\alpha - h)s}$, $s \geq s_0$. By Lemma 1, we get

$$|W(s)| \leq M_{12} s e^{(v_0 - h)s}, \quad s \geq s_0,$$

i.e., $x(t) = O\left(t^{v_0} \ln t\right), t \to +\infty$.

*Case (iii)*: $\alpha > v_0$. We choose

$$h = \alpha + \frac{1}{2}.$$

Then $|G(s)| \leq (M_8 + M_9)e^{(\alpha - h)s}$, $s \geq s_0$ and

$$\gamma \stackrel{\text{df}}{=} h - v_0 > \beta \stackrel{\text{df}}{=} h - \alpha.$$

By Lemma 1, we find $|W(s)| \leq M_{13} e^{(\alpha - h)s}$, $s \geq s_0$, i.e., $x(t) = O\left(t^{\alpha}\right), t \to +\infty$.

Theorem 1 is proved.

**Theorem 2.** *If $a > 0$ and $f(t) = O\left(t^{\alpha}\right), t \to +\infty$, $\alpha \in$ R, then, for each continuously differentiable solution $x(t)$ of Eq. (1), the limit $\lim_{t \to \infty} e^{-at} x(t) \in$ C exists.*

***Proof.*** We now rewrite Eq. (1) in the form

$$\frac{d}{dt}\left\{e^{-at} x(t)\right\} = b e^{-at} x(qt) + c e^{-at} x'(qt) + e^{-at} f(t)$$

and integrate it. As a result, we obtain

$$e^{-at} x(t) = e^{-aq^{-n}} x\left(q^{-n}\right) + c q^{-1}\left\{e^{-a(1-q)t} e^{-aqt} x(qt) - e^{-aq^{-n}(1-q)} e^{-aq^{-(n-1)}} x\left(q^{-(n-1)}\right)\right\}$$

$$+ \left(b + acq^{-1}\right) \int\limits_{q^{-n}}^{t} e^{-a(1-q)s} e^{-aqs} x(qs)\, ds + \int\limits_{q^{-n}}^{t} e^{-as} f(s)\, ds.$$

We define

$$\sup_{t\in[q^{-n+1},q^{-n}]}\left|e^{-at}x(t)\right| \stackrel{\text{df}}{=} J_n.$$

Further, let $q^{-n} \le t \le q^{-n-1}$. We choose a sufficiently large number $n$ such that the inequality $aqt > \alpha \ln t$ is true. This gives

$$\left|e^{-at}x(t)\right| \le \left|e^{-aq^{-n}}x\left(q^{-n}\right)\right| + \left|\frac{c}{q}\right|\left\{e^{-a(1-q)t}\left|e^{-aqt}x(qt)\right|\right.$$

$$\left.+e^{-aq^{-n}(1-q)}\left|e^{-aq^{-(n-1)}}x\left(q^{-(n-1)}\right)\right|\right\}$$

$$+\left|b+acq^{-1}\right|\int_{q^{-n}}^{t}e^{-a(1-q)s}\left|e^{-aqs}x(qs)\right|ds + M_{14}\int_{q^{-n}}^{t}e^{-as}s^{\alpha}\,ds$$

$$\le J_n + 2\left|\frac{c}{q}\right|e^{-a(1-q)q^{-n}}J_n + \left|b+acq^{-1}\right|J_n\frac{e^{-a(1-q)q^{-n}}}{a(1-q)} + M_{14}\frac{e^{-a(1-q)q^{-n}}}{a(1-q)}$$

$$\le \max\{J_n, M_{14}\}\left\{1+\left(2\left|\frac{c}{q}\right| + \frac{\left|b+acq^{-1}\right|+1}{a(1-q)}\right)e^{-a(1-q)q^{-n}}\right\},$$

which yields the inequality

$$\max\{J_{n+1}, M_{14}\} \le \max\{J_n, M_{14}\}\left\{1+\left(2\left|\frac{c}{q}\right| + \frac{\left|b+acq^{-1}\right|+1}{a(1-q)}\right)e^{-a(1-q)q^{-n}}\right\}$$

and the estimate $x(t) = O\left(e^{at}\right)$, $t\to\infty$.

Hence, by using the identity

$$e^{-at_2}x(t_2) - e^{-at_1}x(t_1) = cq^{-1}\left\{e^{-a(1-q)t_2}e^{-aqt_2}x(qt_2) - e^{-a(1-q)t_1}e^{-aqt_1}x(qt_1)\right\}$$

$$+\left(b+acq^{-1}\right)\int_{t_1}^{t_2}e^{-a(1-q)s}e^{-aqs}x(qs)\,ds + \int_{t_1}^{t_2}e^{-as}f(s)ds$$

for some constant $M$ such that

$$\left|e^{-at}x(t)\right| \le M \quad \text{and} \quad |f(t)| \le Mt^{\alpha} \le Me^{aqt}, \quad t \ge q^{-n+1},$$

we arrive at the inequality

$$\left|e^{-at_2}x(t_2) - e^{-at_1}x(t_1)\right| \le \left(|c|q^{-1}\left\{e^{-a(1-q)t_2} + e^{-a(1-q)t_1}\right\}\right.$$

$$\left. + \left\{\left|b + acq^{-1}\right| + 1\right\} \frac{e^{-a(1-q)t_1} - e^{-a(1-q)t_2}}{a(1-q)}\right)M.$$

By the Cauchy principle, the limit $\lim_{t\to\infty} e^{-at}x(t) \in \mathrm{C}$ exists.

Theorem 2 is proved.

If a partial solution of Eq. (1) is known, then the difference between the required and partial solutions is the solution of the homogeneous equation whose properties were studied in [2, 3].

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