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ANALYSIS OF THE PROBLEM OF STABILITY OF THIN SHELLS COMPLIANT TO SHEAR AND COMPRESSION

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The problem of stability of shells compliant to shear and compression is studied by the finite-element method. On the basis of relations of the geometrically nonlinear theory of thin shells compliant to shear and compression (six-mode version), we write the key equations for the determination of their initial postcritical state and formulate the corresponding variational problem. A numerical scheme of the finite-element method is constructed for the solution of the problems of stability of these shells. The order of the rate of convergence of the scheme proposed for the numerical solution of the problems of stability is investigated.

The nonlinear theory of shells [6, 11] gives a key to the explanation of the process of loss of their stability. The problems of stability of thin-walled shells were discussed in [2–5, 10, 12] and remain to be extremely important from the practical viewpoint.

In the present paper, we write the key relations for the determination of the initial postcritical state of flexible shells compliant to shear and compression with the use of the finite-element method. As a specific feature of the mathematical model, we can mention the procedure of semidiscretization of the vector of displacements in elastic bodies with respect to the thickness coordinate performed on the basis of the Timoshenko–Mindlin kinematic hypotheses with preservation of the total vector of rotations of the normal to the median surface.

1. Principal Relations of the Theory of Thin Shells Compliant to Shear and Compression

We consider a shell as a three-dimensional body of constant thickness h. We refer the median surface Ω of the shell to a curvilinear orthogonal coordinate system $\alpha = (\alpha_1, \alpha_2)$ and introduce a variable α_3 orthogonal to the surface and such that $|\alpha_3| \le h/2$. We assume that the coordinate lines of the median surface coincide with the lines of principal curvatures and that the thickness of the shell is much smaller than the other its sizes.

The vector of displacements of an arbitrary point of the shell compliant to shear and compression is completely defined by the components of the vector of displacements $u_i(\alpha)$, $i = 1, 2, 3$, and the vector of angles of rotation of normal to the median surface of the shell $\gamma_i(\alpha)$, $i = 1, 2, 3$. We introduce the following notation:

$u = (u_1, u_2, u_3, \gamma_1, \gamma_2, \gamma_3)^\top$ is the vector of generalized displacements of points of the median surface of the shell,

$e_L = (e_{11}, e_{22}, e_{33}, e_{12}, e_{13}, e_{23}, \kappa_{11}, \kappa_{22}, \kappa_{12}, \kappa_{13}, \kappa_{23})^\top$ is the vector of components of the linear strain tensor,

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$\omega = \left(\overset{0}{\omega}_1, \overset{0}{\omega}_2, \overset{0}{\omega}_3, \overset{1}{\omega}_1, \overset{1}{\omega}_2, \overset{1}{\omega}_3 \right)^\top$ is the vector of components of the tensor of rotations,

and

$\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}, \chi_{11}, \chi_{22}, \chi_{12}, \chi_{13}, \chi_{23})^\top$ is the vector of components of the Green strain tensor.

Then the formulas for the components of the linear strain tensor and the tensor of rotations can be represented in the matrix form as follows (to within $o(h)$):

$$e_L = C_L u\,, \qquad \omega = C_\Omega u\,. \tag{1}$$

The strain relations written for flexible shells with regard for the linear and nonlinear components of strains take the following form:

$$\varepsilon = e_L + e_N\,, \tag{2}$$

where

$$e_N = \frac{1}{2}(C_\Omega u)_{11}^\top E_\Omega (C_\Omega u)\,.$$

Here, C_L and C_Ω are, respectively, the 11×6- and 6×6-dimensional matrices of differential operators and E_Ω is a matrix of the form $E_\Omega = (E_1, E_2, \ldots, E_{11})^\top$, where E_i are 6×6-dimensional matrices. The complete formula for C_L is presented in [1], whereas the expressions for C_Ω and E_i can be found in [9].

Note that relations (1) are determined by the geometric relations of the theory of shells compliant to shear and compression in the linear statement, whereas relations (2) connect the components of the Green strain tensor with displacements in the geometrically nonlinear statement of the problem for the analyzed shells.

The elasticity relations connecting strains with internal forces and moments can be represented in the matrix form as follows:

$$\sigma = B\varepsilon\,,$$

where

$$\sigma = (N_{11}, N_{22}, N_{33}, S, N_{13}, N_{23}, M_{11}, M_{22}, H, M_{13}, M_{23})^\top$$

is the vector of internal (symmetric) forces (moments) and B is a symmetric 11×11-dimensional matrix of elasticity characteristics of the material [1].

The differential equations used to describe the equilibrium of the deformed body and the static boundary conditions imposed on a part Γ_σ of the contour of the median surface of the shell $\Gamma = \Gamma_u \cup \Gamma_\sigma$ follow from the principle of admissible displacements [6] and can be represented in the matrix form as follows:

$$C_\sigma \sigma^* + P = 0, \tag{3}$$

$$G_\sigma \sigma^* \Big|_{\Gamma_\sigma} = \sigma_g . \tag{4}$$

In order to make the system kinematically determined, it is necessary to supplement it with the boundary conditions in displacements:

$$G_u u \Big|_{\Gamma_u} = u_g , \qquad \Gamma_u = \Gamma \setminus \Gamma_\sigma . \tag{5}$$

In relations (3)–(5), we have introduced the following notation:

$P = (P_1, P_2, P_3, m_1, m_2, m_3)^\top$ is the vector of external load,

$\sigma^* = \left(N_{11}^*, N_{22}^*, N_{33}^*, N_{12}^*, N_{21}^*, N_{13}^*, N_{31}^*, N_{23}^*, N_{32}^*, M_{11}^*, M_{22}^*, M_{12}^*, M_{21}^*, M_{13}^*, M_{23}^*\right)^\top$ is the vector of introduced forces (moments),

$\sigma_g = (N_t, N_s, N_n, M_t, M_s, M_n)^\top$ is the vector of boundary forces (moments),

$u_g = (u_t^b, u_s^b, u_n^b, \gamma_t^b, \gamma_s^b, \gamma_n^b)^\top$ is the vector of boundary displacements,

C_σ is a 6×15 matrix of differential operators, and G_σ and G_u are 6×15- and 6×6-dimensional matrices, respectively. The complete formulas for the matrices C_σ, G_σ, and G_u can be found in [9].

The relationship between the symmetric forces (moments) and their characteristics introduced above can be represented in the matrix form as follows:

$$\sigma^* = F\sigma ,$$

where F is a 15×11 matrix whose nonzero elements are presented in [9].

The linear formulation of the equilibrium equations and the corresponding boundary conditions of the theory of shells compliant to shear and compression can be found in [1].

2. Problem of Stability of the Shells Compliant to Shear and Compression

To formulate the problem of stability of the proposed mathematical model of shells, we use the energy criterion of stability and the critical load corresponding to the loss of stability [7].

The condition for the determination of the critical load under which the stable equilibrium state turns into the unstable state has the form

$$\delta^2 \Pi = 0 , \tag{6}$$

where Π is the functional of total potential energy in the geometrically nonlinear theory of shells compliant to shear and compression [7]:

$$\Pi(u) = \frac{1}{2}\iint_{\Omega} \varepsilon^{\top}(u)E_0 B\varepsilon(u)\,d\Omega - \iint_{\Omega} u^{\top}P\,d\Omega - \int_{\Gamma_\sigma} (G_u u)^{\top}\sigma_g\,d\Gamma_\sigma .$$

The functional Π is defined as the difference between the functionals of strain energy U and the work of external forces A:

$$\Pi = U - A .$$

Note that $\delta^2\Pi = \delta^2 U$.

The total displacements u_* in the initial postcritical state are determined as the sum of displacements in the initial (subcritical) state u_0 and the perturbed displacements u:

$$u_* = u_0 + \alpha u .$$

Here, α is a small parameter, $0 < \alpha \ll 1$.

The strain energy U_* in the initial postcritical state is given by the formula

$$\begin{aligned} U_* &= \frac{1}{2}\iint_{\Omega} \varepsilon_*^{\top}(u)E_0 B\varepsilon_*(u)\,d\Omega \\ &= \frac{1}{2}\iint_{\Omega} (e_L(u_0) + \alpha e_L(u) + \alpha^2 e_N(u))^{\top} E_0(\sigma_0 + \alpha\sigma_L(u) + \alpha^2\sigma_N(u))\,d\Omega \\ &= U_0 + \alpha U_1 + \alpha^2 U_2 + \dots . \end{aligned}$$

Here, ε_* are the strains in the initial postcritical state determined as the sums of linear subcritical strains and nonlinear strains caused by the perturbed displacements,

$$U_0 = \frac{1}{2}\iint_{\Omega} e_L^{\top}(u_0)E_0\sigma_0\,d\Omega ,$$

$$U_1 = \iint_{\Omega} e_L^{\top}(u)E_0\sigma_0\,d\Omega ,$$

$$U_2 = \frac{1}{2}\iint_{\Omega} \left(e_L^{\top}(u)E_0\sigma_L(u) + 2e_N^{\top}(u)E_0\sigma_0 \right) d\Omega ,$$

where σ_0 are the stresses in the subcritical state and σ_L and σ_N are the stresses caused by perturbed linear strains e_L and rotations ω, respectively.

The functional U_2 contains the terms quadratic in the perturbed displacements and its second variation yields the stability equation (6).

By using the finite-element approximation

$$u = Nq,$$

where q is the vector of required displacements and rotations at all nodes of the finite elements and N is a block-diagonal matrix of approximating polynomials, we represent the functional U_2 as follows:

$$U_2 = \frac{1}{2}\iint_{\Omega} q^{\top}(C_L N)^{\top} E_0 B (C_L N) q \, d\Omega$$

$$+ \iint_{\Omega} q^{\top}\left(\sum_{k=1}^{11}(E_0\sigma_0)_k (C_\Omega N)_{11}^{\top} E_k C_\Omega N\right)^{\top} q \, d\Omega . \tag{7}$$

Since the subcritical state is determined by the linear theory, the integral characteristics σ_0 are proportional to the loading parameter λ:

$$\sigma_0 = \lambda \sigma_0^*, \tag{8}$$

where σ_0^* is the level of stresses under a given external load.

In view of relation (8), we get the following equation of stability from (7):

$$\iint_{\Omega}(C_L N)^{\top} E_0 B C_L N q \, d\Omega + \lambda \iint_{\Omega}\sum_{k=1}^{11}(E_0\sigma_0)_k (C_\Omega N)_{11}^{\top} E_k C_\Omega N q \, d\Omega = 0 .$$

This equation can be represented in the matrix form as

$$K_T(0)q + \lambda G(q_0) q = 0 , \tag{9}$$

where $K_T(q) = K_U(q) + G(q)$ is the matrix of tangential stiffness,

$$K_U(q) = \int_{\Omega}\big((C_L + (C_\Omega N q)_{11}^{\top} E_\Omega C_\Omega) N\big)^{\top} E_0 B (C_L + (C_\Omega N q)_{11}^{\top} E_\Omega C_\Omega) N \, d\Omega$$

is the matrix of displacements,

$$G(q_0) = \iint_{\Omega}\sum_{k=1}^{11} T_k(Nq_0)(C_\Omega N)_{11}^{\top} E_k C_\Omega N \, d\Omega$$

is the geometric matrix of stiffness or the matrix of initial stresses, q_0 is the required vector of displacements in the linear static problem, and

$$T(q) = (T_1, T_2, \ldots, T_{11})^\top = E_0 B\left(C_L N q + \frac{1}{2}(C_\Omega N q)_{11}^\top E_\Omega C_\Omega N q \right).$$

The least eigenvalue of Eq. (9) determines the critical loading parameter λ^* for which the shell passes from the stable initial equilibrium state into a neighboring equilibrium state.

We now introduce the space of kinematically admissible vectors of displacements

$$V = \left\{ v = (v_1, v_2, v_3, \xi_1, \xi_2, \xi_3) \in \left[W_2^1(\Omega) \right]^6 \middle| \, v = 0 \quad \text{on} \quad \Gamma_u \right\}$$

and the forms

$$a(u, v) = \iint_\Omega (C_L v)^\top E_0 B C_L u \, d\Omega ,$$

$$g(u, v) = \iint_\Omega \sum_{k=1}^{11} c_k (C_\Omega v)_{11}^\top E_k C_\Omega u \, d\Omega , \quad c_k = (E_0 \sigma_0)_k .$$

Further, we formulate the variational problem of stability:

Given:

$\sigma_0 = B e_L(u_0)$, *where* u_0 *is a solution of the linear variational problem.*

It is necessary to find:

a pair $\{u, \lambda\} \in V \times R$, $\|u\|_V = 1$, *such that* $a(u, v) + \lambda g(u, v) = 0 \quad \forall\, u, v \in V$.

The scheme of solution of the problems of stability of shells by the finite-element method is realized in the form of a problem-oriented software.

3. Numerical Example

Consider the problem of determination of the critical loads in the case of axisymmetric buckling of a circular plate with radius R and thickness h restrained on its contour and subjected to the action of radial compressive forces P uniformly distributed along the contour (Fig. 1). It is assumed that the points of the contour may freely move in the plane of the plate and its bent surface is axially symmetric.

We now compare the results of numerical and analytic calculations of the critical load P_{cr} for this problem in the case where Young's modulus of the material of the plate $E = 0.625 \cdot 10^{11}$ N/m^2, its Poisson's ratio $\nu = 0.22$, and $h/R = 1/20$ ($h = 0.5$ m, $R = 10$ m). The analytic value of the critical load determined according to the Kirchhoff–Love theory is given in [2]: $P_{\text{cr}} \cdot 10^{-8} = 1.0043391$.

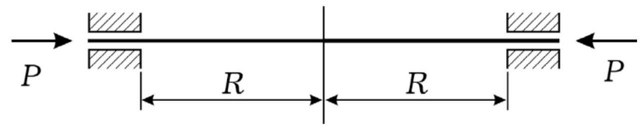

Fig. 1. Circular plate restrained along the contour under the action of radial compression.

Table 1

Partition	$P_{cr} \cdot 10^{-8}$ (five-mode version)	$P_{cr} \cdot 10^{-8}$ (six-mode version)	$k(\varepsilon)$
4×1	1.0222421	1.0271414	2.60
8×1	1.0045389	1.0055348	3.32
16×1	1.0023116	1.0019751	3.77
32×1	1.0021293	1.0016176	3.95
64×1	1.0021170	1.0015914	4.08
128×1	1.0021162	1.0015897	–
256×1	1.0021161	1.0015896	–

In Table 1, we compare the results of numerical calculations of P_{cr} for this problem performed by using a five-mode version of the theory of shells of the Timoshenko–Mindlin type and a six-mode version of the theory of shells compliant to shear and compression.

In order to find the orders of convergence rates, we use the formula

$$k(\varepsilon) = \log_2\left(\frac{f_\varepsilon - f_{\varepsilon/2}}{f_{\varepsilon/2} - f_{\varepsilon/4}}\right),$$

where f_ε, $f_{\varepsilon/2}$, and $f_{\varepsilon/4}$ are the values of approximate solutions obtained on the grids with steps of ε, $\varepsilon/2$, and $\varepsilon/4$, respectively. The computed values of the rates of convergence in the space variable are in good agreement with the corresponding theoretical values presented in [8].

By analyzing the results presented in Table 1, we conclude that the critical load required for the loss of stability of a circular plate decreases if the presence of compression is taken into account.

CONCLUSIONS

The performed numerical experiments, the comparative analysis of the accumulated numerical results obtained with regard for the compliance to shear and compression and the classical results available from the literature, as well as the investigation of the order of convergence rate for the applied method imply that the proposed method aimed at the numerical solution of the problems of stability in the theory of shells compliant to shear and compression enables one to get reliable results. In the future, it is reasonable to perform the analysis of the critical loads for shells with more complicated geometry.

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