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INVERSE SOURCE AND COEFFICIENT PROBLEMS FOR ELLIPTIC AND PARABOLIC EQUATIONS IN HÖLDER AND SOBOLEV SPACES

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We review some results obtained by the authors during the last 15 years. In particular, we present the existence and uniqueness theorems for linear and nonlinear inverse problems of reconstructing unknown coefficients in elliptic and parabolic equations. Bibliography: 48 *titles.*

In memory of Andrei Vasil'evich Bitsadze

The theory of inverse problems for equations in mathematical physics is a rapidly developed in the last decades field of mathematics. Unlike the classical boundary value problems, which are well studied and the solvability conditions for which are known, the situation with inverse problems is much more complicated. Even the formulation of such problems often requires additional investigations, in particular, the study of differential properties of solutions to the direct problems. This is especially evident in the case of nonlinear problems, where for obtaining solvability results it is necessary to trace attentively an exact dependence of differential properties of the solution to the direct problem on the smoothness of coefficients and other data of the problem (cf., for example, [1] in the elliptic case).

In this paper, we give a survey of some results obtained by the authors during the last fifteen years (cf. [2]–[13]). We deal with multidimensional inverse problems for elliptic and parabolic

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equations. Unfortunately, limitations on the length of the paper do not allow us to describe the results of other authors regarding problems in close settings. We also note that the list of references does not pretend to be complete, but more reflects interests of the authors.

We formulate some results on the existence and uniqueness of a solution of the spatial variables to the inverse source problem and the inverse coefficient problem. As a rule, the solvability theory for these problems is more complicated than that for the problems of reconstructing a scalar function of one variable. In the case of linear elliptic and parabolic problems, the inverse problem is equivalently reduced to an operator equation of the second kind with a compact operator. For nonlinear inverse (coefficient) problems the global sufficient conditions for the existence and uniqueness of a solution are obtained in the corresponding classes of functions.

1 Inverse Problems for Elliptic Equations in Hölder Spaces

1.1. The linear problem with source in the elliptic equation. Assume that, in the space $\mathbb{R}^n$ of points $x = (x_1, \dots, x_n)$, we are given a bounded domain D with sufficiently smooth boundary $\partial D \in C^{2,\alpha}$, where $0 < \alpha < 1$ is a fixed constant. Let the space $\mathbb{R}^n$ be embedded into the space $\mathbb{R}^{n+1}$ of points $(y, x) = (y, x_1, \dots, x_n)$. For $q > 0$ and $q_1 < 0 < q_2$ we introduce the cylinders in $\mathbb{R}^{n+1}$

$$Q(q_1, q_2) := \{(y, x) \in \mathbb{R}^{n+1} \mid q_1 < y < q_2, x \in D\}, \quad Q(q) := Q(-q, q)$$

with base D and lateral surfaces

$$\Gamma(q_1, q_2) := \{(y, x) \in \mathbb{R}^{n+1} \mid q_1 < y < q_2, x \in \partial D\}, \quad \Gamma(q) := \Gamma(-q, q).$$

Definition 1. We say that a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies *Condition* (A) if it satisfies the exterior cone condition and there exist numbers p, q, $0 < q < p$, such that $Q(q) \subset \Omega \subset Q(p)$. If, in addition, $Q(q_1, q_2) \subset \Omega$, then we say that Condition (A) holds with the cylinder $Q(q_1, q_2)$ and write $\overline{q} = \min\{|q_1|, q_2\}$.

Definition 2. Let a domain Ω satisfy Condition (A). We define the sets of Hölder functions

$$\mathscr{U}(\Omega) := \{u \in C(\overline{\Omega}) \mid \exists q > 0 \ u \in C^{2,\alpha}(\Omega \cup \Gamma(q))\},$$

$$\mathscr{F}(D) := C^{\alpha}(\overline{D}),$$

$$\mathscr{G}(\Omega) := \{g \in C(\overline{\Omega}) \mid \exists q > 0 \ g \in C^{\alpha}(\Omega) \cap C^{\alpha}(\overline{Q}(q))\},$$

$$\mathscr{M}(\partial\Omega) := \{\mu \in C(\partial\Omega) \mid \exists q > 0 \ \mu \in C^{2,\alpha}(\Gamma(q))\},$$

$$\mathscr{R}(\Omega) := \{(g, \mu, \chi) \mid g \in \mathscr{G}(\Omega), \ \mu \in \mathscr{M}(\partial\Omega), \ \chi \in C^{2,\alpha}(\overline{D}), \ \chi(x) = \mu(0, x), \ x \in \partial D\}.$$

In a domain Ω satisfying Condition (A), we consider the inverse problem for finding a pair of functions $(u, f) \in \mathscr{U}(\Omega) \times \mathscr{F}(D)$ from the conditions

$$(Lu)(y, x) = f(x)h(y, x) + g(y, x), \quad (y, x) \in \Omega, \tag{1}$$

$$u(y, x) = \mu(y, x), \quad (y, x) \in \partial\Omega, \quad u(0, x) = \chi(x), \quad x \in \overline{D}. \tag{2}$$

In Equation (1), L is a uniformly elliptic operator in Ω of the form

$$(Lu)(y,x) = a(y,x)\frac{\partial^2 u}{\partial y^2} + \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(y,x)u. \tag{3}$$

For the problem (1), (2) the Fredholm alternative holds.

Theorem 1. *Let a domain Ω satisfy Condition (A), and let the coefficients of the operator L and the function h satisfy the conditions*

$$a_{ij}, b_i \in C^{\alpha}(\overline{D}), \quad a, a_y, c, c_y, h, h_y \in C^{\alpha}(\overline{\Omega}),$$

$$c(y,x) \leqslant 0, \quad |h(0,x)| \geqslant h_0 > 0, \ (y,x) \in \Omega.$$

Then for the problem (1), (2) *one of the following assertions holds*:

1) *the homogeneous inverse problem* (1), (2), *i.e., the inverse problem* (1), (2) *with $g = \mu = \chi = 0$, has finitely many linearly independent solutions,*

2) *for arbitrary functions $g \in C^{\alpha}(\overline{\Omega})$, $\mu \in C^{2,\alpha}(\partial\Omega)$, $\chi \in C^{2,\alpha}(\overline{D})$, $\chi(x) = \mu(0,x)$, $x \in \partial\Omega$, the inverse problem* (1), (2) *has a unique solution $(u,f) \in \mathscr{U}(\Omega) \times \mathscr{F}(D)$.*

Theorem 1 leads to a number of sufficient unique solvability conditions for inverse problems of the form (1), (2). In what follows, the norm symbol without indices means the sup-norm. We introduce the notation and inequalities used below:

$$\frac{1}{a(x)} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \lambda_0|\xi|^2, \quad x \in D, \tag{4}$$

$$\beta = \frac{1}{\lambda_0}\left\|\frac{b}{a}\right\|; \quad \gamma = \max\left\{\left\|\frac{h(q_1,\cdot)}{h(0,\cdot)}\right\|, \left\|\frac{h(q_2,\cdot)}{h(0,\cdot)}\right\|\right\}. \tag{5}$$

Theorem 2. *Let a domain Ω satisfy Condition (A) with cylinder $Q(q_1,q_2)$, and let $\overline{q} = \min\{|q_1|, q_2\}$. Assume that an operator L of the form* (3) *with coefficients $a = a(x)$, $c = c(x)$ and the function $h(y,x)$ satisfy the conditions*

$$a, a_{ij}, b_i, c \in C^{\alpha}(\overline{D}), \quad h, h_y, h_{yy} \in C^{\alpha}(\overline{\Omega}),$$

$$c(x) \leqslant 0, \quad |h(0,x)| \geqslant h_0 > 0, \ x \in D;$$

moreover, the inequality (4) *holds with a constant $\lambda_0 > 0$ and the constant γ in* (5) *is such that $\gamma < 1$. Let, in addition, at least one of the following two conditions be satisfied*:

1) *the domain D for some $i \in \{1,\ldots,n\}$ lies in the strip $0 < x_i < l_i < l_*$, where*

$$l_* = \frac{1}{\beta+1}\ln\left(1 + \frac{\overline{q}^2(1-\gamma)\lambda_0}{16\left\|\frac{h(\cdot,\cdot)}{h(0,\cdot)}\right\| + \overline{q}^2\left\|\frac{h_{yy}(\cdot,\cdot)}{h(0,\cdot)}\right\|}\right),$$

2) *the coefficient $c(x)$ satisfies the inequality*

$$\frac{c(x)}{a(x)} \leqslant \varkappa < -\frac{1}{\overline{q}^2(1-\gamma)}\left(16\left\|\frac{h(\cdot,\cdot)}{h(0,\cdot)}\right\| + \overline{q}^2\left\|\frac{h_{yy}(\cdot,\cdot)}{h(0,\cdot)}\right\|\right).$$

Then for any triple of functions $(g,\mu,\chi) \in \mathscr{R}(\Omega)$ the inverse problem (1), (2) *has a unique solution $(u,f) \in \mathscr{U}(\Omega) \times \mathscr{F}(D)$.*

In the case where Ω is a cylinder with base D, the following global uniqueness and existence theorem is valid.

Theorem 3. *Let $\Omega = Q(q_1, q_2)$, and let a uniformly elliptic operator L of the form* (3) *with coefficients $a = a(x)$, $c = c(x)$ and the function $h(y, x)$ satisfy the conditions*

$$a, a_{ij}, b_i, c \in C^\alpha(\overline{D}), \quad h, h_y, h_{yy} \in C^\alpha(\overline{\Omega}),$$

$$h(y, x) \geqslant 0, \quad h_{yy}(y, x) \leqslant 0, \ (y, x) \in \Omega,$$

$$c(x) \leqslant 0, \quad h(0, x) \geqslant h_0 > 0, \ x \in D.$$

Then for any triple of functions $(g, \mu, \chi) \in \mathscr{R}(\Omega)$ the inverse problem (1), (2) *has a unique solution $(u, f) \in \mathscr{U}(\Omega) \times \mathscr{F}(D)$.*

To formulate the unique solvability theorem for the inverse problem (1), (2) in the important particular case where Ω is a symmetric cylinder relative to the plane $y = 0$, we introduce the notation. Let $B_R = \{x \in \mathbb{R}^n \mid |x| < R\}$ be a ball in $\mathbb{R}^n$ such that $\overline{D} \subset B_R$. Let $\Omega = (-q, q) \times D$ be a symmetric cylinder in $\mathbb{R}^{n+1}$ with respect to the plane $y = 0$. Denote $\Omega_R = [-q, q] \times B_R$. In this notation, we have the following existence and uniqueness result for the problem (1), (2) with an operator L of the form (3) under the additional symmetry condition on the coefficients of the operator:

$$a(y, x) = a(-y, x), \quad c(y, x) = c(-y, x), \quad (y, x) \in \Omega. \tag{6}$$

An important role in this result is played by a y-even component of the function $h(y, x)$:

$$h_e(y, x) = \frac{h(y, x) + h(-y, x)}{2}, \quad (y, x) \in [-q, q] \times \overline{D}.$$

Theorem 4. *Let the coefficients of a uniformly elliptic operator L of the form* (3) *in Ω and the function $h(y, x)$ satisfy* (6) *and the following conditions*:

$$a, a_y, c, c_y \in C^\alpha(\Omega_R), \quad a_{ij}, b_i, \in C^\alpha(B_R), \quad h, h_y \in C^\alpha(\overline{\Omega}),$$

$$c(y, x) \leqslant 0, \quad (y, x) \in \Omega,$$

$$h_e(y, x)(h_e)_y(y, x) \geqslant 0, \quad c_y(y, x) \geqslant 0, \quad (y, x) \in [-q, 0] \times B_R,$$

$$|h(0, x)| \geqslant h_0 > 0, \quad x \in D.$$

Then for any triple of functions $(g, \mu, \chi) \in \mathscr{R}(\Omega)$ the inverse problem (1), (2) *has a unique solution $(u, f) \in \mathscr{U}(\Omega) \times \mathscr{F}(D)$.*

As a consequence of Theorems 1–4 on the solvability of the inverse problem with overdetermination inside the domain, we formulate the results on the solvability of inverse problems with overdetermination on the boundary. We introduce additional notation and definitions. For $q > 0$ we introduce the sets

$$Q_1(q) = \{(y, x) \in \mathbb{R}^{n+1} \mid -q < y < 0, x \in D\},$$

$$\Gamma_1(q) = \{(y, x) \in \mathbb{R}^{n+1} \mid -q < y \leqslant 0, x \in \partial D\},$$

$$\Gamma_0 = \{(y, x) \in \mathbb{R}^{n+1} \mid y = 0, x \in D\}.$$

Definition 3. We say that a domain $\Omega_- \subset \mathbb{R}^{n+1}$ satisfies Condition (B) if it satisfies the exterior cone condition and there exist numbers p, q, $0 < q < p$, such that $Q_1(q) \subset \Omega_- \subset Q_1(p)$.

Definition 4. Let a domain Ω_- satisfy Condition (B). We define the sets of Hölder functions

$$\mathscr{U}_1(\Omega_-) := \{u \in C(\overline{\Omega}_-) \mid \exists q > 0 \ u \in C^{2,\alpha}(\Omega_- \cup \Gamma_1(q) \cup \Gamma_0)\},$$

$$\mathscr{G}(\Omega_-) := \{g \in C(\overline{\Omega}_-) \mid \exists q > 0 \ g \in C^{\alpha}(\Omega_-) \cap C^{\alpha}(\overline{Q}_1(q))\},$$

$$\mathscr{M}(\partial\Omega_-) := \{\mu \in C(\partial\Omega_-) \mid \exists q > 0 \ \mu \in C^{2,\alpha}(\Gamma_1(q)), \ \mu_y(0,x) = 0, x \in \partial D\},$$

$$\mathscr{R}(\Omega_-) := \{(g, \mu, \chi) \mid g \in \mathscr{G}(\Omega_-), \ \mu \in \mathscr{M}(\partial\Omega_-), \ \chi \in C^{2,\alpha}(\overline{D}), \ \chi(x) = \mu(0,x), \ x \in \partial D\}.$$

We consider the inverse problem of finding a pair of functions $(u, f) \in \mathscr{U}_1(\Omega_-) \times \mathscr{F}(D)$ from the conditions

$$(Lu)(y,x) = f(x)h(y,x) + g(y,x), \quad (y,x) \in \Omega_-, \tag{7}$$

$$\begin{cases} u(y,x) = \mu(y,x), & (y,x) \in \partial\Omega_- \setminus \Gamma_0, \\ u_y(0,x) = 0, & x \in D, \\ u(0,x) = \chi(x), & x \in \overline{D}. \end{cases} \tag{8}$$

If $g = 0$, $\mu = 0$, $\chi = 0$ in (7), (8), the corresponding inverse problem is said to be homogeneous. The following assertion holds (the Fredholm alternative for the problem (7), (8)).

Theorem 5. *Assume that a domain Ω_- satisfies Condition (B) and the coefficients of an operator L of the form* (3) *and the function $h(y,x)$ satisfy the conditions*

$$a_{ij}, b_i \in C^{\alpha}(\overline{D}), \quad a, a_y, c, c_y, h, h_y \in C^{\alpha}(\overline{\Omega}_-),$$

$$c(y,x) \leqslant 0, \quad (y,x) \in \Omega_-,$$

$$|h(0,x)| \geqslant h_0 > 0, \quad a_y(0,x) = c_y(0,x) = h_y(0,x) = 0, \quad x \in \overline{D}.$$

Then for the problem (7), (8) *one of the following two assertions holds*:

1) *the homogeneous inverse problem* (7), (8), *i.e., the inverse problem* (7), (8) *with $g = \mu = \chi = 0$, has finitely many linearly independent solutions,*

2) *for an arbitrary triple of functions $(g, \mu, \chi) \in \mathscr{R}(\Omega_-)$ the inverse problem* (7), (8) *has a unique solution $(u, f) \in \mathscr{U}_1(\Omega_-) \times \mathscr{F}(D)$.*

There are various sufficient conditions for the unique solvability of the problem (7), (8). We formulate one of such conditions in the case where the domain is a cylinder, i.e., $\Omega_- = Q_1(q)$.

Theorem 6. *Let $\Omega_- = Q_1(q)$, and let an operator L of the form* (3) *with $c = c(x)$ and the function $h(y,x)$ satisfy the conditions*

$$a, a_y, a_{yy}, h, h_y, h_{yy} \in C^{\alpha}(\overline{\Omega}_-), \quad a_{ij}, b_i, c \in C^{\alpha}(\overline{D}),$$

$$a_y(0,x) = 0, \quad h_y(0,x) = 0, \quad c(x) \leqslant 0, \quad x \in D,$$

$$a_{yy}(y,x) + c(x) \leqslant 0, \quad (y,x) \in \Omega_-,$$

$$h(y,x) \geqslant 0, \quad h_{yy}(y,x) \leqslant 0, \quad (y,x) \in \Omega_-,$$

$$h(0,x) \geqslant h_0 > 0, \quad x \in D.$$

Then for any triple of functions $(g, \mu, \chi) \in \mathscr{R}(\Omega_-)$ *the inverse problem* (7), (8) *has a unique solution.*

Remark 1. The problems of finding a source in the elliptic equation with overdetermination in the domain are studied in [11, 12]. Similar problems with overdetermination on the boundary of a cylindrical domain were studied in [14]–[23]. Various existence and uniqueness conditions obtained for these inverse problems agree with the assumptions of Theorem 6 and are involved in this theorem as particular cases. A more detailed overview of the background, examples, and applications can be found in [2].

1.2. The inverse coefficient problem. We consider the inverse problem of finding the lower order term in the elliptic equation. Unlike the source problem, this problem is nonlinear and requires different methods.

Definition 5. Let a domain Ω satisfy Condition (A). We introduce the sets of Hölder functions

$$\mathscr{U}_2(\Omega) = \{u \in C(\overline{\Omega}) \mid u \in C^{2,\alpha}(\Omega),\ u_{yy} \in C(\overline{\Omega})\},$$
$$\mathscr{F}_1(D) = \{f \in C^{\alpha}(\overline{D}) \mid f(x) \leqslant 0,\ x \in D\}.$$

For the sake of brevity we consider a particular case where $\Omega = Q(q_1, q_2)$ is a cylindrical domain. In Ω, we consider the problem of finding a pair of functions $(u, f) \in C^{2,\alpha}(\overline{\Omega}) \times \mathscr{F}_1(D)$ from the following conditions:

$$-(Lu)(y, x) = f(x)u(y, x) + g(y, x), \quad (y, x) \in \Omega, \tag{9}$$
$$u(y, x) = \mu(y, x), \quad (y, x) \in \partial\Omega, \quad u(0, x) = \chi(x), \quad x \in \overline{D}. \tag{10}$$

Theorem 7. *Let* $\Omega = Q(q_1, q_2)$, *and let an elliptic operator* L *of the form* (3) *with* $c = c(x)$ *and the functions* g, μ *satisfy the conditions*

$$a, a_y, a_{yy}, g, g_y, g_{yy} \in C^{\alpha}(\Omega) \cap C(\overline{\Omega}),$$
$$a_{ij}, b_i, c \in C^{\alpha}(D) \cap C(\overline{D}), \quad \mu, \mu_y, \mu_{yy} \in C(\overline{\Gamma}(q_1, q_2)),$$
$$c(x) \leqslant 0, \quad x \in D,$$
$$c(x) + a_{yy}(y, x) \leqslant 0, \quad g(y, x) \geqslant 0, \quad g_{yy}(y, x) \leqslant 0, \quad (y, x) \in \Omega,$$
$$\mu(y, x) \geqslant 0, \quad \mu_{yy}(y, x) \leqslant 0, \quad (y, x) \in \Gamma(q_1, q_2);$$

moreover, at least one of the functions g *and* μ *is not equal to zero identically. Then the inverse problem* (9), (10) *cannot have two different solutions in the class of functions* $(u, f) \in C^{2,\alpha}(\overline{\Omega}) \times \mathscr{F}_1(D)$.

As in the linear problem, we separately consider the case of a symmetric cylinder with respect to the plane $y = 0$, i.e., $\Omega = Q(q)$. In this case, we also assume that the operator has the form (3). The inverse problem consists in finding a pair of functions $(u, f) \in C^{2,\alpha}(\overline{\Omega}) \times \mathscr{F}_1(D)$ from the conditions

$$-(Lu)(y, x) = f(x)u(y, x) + g(y, x), \quad (y, x) \in \Omega, \tag{11}$$

$$\begin{cases} u(y,x) = \mu(y,x), & (y,x) \in \overline{\Gamma}(q), \\ u(q,x) = u(-q,x) = 0, & x \in \overline{D}, \end{cases} \tag{12}$$

$$u(0,x) = \chi(x), \quad x \in \overline{D}. \tag{13}$$

For the inverse problem (11)–(13) the following uniqueness theorem holds, where an important role is played by y-even components of the functions g and μ:

$$g_e(y,x) = \frac{g(y,x) + g(-y,x)}{2}, \quad (y,x) \in \Omega_R,$$

$$\mu_e(y,x) = \frac{\mu(y,x) + \mu(-y,x)}{2}, \quad (y,x) \in \overline{\Gamma}(q).$$

Theorem 8. *Let the coefficients of a uniformly elliptic operator L in $\Omega_R = [-q,q] \times B_R$ and the functions g, μ satisfy the conditions*

$$a, a_y, c, c_y \in C^\alpha(\Omega_R), \quad a_{ij}, b_i \in C^\alpha(B_R), \quad \mu \in C^{2,\alpha}(\overline{\Gamma}(q)), \quad g, g_y \in C^\alpha(\Omega_R),$$

$$a(y,x) = a(-y,x), \quad c(y,x) = c(-y,x), \quad c(y,x) \leqslant 0, \quad g_e(y,x) \geqslant 0, \quad (y,x) \in \Omega_R,$$

$$c_y(y,x) \geqslant 0, \quad (g_e)_y(y,x) \geqslant 0, \quad (y,x) \in [-q,0] \times \overline{D},$$

$$\mu_e(y,x) \geqslant 0, \quad (\mu_e)_y(y,x) \geqslant 0, \quad (y,x) \in [-q,0] \times \partial D;$$

moreover, at least one of the functions g_e, μ_e is not equal to zero identically. Then the problem (11)–(13) cannot have two different solutions in the class of functions $(u,f) \in C^{2,\alpha}(\overline{\Omega}) \times \mathscr{F}_1(D)$.

We consider the inverse problem in the domain $\Omega_- = Q_1(q) = (-q,0) \times D$ with data on the boundary. We look for a pair of functions $(u,f) \in \mathscr{U}_2(\Omega_-) \times \mathscr{F}_1(D)$ such that

$$-(Lu)(y,x) = f(x)u(y,x) + g(y,x), \quad (y,x) \in \Omega_-, \tag{14}$$

$$\begin{cases} u(y,x) = \mu(y,x), & (y,x) \in \overline{\Gamma}_1(q), \\ u(-q,x) = 0, \quad u_y(0,x) = 0, & x \in D, \end{cases} \tag{15}$$

$$u(0,x) = \chi(x), \quad x \in \overline{D}. \tag{16}$$

For the inverse problem (14)–(16) the following uniqueness theorem holds.

Theorem 9. *Let the coefficients of a uniformly elliptic operator L in $\Omega_- = (-q,0) \times D$ and the functions g, μ satisfy the conditions*

$$a, a_y, a_{yy}, g, g_y, g_{yy} \in C^\alpha(\Omega_- \cup \Gamma_0) \cap C(\overline{\Omega}_-),$$

$$a_{ij}, b_i, c \in C^\alpha(D) \cap C(\overline{D}), \quad \mu, \mu_y, \mu_{yy} \in C^\alpha(\overline{\Gamma}_1(q)),$$

$$a_y(0,x) = g_y(0,x) = 0, \quad c(x) \leqslant 0, \quad x \in D,$$

$$a_{yy}(y,x) + c(x) \leqslant 0, \quad g(y,x) \geqslant 0, \quad g_{yy}(y,x) \leqslant 0, \quad (y,x) \in \Omega_-,$$

$$\mu(y,x) \geqslant 0, \quad \mu_{yy}(y,x) \geqslant 0, \quad (y,x) \in [-q,0] \times \partial D,$$

$$\mu_y(0,x) = 0, \quad x \in \partial D;$$

moreover, at least one of the functions g, μ is not equal to zero identically. Then the problem (14)–(16) cannot have two different solutions in the class of functions $(u,f) \in \mathscr{U}_2(\Omega_-) \times \mathscr{F}_1(D)$.

In the case where Ω is a cylinder, sufficient conditions for the existence of a solution are obtained for the inverse coefficient problem. In the cylinder $\Omega = Q(q_1, q_2)$, we consider the inverse problem of finding a pair of functions $(u, f) \in \mathscr{U}_2(\Omega) \times \mathscr{F}_1(D)$ from the conditions

$$-(Lu)(y,x) = f(x)u(y,x) + g(y,x), \quad (y,x) \in \Omega, \tag{17}$$

$$\begin{cases} u(y,x) = \mu(y,x), \quad (y,x) \in \overline{\Gamma}(q_1,q_2), \\ u(q_1,x) = u(q_2,x) = 0, \quad u(0,x) = \chi(x), \quad x \in \overline{D}. \end{cases} \tag{18}$$

The operator L in (17) has the form

$$Lu = a(y,x)\frac{\partial^2 u}{\partial y^2} + \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u \equiv a(y,x)\frac{\partial^2 u}{\partial y^2} + L_x u. \tag{19}$$

We say that for the problem (17), (18) the compatibility conditions hold at $y = q_1$, $y = 0$, $y = q_2$ if the functions μ, χ, g satisfy the conditions

$$\mu(q_1,x) = 0, \quad \mu(0,x) = \chi(x), \quad \mu(q_2,x) = 0, \quad x \in \partial D, \tag{20}$$

$$-a(q_1,x)\mu_{yy}(q_1,x) = g(q_1,x), \quad -a(q_2,x)\mu_{yy}(q_2,x) = g(q_2,x), \quad x \in \partial D. \tag{21}$$

To formulate the following theorem, we introduce the set of functions

$$\mathscr{H}(D) := \big\{\chi \in C(\overline{D}) \mid \chi \in C^{2,\alpha}(D), \quad L_x\chi \in C(\overline{D})\big\}$$

and an auxiliary function $\overline{w}$ as the solution to the Dirichlet problem

$$\begin{aligned} &-(L\overline{w})(y,x) - 2a_y(y,x)\overline{w}_y(y,x) - a_{yy}(y,x)\overline{w}(y,x) = (g_{yy})^-(y,x), \quad (y,x) \in \Omega, \\ &\overline{w}(y,x) = (\mu_{yy})^-(y,x), \quad (y,x) \in \overline{\Gamma}(q_1,q_2), \\ &\overline{w}(q_i,x) = \frac{g^+(q_i,x)}{a(q_i,x)}, \quad i = 1,2, \ x \in \overline{D}. \end{aligned}$$

As usual, $g^+(y,x) := \max\{0, g(y,x)\}$ and $g^-(y,x) := \max\{0, -g(y,x)\}$ denote the positive and negative parts of a function g respectively. For the inverse problem (17), (18) the following existence theorem holds.

Theorem 10. *Let $\Omega = Q(q_1,q_2)$, and let an operator L of the form* (19) *and the functions g, μ satisfy the compatibility conditions* (20), (21) *and also the conditions*

$$\begin{aligned} &a, a_y, a_{yy}, g, g_y, g_{yy} \in C^\alpha(\Omega) \cap C(\overline{\Omega}), \\ &a_{ij}, b_i, c \in C^\alpha(D) \cap C(\overline{D}), \quad \mu, \mu_y, \mu_{yy} \in C(\overline{\Gamma}(q_1,q_2)), \\ &\chi \in \mathscr{H}(D), \quad c(x) \leqslant 0, \quad x \in D, \\ &c(x) + a_{yy}(y,x) \leqslant 0, \quad (y,x) \in \Omega, \\ &\chi(x) \geqslant \chi_0 > 0, \quad a(0,x)\overline{w}(0,x) - (L_x\chi)(x) - g(0,x) \leqslant 0, \quad x \in D. \end{aligned}$$

Then the inverse problem (17), (18) *has a solution $(u,f) \in \mathscr{U}_2(\Omega) \times \mathscr{F}_1(D)$.*

We formulate more results concerning the solvability of the inverse problem of finding the coefficient of the equation in the cylinder with overdetermination on the boundary. We fix $q > 0$ and consider in $\Omega_- = Q_1(q)$ the problem of finding a pair of functions $(u, f) \in \mathscr{U}_2(\Omega_-) \times \mathscr{F}_1(D)$ from the relations (14)–(16) with an operator L of the form (19). We say that for the problem (14)–(16) the compatibility conditions are satisfied at $y = -q$ and $y = 0$ if the functions μ, χ, g satisfy the conditions

$$\mu(-q,x) = 0,\ \mu(0,x) = \chi(x),\ \mu_y(0,x) = 0,\ \ -a(-q,x)\mu_{yy}(-q,x) = g(-q,x),\ x \in \partial D. \quad (22)$$

Assume that the coefficients of the operator L and the given functions satisfy the compatibility conditions (22) and the conditions

$$\begin{aligned}
&a, a_y, a_{yy}, g, g_y, g_{yy} \in C^\alpha(\Omega_- \cup \Gamma_0) \cap C(\overline{\Omega}_-),\\
&a_{ij}, b_i, c \in C^\alpha(D) \cap C(\overline{D}), \quad \mu, \mu_y, \mu_{yy} \in C(\overline{\Gamma}_1(q)),\\
&c(x) \leqslant 0, \quad x \in D,\\
&c(x) + a_{yy}(y,x) \leqslant 0, \quad (y,x) \in \Omega_-,\\
&g_y(0,x) = 0, \quad a_y(0,x) = 0, \quad x \in \overline{D}.
\end{aligned}$$

Under these assumptions, we extend a, g, μ as even functions with respect to $y \in [-q, 0]$ Thus, the extended functions, denoted by $\widetilde{a}$, $\widetilde{g}$, $\widetilde{\mu}$, satisfy the inclusions

$$\widetilde{a}, \widetilde{a}_y, \widetilde{a}_{yy}, \widetilde{g}, \widetilde{g}_y, \widetilde{g}_{yy} \in C^\alpha(\Omega) \cap C(\overline{\Omega}), \quad \widetilde{\mu}, \widetilde{\mu}_y, \widetilde{\mu}_{yy} \in C(\overline{\Gamma}(q)).$$

To formulate the existence theorem, we need to define $\overline{w}$ as a solution to the auxiliary problem

$$\begin{aligned}
&-(\widetilde{L}\overline{w})(y,x) - 2\widetilde{a}_y(y,x)\overline{w}_y(y,x) - \widetilde{a}_{yy}(y,x)\overline{w}(y,x) = (\widetilde{g}_{yy})^-(y,x), \quad (y,x) \in \Omega,\\
&\overline{w}(y,x) = (\widetilde{\mu}_{yy})^-(y,x), \quad (y,x) \in \overline{\Gamma}(q),\\
&\overline{w}(q,x) = \frac{\widetilde{g}^+(q,x)}{\widetilde{a}(q,x)}, \quad \overline{w}(-q,x) = \frac{\widetilde{g}^+(-q,x)}{\widetilde{a}(-q,x)}, \quad x \in \overline{D},
\end{aligned}$$

where the operator $\widetilde{L}$ has the form (19), but with $\widetilde{a}$ instead of a. The following existence theorem for the problem (14)–(16) holds.

Theorem 11. *Let $\Omega_- = Q_1(q)$, and let the coefficients of the operator L and the functions g, μ, χ satisfy the compatibility conditions* (22) *and the conditions*

$$\begin{aligned}
&a, a_y, a_{yy}, g, g_y, g_{yy} \in C^\alpha(\Omega_- \cup \Gamma_0) \cap C(\overline{\Omega}_-),\\
&a_{ij}, b_i, c \in C^\alpha(D) \cap C(\overline{D}), \quad \mu, \mu_y, \mu_{yy} \in C(\overline{\Gamma}_1(q)),\\
&\chi \in \mathscr{H}(D), \quad c(x) \leqslant 0, \quad x \in D,\\
&c(x) + a_{yy}(y,x) \leqslant 0, \quad (y,x) \in \Omega_-,\\
&\chi(x) \geqslant \chi_0 > 0, \quad a(0,x)\overline{w}(0,x) - (L_x\chi)(x) - g(0,x) \leqslant 0, \quad x \in D,\\
&g_y(0,x) = 0, \quad a_y(0,x) = 0, \quad x \in \overline{D}.
\end{aligned}$$

Then there exists a solution to the problem (14)–(16) *in the above-indicated class of functions.*

As a consequence of the above existence and uniquenesses theorems, we obtain the existence and uniqueness of a solution to the problem (14)–(16).

Theorem 12. *Let $\Omega_- = Q_1(q)$, and let the coefficients of the operator L and the functions g, μ, χ satisfy the compatibility conditions* (22) *and the conditions*

$$a, a_y, a_{yy}, g, g_y, g_{yy} \in C^\alpha(\Omega_- \cup \Gamma_0) \cap C(\overline{\Omega}_-),$$

$$a_{ij}, b_i, c \in C^\alpha(D) \cap C(\overline{D}), \quad \mu, \mu_y, \mu_{yy} \in C(\overline{\Gamma}_1(q)), \quad \chi \in \mathscr{H}(D),$$

$$c(x) \leqslant 0, \quad x \in D,$$

$$\mu(y,x) \geqslant 0, \quad \mu_{yy}(y,x) \leqslant 0, \quad (y,x) \in \overline{\Gamma}_1(q),$$

$$c(x) + a_{yy}(y,x) \leqslant 0, \quad g(y,x) \geqslant 0, \quad g_{yy}(y,x) \leqslant 0, \quad (y,x) \in \Omega_-,$$

$$\chi(x) \geqslant \chi_0 > 0, \quad a(0,x)\overline{w}(0,x) - (L_x\chi)(x) - g(0,x) \leqslant 0, \quad g_y(0,x) = a_y(0,x) = 0, \quad x \in D.$$

Then the problem (14)–(16) *has a unique solution $(u,f) \in \mathscr{U}_2(\Omega_-) \times \mathscr{F}_1(D)$.*

Remark 2. The inverse problem of finding the coefficient in the elliptic equation with overdetermination in a domain was studied in [12, 13]. A similar problem with overdetermination on the boundary of the cylinder was studied in [21] under the additional assumption that $g = 0$ and the coefficients of the operator L are independent of the variable y. The result of [21] agrees with Theorem 9. In the monograph [24], the question about the existence of a solution to the inverse problem of finding the coefficient in a cylinder with overdetermination on the boundary is formulated as an important problem for the further development of the theory of inverse problems.

2 Inverse Problems for Parabolic Equations in Sobolev Spaces

2.1. The linear problem with source in the equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega \in C^2$, and let $Q = \Omega \times (0,T)$ be a basic cylinder with lateral surface $S = \partial\Omega \times [0,T]$. The general source problem with nonzero functions on the right-hand side of the equation and the initial and boundary conditions, together with the corresponding compatibility and smoothness conditions, is reduced to the problem of finding a pair of functions $\{u(x,t); f(x)\}$ from the relations

$$\rho(x,t)u_t(x,t) - L(t)u(x,t) = h(x,t)f(x), \quad (x,t) \in Q, \tag{23}$$

$$u(x,0) = 0, \quad x \in \Omega, \quad Bu(x,t) = 0, \quad (x,t) \in S, \tag{24}$$

$$l(u) := \int_0^T u(x,t)d\mu(t) = \chi(x), \quad x \in \Omega, \tag{25}$$

where the functions ρ, h, χ, $\mu(t)$ are given, the uniformly elliptic operator $L(t)$ with sufficiently smooth coefficients has the form

$$L(t)u = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i} + d(x,t)u,$$

and the operator of boundary conditions is either of the first or of the third (the second) kind, i.e., $Bu \equiv u$ or $Bu \equiv \dfrac{\partial u}{\partial N} + \sigma(x)u$, where

$$\frac{\partial u}{\partial N} \equiv \sum_{i,j=1}^{n} \cos(\mathbf{n}, x_i) a_{ij}(x) \frac{\partial u}{\partial x_j}$$

denotes the conormal derivative, $\mathbf{n}$ is the outward normal to $\partial\Omega$ at the point $x = (x_1, \dots, x_n)$, $\sigma \in C^1(\partial\Omega)$, $\sigma(x) \geqslant 0$ on $\partial\Omega$. The functions $\chi(x)$ and $h(x,t)$ satisfy the conditions

$$\begin{aligned} &\chi \in W_2^2(\Omega), \quad B\chi(x) = 0, \quad x \in \partial\Omega, \\ &h, h_t \in L_{\infty,2}(Q), \quad |l(h)(x)| \geqslant \delta > 0, \quad x \in \Omega. \end{aligned} \tag{26}$$

The scalar function μ belongs to $BV[0,T]$, and the integral in (25) is understood as the Riemann–Stieltjes integral of continuous functions. We assume that the given functions satisfy the smoothness conditions

$$(A.1) \qquad \begin{aligned} &a_{ij} \in C^1(\overline{\Omega}), \quad \rho, \rho_t \in C(\overline{Q}), \quad b_i, \partial b_i/\partial t, d, d_t \in L_\infty(Q), \quad \mu \in BV[0,T], \\ &\mu(0) = \mu(0+), \quad \mu(t) \not\equiv \text{const} \ \text{ on } [0,T], \quad \rho(x,t) \geqslant \rho_0 > 0, \quad (x,t) \in Q. \end{aligned}$$

By a solution to the inverse problem (23)–(25) we understand a pair of functions $u \in W_2^{2,1}(Q)$, $f \in L_2(\Omega)$ satisfying Equation (23) almost everywhere in Q and the conditions (24), (25). Equivalently, by a solution we sometimes understand a function $f \in L_2(\Omega)$ such that $u = u(x,t;f)$, regarded as a solution to the direct problem (23), (24) with a given f, satisfies the observation condition (25).

The following conditions on the function $\mu(t)$ play an important role:

$$\mu(t) \text{ is a nondecreasing function continuous from the right on } [0,T], \text{ and } \bigvee_0^T(\mu) > 0. \tag{27}$$

Particular cases of the nonlocal observation (25) are the final overdetermination, i.e., $l(u) \equiv u(x,t_1)$, $0 < t_1 \leqslant T < \infty$, where t_1 is fixed, and the integral overdetermination, i.e.,

$$l(u) \equiv \int_0^T u(x,t)\omega(t)dt.$$

The problems with such observation conditions for "stationary" parabolic equations in L_2 were considered in [25]–[27]. The problem with final observation was studied in the Hölder class in [28, 29], where the Fredholm property and uniqueness were proved and the positivty method was proposed (cf. also [30]–[32] and [38, 39]). A more general condition than (25) was considered in [33] (for abstract equations and the semigroup method cf. also [34]–[37]). The results of [25]–[27] are generalized to parabolic equations with coefficients depending on x and t.

It is proved that the problem (23)–(25) is equivalent to a second kind operator equation for the unknown $f(x)$.

Theorem 13. *Let* (26) *and Condition* (A.1) *hold. Then the inverse problem* (23)–(25) *is equivalent to a linear second kind operator equation with compact operator* $\mathscr{B}$ *in* $L_2(\Omega)$.

To formulate the further results, we assume that it is possible to extract the "stationary" part L_0 of the operator $L(t)$:

$$\left.\begin{aligned} &L(t)u = L_0 u + d(x,t)u, \\ &L_0 u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c_0(x)u, \end{aligned}\right\} \tag{28}$$

where the coefficients a_{ij}, b_i, d satisfy Condition $(A.1)$, the functions b_i are now independent of t, and the coefficient $c_0(x)$ satisfies the conditions

$$(E) \qquad \begin{aligned} &c_0 \in L_\infty(\Omega), \quad c_0(x) \leqslant 0, \quad x \in \Omega, \\ &Bu \equiv \frac{\partial u}{\partial N} \text{ implies } c_0(x) \not\equiv 0 \text{ in } \Omega. \end{aligned}$$

Theorem 14. *Assume that the operator $L(t)$ in Equation* (23) *has the form* (28), *the conditions* (26), $(A.1)$, *and* (E) *hold, the function $\mu(t)$ is nondecreasing on $[0,T]$, $|l(h)(x)| \geqslant \delta > 0$ in Ω, $h(x,t)/l(h)(x) \geqslant 0$ in Q, and at least one of the following conditions is satisfied*:

1) $h_t(x,t)/l(h)(x) \geqslant 0$, $d(x,t) \leqslant 0$, $d_t(x,t) \geqslant 0$ *in* Q,

2) $d\mu(t) = \omega(t)dt$, *where* $\omega \in W_1^1(0,T)$ *is such that* $\big(\omega(t)\varrho(x,t)\big)_t' + d(x,t)\omega(t) \leqslant 0$ *in* Q,

3) $d\mu(t) = \omega(t)dt$, *where* $\omega \in BV[0,T]$ *is such that for all* $x \in \Omega$ *the function*

$$\pi(x,t) \equiv \omega(t)\varrho(x,t) + \int\limits_0^t d(x,\tau)\omega(\tau)d\tau$$

is nonincreasing with respect to $t \in [0,T]$.

Then there exists a unique solution $u \in W_2^{2,1}(Q)$, $f \in L_2(\Omega)$ *to the problem* (23)–(25) *and the following stability estimate holds*:

$$\|f\|_{2,\Omega} + \|u\|_{2,Q}^{(2,1)} \leqslant C\|L_0\chi\|_{2,\Omega}.$$

The solution $u(x,t)$ possesses the following additional differential properties:

$$u \in C([0,T]; W_2^2(\Omega)), \quad u_t \in C([0,T]; L_2(\Omega)), \quad u_t \in W_2^{2,1}(Q_\varepsilon), \quad Q_\varepsilon = \Omega \times (\varepsilon, T).$$

Some analogs of such conditions for "stationary" equations within the framework of the semigroup approach and the positivity method were considered in [33]. For "nonstationary" parabolic equations the well-posedness was proved by a new method (cf. [6]), which made it possible to prove the convergence of the corresponding iteration sequence to the solution.

The sufficient uniqueness condition for the linear inverse problem are widely used in the coefficient problems. Since the problem is linear, the proof of the uniqueness of its solution is reduced in a standard way to the proof that the homogeneous inverse problem i.e., the problem (23)–(25) with $\chi = 0$, has no nonzero solutions.

Let $|l(h)(x)| > 0$ almost everywhere in Ω. We consider the function $\varphi_0(x) := \operatorname{sgn} l(h)(x)$ such that $\varphi_0^2(x) = 1$ almost everywhere in Ω.

Theorem 15. *Assume that the operator $L(t)$ in Equation (23) has the form (28), Conditions (A.1) and (E) hold, $h, h_t \in L_{\infty,2}(Q)$, the function $\mu(t)$ satisfies (27), $|l(h)(x)| > 0$ in Ω, and $h(x,t)\varphi_0(x) \geqslant 0$ in Q. In addition, assume that at least one of the following conditions holds*:

1) $h_t(x,t)\varphi_0(x) \geqslant 0$, $d(x,t) \leqslant 0$, $d_t(x,t) \geqslant 0$ *in* Q,

2) $d\mu(t) = \omega(t)dt$, *where* $\omega \in BV[0,T]$ *is such that for all* $x \in \Omega$ *the function*

$$\pi(x,t) \equiv \omega(t)\varrho(x,t) + \int_0^t d(x,\tau)\omega(\tau)d\tau$$

is nonincreasing with respect to $t \in [0,T]$.

Then the problem (23)–(25) with $\chi = 0$ *has only zero solution* $u = 0$, $f = 0$.

In the study of any problem it is important to obtain necessary and sufficient conditions of the existence and uniqueness of a solution. We obtain the necessary and sufficient conditions for uniqueness and well-posedness of parabolic inverse problems related to the completeness and basis property, respectively, of some system of functions in $L_2(\Omega)$. These questions were treated by using an example of the inverse problem of finding a pair of functions $\{u(x,t); f(x)\}$ from the relations

$$u_t(x,t) - \mathscr{L}_0 u(x,t) = h(x,t)f(x), \quad (x,t) \in Q, \tag{29}$$

$$u(x,0) = 0, \quad x \in \Omega, \quad Bu(x,t) = 0, \quad (x,t) \in S, \tag{30}$$

$$l(u) = \chi(x), \quad x \in \Omega, \tag{31}$$

where $\chi \in W_2^2(\Omega)$, $B\chi(x) = 0$ on $\partial\Omega$, and $\mathscr{L}_0$ is a stationary uniformly elliptic (symmetric) operator of the form

$$\mathscr{L}_0 u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + c_0(x)u$$

with coefficients $a_{ij} \in C^1(\overline{\Omega})$, $c_0 \in L_\infty(\Omega)$, and Condition (E) holds.

The eigenfunctions and eigenvalues of the problem

$$-\mathscr{L}_0 v(x) = \lambda v(x), \ x \in \Omega, \quad Bv(x) = 0, \ x \in \partial\Omega,$$

are denoted by $\{e_k(x)\}$ and $\{\lambda_k\}$ respectively, where λ_k are enumerated in ascending order of the absolute value (with multiplicity taken into account) and $\|e_k\|_{2,\Omega} = 1$ for all $k = 1, 2, \dots$. As is known, $e_k \in W_2^2(\Omega)$, $\lambda_k \in \mathbb{R}$, and $\lambda_k \to +\infty$. In our case, $\lambda_1 > 0$ and the system $\{e_k(x)\}$ is an orthonormal basis in the space $L_2(\Omega)$. We introduce the system of functions

$$\psi_k(x) := \lambda_k \int_0^T \bigg(\int_0^t e^{-\lambda_k(t-\tau)} h(x,\tau)d\tau \bigg) d\mu(t) e_k(x) \equiv \beta_k(x)e_k(x), \quad k \in \mathbb{N}. \tag{32}$$

For this system a completeness criterion holds.

Theorem 16. *Assume that $h, h_t \in L_{\infty,2}(Q)$, $\mu \in BV[0,T]$, and the operators $\mathscr{L}_0$ and B satisfy the above conditions. Then the system $\{\psi_k(x)\}$ in (32) is complete in $L_2(\Omega)$ if and only if the solution to the inverse problem (29)–(31) is unique.*

For the system $\{\psi_k\}$ the moment problem can be stated, i.e., the problem of finding a function $f(x) \in L_2(\Omega)$ satisfying the equalities

$$(\psi_k, f) := \int\limits_{\Omega} \psi_k(x) f(x) dx = \alpha_k \quad \forall\ k = 1, 2, \ldots, \tag{33}$$

where the number sequence $\alpha = \{\alpha_k\}$ is given. As usual, the condition $\alpha \in l_2$ means that

$$\sum_k |\alpha_k|^2 < \infty.$$

As proved in [40], the solvability of the inverse problem with final observation is equivalent to the solvability of the corresponding moment problem. For a more general system of functions defined in (32) the following result holds.

Theorem 17. *Assume that $h, h_t \in L_{\infty,2}(Q)$, $\mu \in BV[0,T]$, the operators $\mathscr{L}_0$ and B satisfy the above conditions, and the system $\{\psi_k(x)\}$ is introduced by* (32). *The solvability of the moment problem* (33) *is equivalent to the solvability of the inverse problem* (29)–(31); *namely, the following two assertions are valid.*

1. *If $\alpha \in l_2$ and* (33) *is solvable, then the function*

$$\chi(x) := \sum_k (\alpha_k/\lambda_k)\, e_k(x) \in W_2^2(\Omega);$$

moreover, $B\chi(x) = 0$, $x \in \partial\Omega$, and the inverse problem (29)–(31) *is solvable.*

2. *Let the inverse problem* (29)–(31) *with $l(u) = \chi(x) \in W_2^2(\Omega)$ such that $B\chi(x) = 0$, $x \in \partial\Omega$, be solvable. Then the moment problem* (33) *with $\alpha_k = \lambda_k(\chi, e_k)$ is solvable and $\alpha \in l_2$.*

Definition 6. The inverse problem (29)–(31) is said to be *well posed* if for any function $\chi \in W_2^2(\Omega)$ such that $B\chi(x) = 0$ on $\partial\Omega$ there exists a unique function $f \in L_2(\Omega)$ such that the solution $u(x,t;f)$ to the direct problem (29), (30) satisfies the observation condition (31) and the following stability estimate holds:

$$\|f\|_{2,\Omega} \leqslant C\|\mathscr{L}_0\chi\|_{2,\Omega}.$$

It is found that the unique solvability of the inverse problem (29)–(31) is closely connected with the basis property of the Riesz system $\{\psi_k(x)\}$.

Theorem 18. *Assume that $h, h_t \in L_{\infty,2}(Q)$, $\mu \in BV[0,T]$, the operators $\mathscr{L}_0$ and B satisfy the above conditions, and $\{\psi_k(x)\}$ is defined by* (32). *Then $\{\psi_k(x)\}$ is the Riesz basis in the space $L_2(\Omega)$ if and only if the inverse problem* (29)–(31) *is well posed.*

As a consequence of Theorems 14 and 15, it is possible to obtain the completeness and Riesz basis property for a large class of such systems in the multidimensional case.

Corollary 1. *Assume that $h, h_t \in L_{\infty,2}(Q)$, the operators $\mathscr{L}_0$ and B satisfy the above conditions, the function μ satisfies* (27), *$|l(h)(x)| > 0$ in Ω, and $h(x,t)\varphi_0(x) \geqslant 0$ in Q. In addition, at least one of the following assertions holds*:

1) *$h_t(x,t)\varphi_0(x) \geqslant 0$ in Q,*

2) *$d\mu(t) = \omega(t)dt$, where $\omega \in BV[0,T]$ is nonincreasing on $[0,T]$.*

Then the system $\{\psi_k(x)\}$ introduced in (32) *is complete in $L_2(\Omega)$.*

If we reinforce the condition $|l(h)(x)| > 0$ in Ω in this corollary by requiring the inequality $|l(h)(x)| \geqslant \delta > 0$ in Ω, then the system $\{\psi_k(x)\}$ is a Riesz basis in the space $L_2(\Omega)$.

2.2. The inverse coefficient problem. We consider the inverse problem of reconstructing the coefficient at u in a parabolic equation. In the cylinder $Q = \Omega \times (0,T)$ with lateral surface $S = \partial\Omega \times [0,T]$, we study the problem of finding functions $\{u(x,t); c(x)\}$ satisfying the conditions

$$\rho(x,t)u_t - L_0 u - d(x,t)u = c(x)u + g(x,t), \quad (x,t) \in Q, \tag{34}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \quad Bu = \beta(x,t), \quad (x,t) \in S, \tag{35}$$

$$\int_0^T u(x,t)d\mu(t) = \chi(x), \quad x \in \Omega. \tag{36}$$

Here, ρ, d, g, u_0, β, μ, χ are given and the uniformly elliptic operator L_0 has the form

$$L_0 u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c_0(x)u, \quad c_0(x) \leqslant 0,$$

where $c_0(x)$ additionally satisfies Condition (E). The given functions in (34)–(36) satisfy the conditions (for some fixed $p \geqslant n+1$)

$$(A.2) \qquad \begin{aligned} & a_{ij}(x) \in C^1(\overline{\Omega}), \quad \rho, \rho_t \in C(\overline{Q}), b_i \in L_\infty(\Omega), \quad c_0 \in L_p(\Omega), \quad d, d_t \in L_\infty(Q), \\ & \mu \in BV[0,T], \quad \mu(0) = \mu(0+), \quad \mu(t) \not\equiv \text{const on } [0,T], \\ & \rho(x,t) \geqslant \rho_0 > 0, \quad (x,t) \in Q, \quad c_0(x) \leqslant 0 \quad x \in \Omega, \quad \rho_0 = \text{const}. \end{aligned}$$

Assume that there is a function $\Phi(x,t)$ given in the entire cylinder $\overline{Q}$ and such that $\Phi, \Phi_t \in W_p^{2,1}(Q)$; moreover, $\Phi(x,0) = u_0(x)$ in Ω, whereas $B\Phi = \beta(x,t)$ on S. Thus, we assume that

$$(B.2) \qquad \begin{aligned} & g, g_t \in L_p(Q), \quad u_0 \in W_p^2(\Omega), \quad \exists \Phi(x,t): \quad \Phi, \Phi_t \in W_p^{2,1}(Q), \\ & \Phi(x,0) = u_0(x) x \in \Omega, \quad B\Phi(x,t) = \beta(x,t), \quad (x,t) \in S. \end{aligned}$$

The solution to the direct problem (34), (35) with $c(x) = 0$ is denoted by $u^0(x,t)$. We also impose the smoothness and compatibility conditions on the function $\chi(x)$:

$$(C.2) \qquad \chi \in W_p^2(\Omega), \quad B\chi(x) = l(\beta)(x), \quad x \in \partial\Omega.$$

If Conditions $(A.2)$, $(B.2)$, $(C.2)$ hold, then the solution to this problem is looked for in the class of functions $u \in W_p^{2,1}(Q)$, $c \in E_- := \{v \in L_p(\Omega) \mid v(x) \leqslant 0 \text{ in } \Omega\}$ with fixed $p \geqslant n+1$. Earlier, the coefficient $c(x)$ in such a problem was looked for in the class of bounded or even smoother functions. Some results for Hölder solutions and a particular case of final observation can be found in [41, 30, 24].

We consider the case of an absolutely continuous measure $d\mu(t) = \omega(t)dt$, where $\omega \in BV[0,T]$ satisfies the conditions

$$\begin{aligned} & d\mu(t) = \omega(t)dt, \quad \omega \in BV[0,T], \quad \omega(t) \geqslant 0 \text{ on } [0,T], \\ & \forall x \in \Omega \ \ \pi_0(x,t) := \rho(x,t)\omega(t) + \int_0^t d(x,\xi)\omega(\xi)d\xi \text{ is nonincreasing in } t \in [0,T]. \end{aligned} \tag{37}$$

Theorem 19. *Let Conditions* (*A*.2), (*B*.2), (*C*.2), (37) *hold, and let*

$$g(x,t) \geqslant 0 \text{ in } Q, \quad u_0(x) \geqslant 0 \text{ in } \Omega, \quad \beta(x,t) \geqslant 0 \text{ on } S, \quad \chi(x) > 0 \text{ in } \Omega.$$

Then the solution to the inverse problem (34)–(36) *is unique in the class of functions* $u \in W_p^{2,1}(Q)$, $c \in E_-$ *for* $p = n+1$.

Theorem 20. *Let the assumptions of Theorem* 19 *hold, and let*

$$\chi(x) \geqslant \delta > 0, \quad L_0[l(u^0) - \chi](x) \leqslant 0 \text{ in } \Omega.$$

Then there exists a unique pair of functions $\{u; c\}$ *solving the inverse problem* (34)–(36); *moreover, this solution possesses the properties*

$$u_t \in C([0,T]; L_p(\Omega)) \cap W_p^{2,1}(Q_\varepsilon), \quad u \in C([0,T]; W_p^2(\Omega)), \tag{38}$$

where $Q_\varepsilon = \Omega \times (\varepsilon, T)$ *and* $c(x) \geqslant -v_0(x)/\chi(x)$, $v_0(x) \equiv |L_0\chi| + l(g) + \rho(x,0)\omega(0)u_0(x)$.

The following two theorems are devoted to the case of a general nonlocal observation, i.e., the case where, regarding the function $\mu(t)$, it is only known that μ belongs to $BV[0,T]$ and satisfies (27).

Theorem 21. *Assume that Conditions* (*A*.2), (*B*.2), (*C*.2) *hold,* μ *satisfies* (27), *and the following inequalities hold*:

$$\begin{aligned} &g(x,t) \geqslant 0, \quad g_t(x,t) \geqslant 0, \quad d(x,t) \leqslant 0, \quad d_t(x,t) \geqslant 0 \quad \text{in } Q, \\ &u_0(x) \geqslant 0, \quad \chi(x) > 0 \quad \text{in } \Omega, \\ &\beta(x,t) \geqslant 0, \quad \beta_t(x,t) \geqslant 0 \quad \text{on } S. \end{aligned}$$

If for some $v_1 \in E_-$

$$L_0 u_0 + (d(x,0) + v_1(x))u_0(x) + g(x,0) \geqslant 0, \quad x \in \Omega,$$

and the inverse problem (34)–(36) *has a solution* $u \in W_p^{2,1}(Q)$, $c \in E_-$ *satisfying the inequality* $c \geqslant v_1$, *then this solution is unique in the class of functions* $u \in W_p^{2,1}(Q)$, $c \in E_-$ *for* $p = n+1$.

Theorem 22. *Assume that Conditions* (*A*.2), (*B*.2), (*C*.2) *hold,* μ *satisfies* (27), *and the following inequalities hold*:

$$\begin{aligned} &g(x,t) \geqslant 0, \quad g_t(x,t) \geqslant 0, \quad d(x,t) \leqslant 0, \quad d_t(x,t) \geqslant 0, \quad \text{in } Q, \\ &u_0(x) \geqslant 0, \quad \chi(x) \geqslant \delta > 0 \quad \text{in } \Omega, \\ &\beta(x,t) \geqslant 0, \quad \beta_t(x,t) \geqslant 0 \quad \text{on } S, \\ &L_0 u_0 + [d(x,0) - v_0(x)\chi^{-1}(x)]u_0(x) + g(x,0) \geqslant 0, \quad L_0[l(u^0) - \chi](x) \leqslant 0 \quad \text{in } \Omega, \end{aligned}$$

where $v_0(x) \equiv |L_0\chi| + l(g)$. *Then there exists a unique pair* $\{u; c\}$ *that is a solution to the inverse problem* (34)–(36); *moreover,* $u(x,t)$ *possesses the differential properties* (38) *and the function* $c(x)$ *satisfies the inequality* $c(x) \geqslant -v_0(x)/\chi(x)$ *in* Ω.

Remark 3. The assumptions of these theorems contain no restrictions on the norms of the given functions, but have the form of one-sided inequalities of positivity and monotonicity type which can be rather easily verified. Under the assumptions of Theorem 20 or 22, an iteration process for finding a solution $c(x)$ is proposed and the convergence of this process is justified, which can be useful in applications. A series of examples shows that the class of problem for which the assumptions of the proved theorem hold is rather large. There is an example of an inverse coefficient problem possessing a nonunique solution (cf. [10]). The problem in close settings was studied in [42]–[48].

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