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ON CONTACT BETWEEN A THIN OBSTACLE AND A PLATE CONTAINING A THIN INCLUSION

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We consider problems governing a contact between an elastic plate with a thin elastic inclusion and a thin elastic obstacle and study the equilibrium of the plate with or without cuts. We discuss various statements and establish the existence of a solution. We analyze the limit problem as the rigidity parameter of the elastic inclusion tends to infinity. Bibliography: 14 *titles. Illustrations*: 5 *figures.*

At present, the solvabililty of the equilibrium problem is established for various models describing cracks in solids (cf. [1]–[3] for details). In particular, various cases of thin inclusions in elastic bodies were studied in [4]–[11]. There is a huge literature devoted to contact problems, in particular, obstacle problems. The obstacle problems for a plate were analyzed in [12, 13]. The contact problem for a plate with a beam being a thin obstacle was first studied in [14].

In this paper, we are interested in the equilibrium of a plate (the Kirchhoff–Love model) with an obstacle and an inclusion (the Bernoulli–Euler beam model).

The paper is organized as follows. A plate without cuts (Figure 1, I) is considered in Section 1. We formulate the equilibrium problem and establish its unique solvability by using the variational approach. In Section 2, we study the equilibrium problem for a plate containing a cut (Figure 1, II). In Section 3, we study the dependence of solutions to the problem about a plate without cuts on the rigidity parameter of the inclusion. Then we pass to the limit as the rigidity parameter tends to infinity and find a complete collection of the boundary contact conditions for the limit problem.

1 Plate without Cuts

Assume that x_1, x_2, z denote the Cartesian coordinates and consider a plate occupying a bounded domain $\Omega \subset \mathbb{R}^2$ in the x_1x_2-plane (Figure 2). We assume that the boundary Γ of

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Ω is smooth. Assume that γ_{ob} and γ_{in} are segments in Ω intersecting under a nonzero angle. We also assume that the sets γ_{ob} and γ_{in} are open and have no common points with Γ. In the problem under consideration, the sets γ_{ob} and γ_{in} are interpreted as an obstacle and an inclusion respectively. We introduce the notation: $\gamma = \gamma_{ob} \cup \gamma_{in}$, $\Omega_g = \Omega \setminus \overline{\gamma}$, $\mathring{\gamma}_{ob} = \gamma_{ob} \setminus \gamma_{in}$, $\mathring{\gamma}_{in} = \gamma_{in} \setminus \gamma_{ob}$, $\mathring{\gamma} = \mathring{\gamma}_{ob} \cup \mathring{\gamma}_{in}$. We denote by $n = (n_1, n_2)$ the outward unit normal to Γ and by $\nu = (\nu_1, \nu_2)$ the unit normal to $\mathring{\gamma}$. The functions $w = w(x_1, x_1)$, $u = u(\eta_1)$, $v = v(\eta_2)$ are unknown. They are defined on the sets Ω, γ_{ob}, γ_{in} and characterize displacements of the plate, obstacle, inclusion along the z-axis respectively.

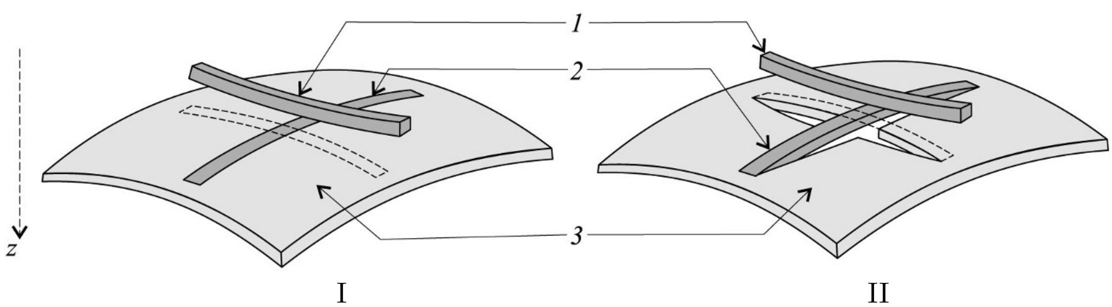

FIGURE 1. Contacting bodies: a thin obstacle – 1, a thin inclusion – 2, a plate – 3.

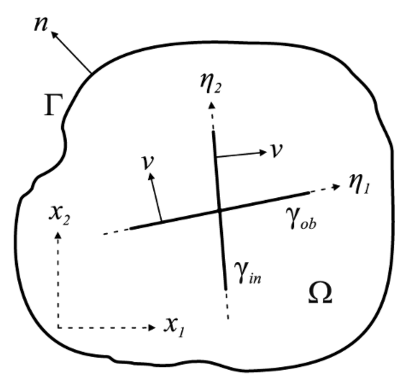

FIGURE 2. Geometry of the problem: the plate Ω, the obstacle γ_{ob}, and the inclusion γ_{in}.

We study the boundary value problem

$$(b_{ijkl} w_{,kl})_{,ij} = f \quad \text{in } \Omega_g, \tag{1}$$

$$w - u \geqslant 0 \text{ on } \gamma_{ob}, \quad w = v \text{ on } \gamma_{in}, \quad [w] = [w_{,\nu}] = 0 \text{ on } \mathring{\gamma}, \tag{2}$$

$$[m_\nu] = 0 \text{ on } \mathring{\gamma}, \quad \int\limits_{\mathring{\gamma}} [t^\nu] w + \int\limits_{\gamma_{ob}} u_{,11} u_{,11} + \int\limits_{\gamma_{in}} v_{,22} v_{,22} = 0, \tag{3}$$

$$\int\limits_{\mathring{\gamma}} [t^\nu] \overline{w} + \int\limits_{\gamma_{ob}} u_{,11} \overline{u}_{,11} + \int\limits_{\gamma_{in}} v_{,22} \overline{v}_{,22} \geqslant 0 \quad \forall\ (\overline{w}, \overline{u}, \overline{v}) \in K, \tag{4}$$

$$w = w_{,n} = 0 \text{ on } \Gamma, \quad u = u_{,1} = 0 \text{ on } \partial\gamma_{ob}, \tag{5}$$

where

$$f \in L^2(\Omega), \quad b_{ijkl} \in L^\infty(\Omega), \tag{6}$$

$$b_{ijkl} = b_{jikl} = b_{klij}, \quad b_{ijkl}\vartheta_{kl}\vartheta_{ij} \geqslant c_0|\vartheta|^2, \quad c_0 > 0, \quad \forall \vartheta_{ij} : \vartheta_{ij} = \vartheta_{ji}, \quad i,j,k,l = 1,2, \tag{7}$$

$$w_{,ij} = \frac{\partial^2 w}{\partial x_i \partial x_j}, \quad w_{,\nu} = \frac{dw}{d\nu}, \quad u_{,1} = \frac{du}{d\eta_1}, \quad v_{,2} = \frac{dv}{d\eta_2}; \quad [w] = w\big|_{\mathring{\gamma}^+} - w\big|_{\mathring{\gamma}^-},$$

$$m_\nu(w) = -m_{ij}(w)\nu_j\nu_i, \quad t^\nu(w) = -m_{ij,j}(w)\nu_i - m_{ij,k}(w)\tau_k\tau_j\nu_i,$$

$$m_{ij}(w) = -b_{ijkl}w_{,kl} \text{ in } \Omega, \quad (\tau_1, \tau_2) = (-\nu_2, \nu_1).$$

Equation (1) is the equilibrium equation for a plate. We will use the standard convention about summation over repeated indices. In (6) and (7), f is the distributed load and b_{ijkl} are the moduli of elasticity of the plate. We note that the equilibrium equation does not hold on the set γ, where the contact conditions (in particular, (2)) are given. The conditions (3) and (4) for the bending moment m_ν and shearing force t^ν express the virtual work principle. The set of admissible displacements is defined by

$$K = \{(w,u,v) \in H_0^2(\Omega) \times H_0^2(\gamma_{ob}) \times H^2(\gamma_{in}) : \; w - u \geqslant 0 \text{ on } \gamma_{ob}, \; w = v \text{ on } \gamma_{in}\}.$$

The conditions (5) mean that the plate is fixed on the edge and the obstacle is fixed at the endpoints.

1.1. Variational statement of the problem. The boundary value problem (1)–(5) can be formulated as the variational problem

$$\inf_{(w,u,v)\in K} \Pi(w,u,v) \tag{8}$$

with the total potential energy functional

$$\Pi(w,u,v) = \frac{1}{2}\int\limits_\Omega b_{ijkl}w_{,kl}w_{,ij} - \int\limits_\Omega fw + \frac{1}{2}\int\limits_{\gamma_{ob}} (u_{,11})^2 + \frac{1}{2}\int\limits_{\gamma_{in}} (v_{,22})^2. \tag{9}$$

We note that the functional Π is convex and differentiable; moreover, it is minimized over a convex set. Therefore, the problem (8) is equivalent to the variational inequality

$$(w,u,v) \in K, \quad \forall (\overline{w}, \overline{u}, \overline{v}) \in K :$$

$$\int\limits_\Omega b_{ijkl}w_{,kl}(\overline{w} - w)_{,ij} - \int\limits_\Omega f(\overline{w} - w) + \int\limits_{\gamma_{ob}} u_{,11}(\overline{u} - u)_{,11} + \int\limits_{\gamma_{in}} v_{,22}(\overline{v} - v)_{,22} \geqslant 0. \tag{10}$$

We show that the problems (1)–(5) and (10) are equivalent for smooth solutions. For this purpose we use the Green formula

$$\int\limits_{\Omega_g} b_{ijkl}w_{,kl}\widetilde{w}_{,ij} - \int\limits_{\Omega_g} (b_{ijkl}w_{,kl})_{,ij}\widetilde{w} = -\int\limits_{\mathring{\gamma}} [m_\nu \widetilde{w}_{,\nu}] + \int\limits_{\mathring{\gamma}} [t^\nu \widetilde{w}] + \int\limits_\Gamma m_n \widetilde{w}_{,n} - \int\limits_\Gamma t^n \widetilde{w}, \tag{11}$$

valid for w, $m_{ij}(w)$, $\widetilde{w} \in H^2(\Omega_g)$.

Let functions w, u, v satisfy (1)–(5) almost everywhere on the corresponding sets, and let these functions be sufficiently smooth so that $u \in H^2(\gamma_{ob})$, $v \in H^2(\gamma_{in})$ and the Green formula (11) is valid for w. We prove that the triple (w, u, v) is a solution of the variational inequality (10). By (2) and (5), we have $(w, u, v) \in K$. We choose arbitrarily $(\overline{w}, \overline{u}, \overline{v}) \in K$. Multiplying both sides of (1) by $\overline{w} - w$ and integrating over Ω_g, we find

$$\int\limits_{\Omega_g} (b_{ijkl} w_{,kl})_{,ij} (\overline{w} - w) = \int\limits_{\Omega_g} f(\overline{w} - w).$$

Taking into account the Green formula (11) and the first condition in (5), we get

$$\int\limits_{\Omega_g} b_{ijkl} w_{,kl} (\overline{w} - w)_{,ij} - \int\limits_{\Omega_g} f(\overline{w} - w) = -\int\limits_{\mathring{\gamma}} [m_\nu (\overline{w} - w)_{,\nu}] + \int\limits_{\mathring{\gamma}} [t^\nu (\overline{w} - w)].$$

Using the last condition in (2), we obtain the identity

$$\int\limits_{\Omega} b_{ijkl} w_{,kl} (\overline{w} - w)_{,ij} - \int\limits_{\Omega} f(\overline{w} - w) = -\int\limits_{\mathring{\gamma}} [m_\nu] (\overline{w} - w)_{,\nu} + \int\limits_{\mathring{\gamma}} [t^\nu] (\overline{w} - w).$$

From (3) and (4) it follows that

$$-\int\limits_{\mathring{\gamma}} [m_\nu] (\overline{w} - w)_{,\nu} + \int\limits_{\mathring{\gamma}} [t^\nu] (\overline{w} - w) = \int\limits_{\mathring{\gamma}} [t^\nu] (\overline{w} - w) \geqslant -\int\limits_{\gamma_{ob}} u_{,11} (\overline{u} - u)_{,11} - \int\limits_{\gamma_{in}} v_{,22} (\overline{v} - v)_{,22}.$$

Thus, the variational inequality (10) is valid.

Let (w, u, v) be a solution of the variational inequality (10). It is obvious that it satisfies (2) and (5). Let us prove (1) and (3), (4). Substituting $(\overline{w}, \overline{u}, \overline{v}) = (w \pm \varphi, u, v)$, $\varphi \in C_0^\infty(\Omega_g)$ for test functions into (10), we get

$$\int\limits_{\Omega_g} b_{ijkl} w_{,kl} \varphi_{,ij} = \int\limits_{\Omega_g} f\varphi,$$

which means that w satisfies the equilibrium equation (1) in the sense of the theory of distributions. Taking into account this equation, we can apply the Green formula (11) to (10) and obtain the inequality

$$-\int\limits_{\mathring{\gamma}} [m_\nu] (\overline{w} - w)_{,\nu} + \int\limits_{\mathring{\gamma}} [t^\nu] (\overline{w} - w) + \int\limits_{\gamma_{ob}} u_{,11} (\overline{u} - u)_{,11} + \int\limits_{\gamma_{in}} v_{,22} (\overline{v} - v)_{,22} \geqslant 0 \quad \forall (\overline{w}, \overline{u}, \overline{v}) \in K. \tag{12}$$

To prove the first condition in (3), we substitute $(\overline{w}, \overline{u}, \overline{v}) = (w \pm \phi_A, u, v)$, where $\phi_A = 0$ on $\mathring{\gamma}$, for test functions into (12), where

$$\phi_A \in H^2(\Omega), \quad \text{supp } \phi_A \subset A, \quad A \subset \Omega, \quad A = \overline{A}, \tag{13}$$

and for A we take the set shown in Figure 3. Then we obtain the equality

$$\int\limits_{\mathring{\gamma}} [m_\nu] (\phi_A)_{,\nu} = 0.$$

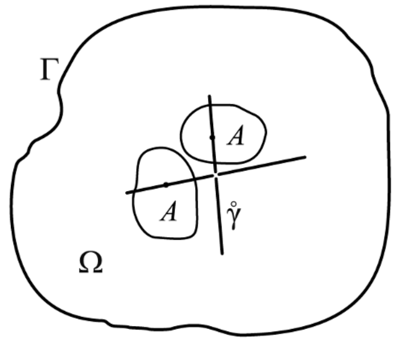

FIGURE 3. Neighborhoods of points of the set $\mathring{\gamma}$.

By the arbitrariness of ϕ_A, we obtain the required condition in (3). Finally, taking $(\overline{w},\overline{u},\overline{v}) = (0,0,0)$ and the $(\overline{w},\overline{u},\overline{v}) = 2(w,u,v)$ for test functions in (12), we find

$$-\int\limits_{\mathring{\gamma}}[m_\nu]w_{,\nu} + \int\limits_{\mathring{\gamma}}[t^\nu]w + \int\limits_{\gamma_{ob}} u_{,11}u_{,11} + \int\limits_{\gamma_{in}} v_{,22}v_{,22} = 0,$$

$$-\int\limits_{\mathring{\gamma}}[m_\nu]\overline{w}_{,\nu} + \int\limits_{\mathring{\gamma}}[t^\nu]\overline{w} + \int\limits_{\gamma_{ob}} u_{,11}\overline{u}_{,11} + \int\limits_{\gamma_{in}} v_{,22}\overline{v}_{,22} \geqslant 0 \quad \forall(\overline{w},\overline{u},\overline{v}) \in K.$$

Taking into account the proved condition in (3), we obtain the remaining relations in (3) and (4). Thus, a smooth solution of the variational inequality (10) satisfies all the conditions (1)–(5) of the boundary value problem. Thus the problems (1)–(5) and (10) are equivalent on the class of sufficiently smooth solutions.

Since the variational inequality (10) and the minimization problem (8) are equivalent, the problems (1)–(5) and (8) are equivalent on the class of sufficiently smooth solutions.

1.2. Unique solvability of the problem. We prove the solvability of the minimization problem (8). The energy functional Π is weakly lower semicontinuous since it is convex and continuous. We note that if

$$(w^n,u^n,v^n) \in K, \quad (w^n,u^n,v^n) \longrightarrow (w,u,v) \quad \text{strongly in } H_0^2(\Omega) \times H_0^2(\gamma_{ob}) \times H^2(\gamma_{in}),$$

then

$$(w^n\big|_\gamma, u^n, v^n) \longrightarrow (w\big|_\gamma, u, v) \quad \text{strongly in } \ L^2(\gamma) \times L^2(\gamma_{ob}) \times L^2(\gamma_{in})$$

in view of the embedding theorem. Passing to a subsequence, if necessary, we can set

$$(w^n\big|_\gamma, u^n, v^n) \longrightarrow (w\big|_\gamma, u, v) \quad \text{a.e. on } \gamma \times \gamma_{ob} \times \gamma_{in}.$$

Since $w^n - u^n \geqslant 0$ almost everywhere on γ_{ob} and $w^n = v^n$ almost everywhere on γ_{in}, we have $w - u \geqslant 0$ almost everywhere on γ_{ob} and $w = v$ almost everywhere on γ_{in}. Hence the set K is closed. Since K is convex, we conclude that K is weakly closed. Consequently, to prove the solvability of the problem (8), it suffices to verify that the functional Π is coercive. For this purpose we prove the following assertion.

Lemma 1. *There exists a constant $c > 0$ such that*

$$\int_{\Omega} b_{ijkl} w_{,kl} w_{,ij} + \int_{\gamma_{ob}} (u_{,11})^2 + \int_{\gamma_{in}} (v_{,22})^2 \geqslant c\|(w,u,v)\|^2 \quad \forall (w,u,v) \in K,$$

where $\|(w,u,v)\|^2 = \|w\|^2_{H^2_0(\Omega)} + \|u\|^2_{H^2_0(\gamma_{ob})} + \|v\|^2_{H^2(\gamma_{in})}$.

Proof. By the conditions (7) and the Poincaré–Friedrichs inequality, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\int_{\Omega} b_{ijkl} w_{,kl} w_{,ij} + \int_{\gamma_{ob}} (u_{,11})^2 \geqslant c_1 \|w\|^2_{H^2_0(\Omega)} + c_2 \|u\|^2_{H^2_0(\gamma_{ob})}.$$

Using the condition $w = v$ on γ_{in} and the embedding theorem, we can choose a constant $c_3 > 0$ such that

$$\frac{c_1}{2} \|w\|^2_{H^2_0(\Omega)} - c_3 \int_{\gamma_{in}} v^2 \geqslant 0.$$

According to the theory of Sobolev spaces,

$$c_3 \int_{\gamma_{in}} v^2 + \int_{\gamma_{in}} (v_{,22})^2 \geqslant c_4 \|v\|^2_{H^2(\gamma_{in})},$$

where $c_4 > 0$ is a constant. Adding the above inequalities, we find

$$\int_{\Omega} b_{ijkl} w_{,kl} w_{,ij} + \int_{\gamma_{ob}} (u_{,11})^2 + \int_{\gamma_{in}} (v_{,22})^2 \geqslant \frac{c_1}{2} \|w\|^2_{H^2_0(\Omega)} + c_2 \|u\|^2_{H^2_0(\gamma_{ob})} + c_4 \|v\|^2_{H^2(\gamma_{in})}.$$

For the constant $c > 0$ we can take the least of the constants $c_1/2$, c_2, and c_4. □

By Lemma 1, it is easy to see that

$$\Pi(w,u,v) \geqslant \frac{c}{2} \|(w,u,v)\|^2 - c_5 \|w\|_{H^2_0(\Omega)} \geqslant \frac{c}{2} \|(w,u,v)\|^2 - c_5 \|(w,u,v)\|, \quad c_5 > 0.$$

Hence $\Pi(w,u,v) \longrightarrow \infty$ as $\|(w,u,v)\| \longrightarrow \infty$. Thus, the functional Π is coercive on the set K and, consequently, the minimization problem (8) is solvable.

To prove the uniqueness of a solution to the problem (8), we assume that there are two different solutions (w^1, u^1, v^1) and (w^2, u^2, v^2). Then

$$\int_{\Omega} b_{ijkl} w^1_{,kl} (\overline{w} - w^1)_{,ij} + \int_{\Omega} f(\overline{w} - w^1) + \int_{\gamma_{ob}} u^1_{,11} (\overline{u} - u^1)_{,11} + \int_{\gamma_{in}} v^1_{,22} (\overline{v} - v^1)_{,22} \geqslant 0,$$

$$\int_{\Omega} b_{ijkl} w^2_{,kl} (\overline{w} - w^2)_{,ij} + \int_{\Omega} f(\overline{w} - w^2) + \int_{\gamma_{ob}} u^2_{,11} (\overline{u} - u^2)_{,11} + \int_{\gamma_{in}} v^2_{,22} (\overline{v} - v^2)_{,22} \geqslant 0,$$

where $(\overline{w}, \overline{u}, \overline{v}) \in K$ is arbitrary. Substituting $(\overline{w}, \overline{u}, \overline{v}) = (w^2, u^2, v^2)$ into the first inequality and $(\overline{w}, \overline{u}, \overline{v}) = (w^1, u^1, v^1)$ into the second one, we summarize the results and obtain the estimate

$$\int_{\Omega} b_{ijkl} \widetilde{w}_{,kl} \widetilde{w}_{,ij} + \int_{\gamma_{ob}} (\widetilde{u}_{,11})^2 + \int_{\gamma_{in}} (\widetilde{v}_{,22})^2 \leqslant 0,$$

where $(\widetilde{w},\widetilde{u},\widetilde{v}) = (w^2 - w^1, u^2 - u^1, v^2 - v^1)$. Moreover, we have the estimate

$$c\|(\widetilde{w},\widetilde{u},\widetilde{v})\|^2 \leqslant \int\limits_{\Omega} b_{ijkl}\widetilde{w}_{,kl}\widetilde{w}_{,ij} + \int\limits_{\gamma_{ob}} (\widetilde{u}_{,11})^2 + \int\limits_{\gamma_{in}} (\widetilde{v}_{,22})^2$$

which is proved in the same way as Lemma 1 (the key role is played by the condition $\widetilde{w} = \widetilde{v}$ on γ_{in} which follows from the construction of $(\widetilde{w},\widetilde{u},\widetilde{v})$). Hence $\|(\widetilde{w},\widetilde{u},\widetilde{v})\| \leqslant 0$ and, consequently, $(\widetilde{w},\widetilde{u},\widetilde{v}) = (0,0,0)$. Thus, the solutions (w^1, u^1, v^1) and (w^2, u^2, v^2) are equal, which contradicts our assumption. Thereby we have proved that the problem (8) is uniquely solvable.

2 Plate Containing a Cut

In this section, we study the case where displacements of the plate can be discontinuous on a given line. We introduce the set of admissible displacements

$$K_c = \{(w,u,v) \in H^2_\Gamma(\Omega_g) \times H^2_0(\gamma_{ob}) \times H^2(\gamma_{in}) : \; w - u \geqslant 0 \text{ on } \mathring{\gamma}^+_{ob}, \; w = v \text{ on } \mathring{\gamma}^+_{in}\},$$

where $H^2_\Gamma(\Omega_g) = \{w \in H^2(\Omega_g) : \; w = w_{,n} = 0 \text{ on } \Gamma\}$, and consider the equilibrium problem for finding w, u, v such that

$$(b_{ijkl}w_{,kl})_{,ij} = f \quad \text{in } \Omega_g, \tag{14}$$

$$w - u \geqslant 0 \text{ on } \mathring{\gamma}^+_{ob}, \quad w = v \text{ on } \mathring{\gamma}^+_{in}, \tag{15}$$

$$m_\nu = 0 \text{ on } \mathring{\gamma}^+, \quad m_\nu = t^\nu = 0 \text{ on } \mathring{\gamma}^-, \tag{16}$$

$$\int\limits_{\mathring{\gamma}^+} t^\nu w + \int\limits_{\gamma_{ob}} u_{,11}u_{,11} + \int\limits_{\gamma_{in}} v_{,22}v_{,22} = 0, \tag{17}$$

$$\int\limits_{\mathring{\gamma}^+} t^\nu \overline{w} + \int\limits_{\gamma_{ob}} u_{,11}\overline{u}_{,11} + \int\limits_{\gamma_{in}} v_{,22}\overline{v}_{,22} \geqslant 0 \quad \forall\; (\overline{w},\overline{u},\overline{v}) \in K_c, \tag{18}$$

$$w = w_{,n} = 0 \text{ on } \Gamma, \quad u = u_{,1} = 0 \text{ on } \partial\gamma_{ob}. \tag{19}$$

The contact conditions on $\mathring{\gamma}^+$ are described by (15), (17), (18) and the first relation in (16). The second relation in (16) means that the plate edge is free on the other cut side.

The boundary value problem (14)–(19) admits the variational statement

$$\inf_{(w,u,v)\in K_c} \Big\{ \frac{1}{2} \int\limits_{\Omega_g} b_{ijkl}w_{,kl}w_{,ij} - \int\limits_{\Omega_g} fw + \frac{1}{2}\int\limits_{\gamma_{ob}} (u_{,11})^2 + \frac{1}{2}\int\limits_{\gamma_{in}} (v_{,22})^2 \Big\}. \tag{20}$$

The problem (20) has a unique solution satisfying the variational inequality

$$(w,u,v) \in K_c \quad \forall\; (\overline{w},\overline{u},\overline{v}) \in K_c :$$

$$\int\limits_{\Omega_g} b_{ijkl}w_{,kl}(\overline{w} - w)_{,ij} - \int\limits_{\Omega_g} f(\overline{w} - w) + \int\limits_{\gamma_{ob}} u_{,11}(\overline{u} - u)_{,11} + \int\limits_{\gamma_{in}} v_{,22}(\overline{v} - v)_{,22} \geqslant 0. \tag{21}$$

Indeed, the functional in (20) is weakly lower semicontinuous and the set K_c is weakly closed. Let us show that the functional is coercive. Assume that the set γ can be extended to the exterior boundary Γ, dividing Ω_g into the subdomains Ω_i, $\operatorname{mes}(\partial\Omega_i \cap \Gamma) \neq 0$, $i = 1, \dots, 4$ (cf. Figure 4). For the restrictions of w on Ω_i, and their generalized derivatives the Poincaré–Friedrichs inequality holds. Hence there exist constants $c_6 > 0$ and $c_2 > 0$ such that

$$\int\limits_{\Omega_g} b_{ijkl} w_{,kl} w_{,ij} + \int\limits_{\gamma_{ob}} (u_{,11})^2 + \int\limits_{\gamma_{in}} (v_{,22})^2 \geqslant c_6 \|w\|^2_{H^2_\Gamma(\Omega_g)} + c_2 \|u\|^2_{H^2_0(\gamma_{ob})} + \int\limits_{\gamma_{in}} (v_{,22})^2.$$

Since $w = v$ on $\mathring{\gamma}^+_{in}$, we can choose a constant $c_7 > 0$ such that

$$\int\limits_{\Omega_g} b_{ijkl} w_{,kl} w_{,ij} + \int\limits_{\gamma_{ob}} (u_{,11})^2 + \int\limits_{\gamma_{in}} (v_{,22})^2 \geqslant \frac{c_6}{2} \|w\|^2_{H^2_\Gamma(\Omega_g)} + c_2 \|u\|^2_{H^2_0(\gamma_{ob})} + c_7 \|v\|^2_{H^2(\gamma_{in})}.$$

This estimate, together with the inequality

$$-\int\limits_{\Omega_g} f w \geqslant -c_5 \|w\|_{H^2_\Gamma(\Omega_g)}, \quad c_5 > 0,$$

implies the coercivity of the functional on K_c. Hence the problem (20) has a solution. The uniqueness of a solution can be easily proved by contradiction.

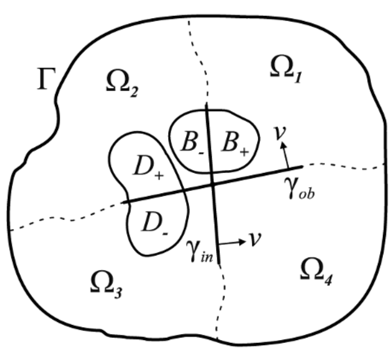

FIGURE 4. Geometry of the problem about a plate containing a cut.

Let us show the equivalence of the problems (14)–(19) and (20), (21) on the class of sufficiently smooth solutions. Assume that w, u, v satisfy (14)–(19), $u \in H^2(\gamma_{ob})$, $v \in H^2(\gamma_{in})$, and w is sufficiently smooth so that the Green formula (11) holds. We show that (w, u, v) satisfies (21). It is obvious that $(w, u, v) \in K_c$. Let $(\overline{w}, \overline{u}, \overline{v}) \in K_c$. Multiplying both sides of (14) by $\overline{w} - w$, integrating over Ω_g, and using the Green formula (11) and the conditions (19), we get

$$\int\limits_{\Omega_g} b_{ijkl} w_{,kl} (\overline{w} - w)_{,ij} - \int\limits_{\Omega_g} f(\overline{w} - w) = -\int\limits_{\mathring{\gamma}} [m_\nu (\overline{w} - w)_{,\nu}] + \int\limits_{\mathring{\gamma}} [t^\nu (\overline{w} - w)].$$

Applying (16), we obtain the identity

$$\int\limits_{\Omega_g} b_{ijkl} w_{,kl} (\overline{w} - w)_{,ij} - \int\limits_{\Omega_g} f(\overline{w} - w) = \int\limits_{\mathring{\gamma}^+} t^\nu (\overline{w} - w)$$

which implies (21) in view of (17) and (18). On the other hand, for a smooth solution to the problem (21) all the relations (14)–(19) hold. Indeed, (15) and (19) are valid by the definition of the set K_c. The equilibrium equation (14) in the sense of the theory of distributions can be easily obtained by taking $(\overline{w}, \overline{u}, \overline{v}) = (w \pm \varphi, u, v)$ in (21), where $\varphi \in C_0^\infty(\Omega_g)$. By the Green formula (11) and (19), from (21) we find

$$-\int\limits_{\mathring{\gamma}} [m_\nu(\overline{w} - w)_{,\nu}] + \int\limits_{\mathring{\gamma}} [t^\nu(\overline{w} - w)] + \int\limits_{\gamma_{ob}} u_{,11}(\overline{u} - u)_{,11} + \int\limits_{\gamma_{in}} v_{,22}(\overline{v} - v)_{,22} \geqslant 0 \quad \forall (\overline{w}, \overline{u}, \overline{v}) \in K_c.$$

Taking the test functions $(\overline{w}, \overline{u}, \overline{v}) = (0, 0, 0)$ and $(\overline{w}, \overline{u}, \overline{v}) = 2(w, u, v)$, we obtain the relations

$$\begin{aligned} &-\int\limits_{\mathring{\gamma}} [m_\nu w_{,\nu}] + \int\limits_{\mathring{\gamma}} [t^\nu w] + \int\limits_{\gamma_{ob}} u_{,11} u_{,11} + \int\limits_{\gamma_{in}} v_{,22} v_{,22} = 0, \\ &-\int\limits_{\mathring{\gamma}} [m_\nu \overline{w}_{,\nu}] + \int\limits_{\mathring{\gamma}} [t^\nu \overline{w}] + \int\limits_{\gamma_{ob}} u_{,11} \overline{u}_{,11} + \int\limits_{\gamma_{in}} v_{,22} \overline{v}_{,22} \geqslant 0 \quad \forall (\overline{w}, \overline{u}, \overline{v}) \in K_c. \end{aligned} \tag{22}$$

Let us prove (16) and thereby prove (17) and (18). For this purpose we take

$$\begin{aligned} &(\overline{w}, \overline{u}, \overline{v}) = (\pm\phi_{B_+}, 0, 0), \quad (\overline{w}, \overline{u}, \overline{v}) = (\pm\phi_{D_+}, 0, 0), \quad \phi_{B_+}\big|_{\mathring{\gamma}^+} = \phi_{D_+}\big|_{\mathring{\gamma}^+} = 0, \\ &(\overline{w}, \overline{u}, \overline{v}) = (\pm\phi_{B_-}, 0, 0), \quad (\overline{w}, \overline{u}, \overline{v}) = (\pm\phi_{D_-}, 0, 0) \end{aligned}$$

for test functions in (22), where $\phi_A \in H^2(\Omega_g)$, supp $\phi_A \subset A$, $A \subset \Omega$, $A = \overline{A}$, and the sets $B_\pm$ and $D_\pm$ are as shown in Figure 4. Then

$$\int\limits_{\mathring{\gamma}_{in}^+} m_\nu (\phi_{B_+})_{,\nu} = 0, \quad \int\limits_{\mathring{\gamma}_{ob}^+} m_\nu (\phi_{D_+})_{,\nu} = 0,$$

$$\int\limits_{\mathring{\gamma}_{in}^-} m_\nu (\phi_{B_-})_{,\nu} - \int\limits_{\mathring{\gamma}_{in}^-} t^\nu \phi_{B_-} = 0, \quad \int\limits_{\mathring{\gamma}_{ob}^-} m_\nu (\phi_{D_-})_{,\nu} - \int\limits_{\mathring{\gamma}_{ob}^-} t^\nu \phi_{D_-} = 0,$$

which implies (16) because $(\phi_{B_+})_{,\nu}$, $(\phi_{D_+})_{,\nu}$, $(\phi_{B_-})_{,\nu}$, $(\phi_{D_-})_{,\nu}$, ϕ_{D_-}, ϕ_{B_-} are arbitrary. Thus, a smooth solution of the variational inequality (21) satisfies (14)–(19). Hence we proved the equivalence of the problems (14)–(19) and (21) on the class of sufficiently smooth solutions.

3 Limit Problem as the Rigidity Parameter Tends to Infinity

We come back to the case of a plate without cuts, studied in Section 1. Now, we consider the equilibrium problem with the positive parameter λ:

$$(b_{ijkl} w^\lambda_{,kl})_{,ij} = f \quad \text{in } \Omega_g, \tag{23}$$

$$w^\lambda - u^\lambda \geqslant 0 \text{ on } \gamma_{ob}, \quad w^\lambda = v^\lambda \text{ on } \gamma_{in}, \quad [w^\lambda] = [w^\lambda_{,\nu}] = 0 \text{ on } \mathring{\gamma}, \tag{24}$$

$$[m_\nu(w^\lambda)] = 0 \text{ on } \mathring{\gamma}, \quad \int\limits_{\mathring{\gamma}} [t^\nu(w^\lambda)] w^\lambda + \int\limits_{\gamma_{ob}} u^\lambda_{,11} u^\lambda_{,11} + \lambda \int\limits_{\gamma_{in}} v^\lambda_{,22} v^\lambda_{,22} = 0, \tag{25}$$

$$\int\limits_{\mathring{\gamma}} [t^\nu(w^\lambda)] \overline{w} + \int\limits_{\gamma_{ob}} u^\lambda_{,11} \overline{u}_{,11} + \lambda \int\limits_{\gamma_{in}} v^\lambda_{,22} \overline{v}_{,22} \geqslant 0 \quad \forall (\overline{w}, \overline{u}, \overline{v}) \in K, \tag{26}$$

$$w^\lambda = w^\lambda_{,n} = 0 \text{ on } \Gamma, \quad u^\lambda = u^\lambda_{,1} = 0 \text{ on } \partial\gamma_{ob}, \tag{27}$$

The parameter λ characterizes the inclusion rigidity. Indeed, taking the test functions $(\overline{w}, \overline{u}, \overline{v}) = (\pm\phi, 0, \pm\varphi)$, where $\varphi = \phi$ on γ_{in}, $\phi \in C_0^\infty(\Omega \setminus \gamma_{ob} \setminus \partial\gamma_{in})$, in (26), we get

$$\int\limits_{\mathring{\gamma}_{in}} [t^\nu(w^\lambda)]\varphi + \lambda \int\limits_{\mathring{\gamma}_{in}} v^\lambda_{,22}\varphi_{,22} = 0 \quad \forall\, \varphi \in C_0^\infty(\mathring{\gamma}_{in}).$$

This means that a solution to the problem (23)–(27) satisfies the equation

$$-\lambda v^\lambda_{,2222} = [t^\nu(w^\lambda)] \quad \text{on } \mathring{\gamma}_{in}$$

in the sense of the theory of distributions. Within the framework of the theory of Bernoulli–Euler beams, the equation obtained expresses the connection between the beam bending and the load on the beam. With each fixed $\lambda \in (0, \infty)$ we can associate a unique solution $(w^\lambda, u^\lambda, v^\lambda)$ to the boundary value problem (23)–(27). Indeed, this problem can be formulated as the minimization problem

$$\inf_{(w^\lambda, u^\lambda, v^\lambda) \in K} \Big\{ \frac{1}{2} \int\limits_\Omega b_{ijkl} w^\lambda_{,kl} w^\lambda_{,ij} - \int\limits_\Omega f w^\lambda + \frac{1}{2} \int\limits_{\gamma_{ob}} (u^\lambda_{,11})^2 + \frac{\lambda}{2} \int\limits_{\gamma_{in}} (v^\lambda_{,22})^2 \Big\} \tag{28}$$

and as the variational inequality

$$(w^\lambda, u^\lambda, v^\lambda) \in K, \quad \forall (\overline{w}, \overline{u}, \overline{v}) \in K :$$

$$\int\limits_\Omega b_{ijkl} w^\lambda_{,kl} (\overline{w} - w^\lambda)_{,ij} - \int\limits_\Omega f(\overline{w} - w^\lambda) + \int\limits_{\gamma_{ob}} u^\lambda_{,11}(\overline{u} - u^\lambda)_{,11} + \lambda \int\limits_{\gamma_{in}} v^\lambda_{,22}(\overline{v} - v^\lambda)_{,22} \geqslant 0. \tag{29}$$

The equivalence of the problems (23)–(27) and (29) on the class of sufficiently smooth solutions is proved in the same way as in Subsection 1.1. Furthermore, the minimization problem (28) is uniquely solvable. The proof is the same as in Subsection 1.2.

3.1. The limit as $\lambda \to \infty$. We study the behavior of the solution $(w^\lambda, u^\lambda, v^\lambda)$ to the problem (29) as $\lambda \to \infty$. We show that it is possible to extract a converging subsequence from the family $(w^\lambda, u^\lambda, v^\lambda)_{\lambda \in (0,\infty)}$. Substituting the test functions $(\overline{w}, \overline{u}, \overline{v}) = (0, 0, 0)$ and $(\overline{w}, \overline{u}, \overline{v}) = 2(w^\lambda, u^\lambda, v^\lambda)$ into (29), we obtain the equality

$$\int\limits_\Omega b_{ijkl} w^\lambda_{,kl} w^\lambda_{,ij} - \int\limits_\Omega f w^\lambda + \int\limits_{\gamma_{ob}} (u^\lambda_{,11})^2 + \lambda \int\limits_{\gamma_{in}} (v^\lambda_{,22})^2 = 0. \tag{30}$$

Applying Lemma 1 and the Cauchy–Bunyakowsky inequality to the left-hand side of (30), we obtain the uniform estimate

$$\|w^\lambda\|^2_{H_0^2(\Omega)} - \|w^\lambda\|_{H_0^2(\Omega)} + \|u^\lambda\|^2_{H_0^2(\gamma_{ob})} + \|v^\lambda\|^2_{H^2(\gamma_{in})} \leqslant 0$$

with respect to $\lambda \in [\lambda_0, \infty)$, where $\lambda_0 > 0$. This estimate implies the boundedness of the solution:

$$\|w^\lambda\|_{H_0^2(\Omega)} \leqslant c_8, \quad \|u^\lambda\|_{H_0^2(\gamma_{ob})} \leqslant c_9, \quad \|v^\lambda\|_{H^2(\gamma_{in})} \leqslant c_{10}, \quad c_8, c_9, c_{10} > 0. \tag{31}$$

In turn, from (30) and (7) it follows that

$$\lambda \int_{\gamma_{in}} (v^\lambda_{,22})^2 \leqslant \int_\Omega f w^\lambda,$$

which implies the estimate

$$\int_{\gamma_{in}} (v^\lambda_{,22})^2 \leqslant \frac{c_{11}}{\lambda}, \quad c_{11} > 0. \tag{32}$$

By the estimates (31) and (32), there exists a subsequence such that

$$(w^\lambda, u^\lambda, v^\lambda) \longrightarrow (w, u, v) \quad \text{weakly in } H^2_0(\Omega) \times H^2_0(\gamma_{ob}) \times H^2(\gamma_{in}), \quad \lambda \longrightarrow \infty, \tag{33}$$

$$v_{,22} = 0 \quad \text{on } \gamma_{in}. \tag{34}$$

We clarify some properties of the limit functions. Since the set K is weakly closed, from (33) we have $w - u \geqslant 0$ on γ_{ob} and $w = v$ on γ_{in}. The relation (34) means that the function $v(\eta_2)$ is affine on γ_{in}. By the linear connection between the coordinates η_2 and x_1, x_2, the restriction $w\big|_{\gamma_{in}}$ belongs to the space of rigid displacements

$$L(\gamma_{in}) = \{l : l(x_1, x_2) = a_0 + a_1 x_1 + a_2 x_2 \text{ on } \gamma_{in}, a_i \in \mathbb{R}, i = 0, 1, 2\}.$$

The space $L(\gamma_{in})$ consists of all functions possessing the affine structure on the set γ_{in}. Thus, for the limit functions we have $(w, u) \in K_r$, where

$$K_r = \big\{(w, u) \in H^2_0(\Omega) \times H^2_0(\gamma_{ob}) : w - u \geqslant 0 \text{ on } \gamma_{ob}, w\big|_{\gamma_{in}} \in L(\gamma_{in})\big\}.$$

We deduce the variational problem corresponding to the limit case. For this purpose we pass to the limit in (29). Let $(\overline{w}, \overline{u}) \in K_r$ be arbitrary. It is obvious that $(\overline{w}, \overline{u}, \overline{w}\big|_{\gamma_{in}}) \in K$. Therefore, (29) implies

$$\int_\Omega b_{ijkl} w^\lambda_{,kl} \overline{w}_{,ij} + \int_{\gamma_{ob}} u^\lambda_{,11} \overline{u}_{,11} - \int_\Omega f(\overline{w} - w^\lambda) \geqslant \int_\Omega b_{ijkl} w^\lambda_{,kl} w^\lambda_{,ij} + \int_{\gamma_{ob}} (u^\lambda_{,11})^2 + \lambda \int_{\gamma_{in}} (v^\lambda_{,22})^2.$$

Taking into account (33) and (34), we pass to the lower limit on both sides of the obtained inequality. Then

$$\liminf_{\lambda\to\infty} \left\{ \int_\Omega b_{ijkl} w^\lambda_{,kl} \overline{w}_{,ij} + \int_{\gamma_{ob}} u^\lambda_{,11} \overline{u}_{,11} - \int_\Omega f(\overline{w} - w^\lambda) \right\}$$

$$= \int_\Omega b_{ijkl} w_{,kl} \overline{w}_{,ij} + \int_{\gamma_{ob}} u_{,11} \overline{u}_{,11} - \int_\Omega f(\overline{w} - w).$$

On the other hand,

$$\liminf_{\lambda\to\infty} \left\{ \int_\Omega b_{ijkl} w^\lambda_{,kl} w^\lambda_{,ij} + \int_{\gamma_{ob}} (u^\lambda_{,11})^2 + \lambda \int_{\gamma_{in}} (v^\lambda_{,22})^2 \right\}$$

$$\geqslant \liminf_{\lambda\to\infty} \int_\Omega b_{ijkl} w^\lambda_{,kl} w^\lambda_{,ij} + \liminf_{\lambda\to\infty} \int_{\gamma_{ob}} (u^\lambda_{,11})^2 + \lambda_0 \liminf_{\lambda\to\infty} \int_{\gamma_{in}} (v^\lambda_{,22})^2$$

$$\geqslant \int_\Omega b_{ijkl} w_{,kl} w_{,ij} + \int_{\gamma_{ob}} (u_{,11})^2.$$

As a result,

$$(w,u)\in K_r,\quad \forall(\overline{w},\overline{u})\in K_r:$$
$$\int\limits_{\Omega} b_{ijkl}w_{,kl}(\overline{w}-w)_{,ij}-\int\limits_{\Omega} f(\overline{w}-w)+\int\limits_{\gamma_{ob}} u_{,11}(\overline{u}-u)_{,11}\geqslant 0. \tag{35}$$

Thus, we obtain the variational inequality corresponding to the limit case. We note that it is equivalent to the minimization problem

$$\inf_{(w,u)\in K_r}\Big\{\frac{1}{2}\int\limits_{\Omega} b_{ijkl}w_{,kl}w_{,ij}-\int\limits_{\Omega} fw+\frac{1}{2}\int\limits_{\gamma_{ob}}(u_{,11})^2\Big\},$$

the solvability of which can be proved independently from the above arguments since the functional is coercive.

The boundary value problem corresponding to the variational inequality (35) is to find functions w, u and constants $a_0,a_1,a_2\in\mathbb{R}$ such that

$$(b_{ijkl}w_{,kl})_{,ij}=f\quad\text{in }\Omega_g, \tag{36}$$
$$w-u\geqslant 0\text{ on }\gamma_{ob},\quad [w]=[w_{,\nu}]=[m_\nu]=0\text{ on }\mathring{\gamma}, \tag{37}$$
$$w=a_0+a_1x_1+a_2x_2\quad\text{on }\gamma_{in}, \tag{38}$$
$$\int\limits_{\mathring{\gamma}}[t^\nu]w+\int\limits_{\gamma_{ob}}u_{,11}u_{,11}=0, \tag{39}$$
$$\int\limits_{\mathring{\gamma}}[t^\nu]\overline{w}+\int\limits_{\gamma_{ob}}u_{,11}\overline{u}_{,11}\geqslant 0\quad\forall(\overline{w},\overline{u})\in K_r, \tag{40}$$
$$w=w_{,n}=0\text{ on }\Gamma,\quad u=u_{,1}=0\text{ on }\partial\gamma_{ob}. \tag{41}$$

The equivalence of this problem and the variational inequality (35) on the class of sufficiently smooth solutions is proved in the same way as in Subsection 1.1.

The system (36)–(41) describes a contact of a thin elastic obstacle with a plate containing a thin rigid inclusion. The inclusion in the plate is described by the condition (38) which can be interpreted as the fact that the plate displacements on the set γ_{in} have a certain structure.

3.2. Differential statement of the limit problem. The problem (36)–(41) obtained as the limit of the problems (23)–(27) as $\lambda\to\infty$ is equivalent to the following problem: Find functions w, u and constants $a_0,a_1,a_2\in\mathbb{R}$ such that

$$(b_{ijkl}w_{,kl})_{,ij}=f\quad\text{in }\Omega_g, \tag{42}$$
$$w-u\geqslant 0\text{ on }\gamma_{ob},\quad [w]=[w_{,\nu}]=[m_\nu]=0\text{ on }\mathring{\gamma}, \tag{43}$$
$$w=a_0+a_1x_1+a_2x_2\quad\text{on }\gamma_{in}, \tag{44}$$
$$[t^\nu]=-u_{,1111},\quad [t^\nu]\geqslant 0,\quad [t^\nu](w-u)=0\text{ on }\mathring{\gamma}_{ob}, \tag{45}$$
$$[u(0)]=[u_{,1}(0)]=[u_{,11}(0)]=0, \tag{46}$$

$$\int_{\mathring{\gamma}_{in}} [t^\nu] + [u_{,111}(0)] = 0, \quad \int_{\mathring{\gamma}_{in}} [t^\nu] x_i + [u_{,111}(0)]\varkappa_i = 0, \quad i = 1, 2, \tag{47}$$

$$[u_{,111}(0)] \leqslant 0, \quad [u_{,111}(0)](w(\varkappa_1, \varkappa_2) - u(0)) = 0, \tag{48}$$

$$w = w_{,n} = 0 \text{ on } \Gamma, \quad u = u_{,1} = 0 \text{ on } \partial\gamma_{ob}, \tag{49}$$

where $\gamma_{ob} \cap \gamma_{in} = \{(\varkappa_1, \varkappa_2)\}$, and $[u(0)] = u(0+) - u(0-)$. The contact conditions on the plate, obstacle, and rigid inclusion are given on the set γ. In particular, the first condition in (45) is the equilibrium equation for the obstacle. The load on the obstacle is realized by the jump of the shearing force $[t^\nu]$ on $\mathring{\gamma}_{ob}$. At the same time, by the remaining conditions in (45), the jump $[t^\nu]$ vanishes at points where there is no contact between the plate and obstacle. The conditions (47) mean the equilibrium of the rigid inclusion. The inclusion is subject to the action of $[t^\nu]$ on $\mathring{\gamma}_{in}$ from the side of the plate and the action of $[u_{,111}(0)]$ at the point $(\varkappa_1, \varkappa_2)$ from the side of the obstacle. From the mechanical point of view, the conditions (47) mean that the principal vector of the above forces and the principal momentum vanish. By the condition (48) the jump $[u_{,111}(0)]$ vanishes if there is no contact between the obstacle and inclusion.

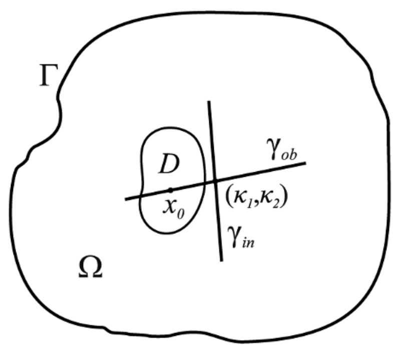

FIGURE 5. Neighborhood of a point of the set $\mathring{\gamma}_{ob}$.

We show that the problems (36)–(41) and (42)–(49) are equivalent on the class of sufficiently smooth solutions. For this purpose we assume that w, $m_{ij}(w) \in H^2(\Omega_g)$, $u \in H^4(\gamma_{ob})$. Let w and u be smooth solutions to the problem (36)–(41). We prove that they satisfy (42)–(49). It suffices to obtain the conditions (45)–(48) since the remaining ones are involved in the formulation of the problem (36)–(41). We substitute the test functions $(\overline{w}, \overline{u}) = (\pm\phi_D, \pm\phi_D\big|_{\gamma_{ob}})$ into (40), where D is the closure of a small neighborhood of some point in $\mathring{\gamma}_{ob}$ (functions of the form ϕ_A are defined by (13) and the set D is represented in Figure 5). Integrating by parts, we get

$$\int_{\mathring{\gamma}_{ob}} ([t^\nu] + u_{,1111})\phi_D = 0,$$

where the values of ϕ_D are arbitrary on $\mathring{\gamma}_{ob}$. Thus, we obtain the first condition in (45). To prove the last condition in (45), we assume that $w > u$ at some point $x_0 \in \mathring{\gamma}_{ob}$. For D we take the closure of a small neighborhood of this point and choose a small parameter $\varepsilon_\phi > 0$ such that $(\overline{w}, \overline{u}) = (w \pm \varepsilon_\phi\phi_D, u) \in K_r$, where $\phi_D \geqslant 0$ on $\mathring{\gamma}_{ob}$. Then (40) implies

$$\int_{\mathring{\gamma}} [t^\nu] w \pm \varepsilon_\phi \int_{\mathring{\gamma}_{ob}} [t^\nu]\phi_D + \int_{\gamma_{ob}} u_{,11}u_{,11} \geqslant 0.$$

The last inequality, together with (39), leads to the equality

$$\int\limits_{\mathring{\gamma}_{ob}} [t^\nu]\phi_D = 0 \tag{50}$$

which means that $[t^\nu] = 0$ near the point x_0. At the same time, if $[t^\nu] > 0$ at x_0, then $w = u$ at x_0. Indeed, assuming that $[t^\nu] > 0$ and $w > u$ at the point x_0 and repeating the above procedure, we obtain the identity (50) which contradicts our assumption. Thereby the last condition in (45) is proved.

Taking the test functions $(\overline{w}, \overline{u}) = (\pm\phi, \pm\varphi)$, $(\phi, \varphi) \in K_r$, where $\phi = \varphi$ on γ_{ob}, $\phi = \widehat{a}_0 + \widehat{a}_1 x_1 + \widehat{a}_2 x_2$ on γ_{in}, $\widehat{a}_0, \widehat{a}_1, \widehat{a}_2 \in \mathbb{R}$ in (40), we get

$$\int\limits_{\mathring{\gamma}_{in}} [t^\nu]\phi + \int\limits_{\mathring{\gamma}_{ob}} [t^\nu]\varphi + \int\limits_{\mathring{\gamma}_{ob}} u_{,11}\varphi_{,11} = 0.$$

Integrating by parts the last term and taking into account the first condition in (45), we find

$$\widehat{a}_0 \int\limits_{\mathring{\gamma}_{in}} [t^\nu] + \widehat{a}_1 \int\limits_{\mathring{\gamma}_{in}} [t^\nu]x_1 + \widehat{a}_2 \int\limits_{\mathring{\gamma}_{in}} [t^\nu]x_2 - [u_{,11}(0)]\varphi_{,1}(0) + [u_{,111}(0)]\varphi(0) = 0.$$

By the choice of the test functions, we have $\varphi(0) = \widehat{a}_0 + \widehat{a}_1\varkappa_1 + \widehat{a}_2\varkappa_2$. Therefore,

$$\widehat{a}_0\Big(\int\limits_{\mathring{\gamma}_{in}} [t^\nu] + [u_{,111}(0)]\Big) + \widehat{a}_i\Big(\int\limits_{\mathring{\gamma}_{in}} [t^\nu]x_i + [u_{,111}(0)]\varkappa_i\Big) - [u_{,11}(0)]\varphi_{,1}(0) = 0.$$

Since $\widehat{a}_0$, $\widehat{a}_1$, $\widehat{a}_2$, $\varphi_{,1}(0)$ are independent and arbitrary, the conditions (47) and (46) hold.

At the next step, we take the test functions $(\overline{w}, \overline{u}) = (\zeta, 0)$, $(\zeta, 0) \in K_r$, where $\zeta \geqslant 0$ on γ_{ob}, in (40). Then

$$\int\limits_{\mathring{\gamma}_{in}} [t^\nu]\zeta + \int\limits_{\mathring{\gamma}_{ob}} [t^\nu]\zeta \geqslant 0.$$

Moreover, $\zeta\big|_{\gamma_{in}} \in L(\gamma_{in})$. Hence, taking into account (47), we obtain the inequality

$$-[u_{,111}(0)]\zeta(\varkappa_1, \varkappa_2) + \int\limits_{\mathring{\gamma}_{ob}} [t^\nu]\zeta \geqslant 0.$$

The test functions can be taken in such a way that $\zeta(\varkappa_1, \varkappa_2) = 0$. Therefore, the second condition in (45) is valid. Let us prove the first condition in (48) by contradiction. We assume that $\zeta(\varkappa_1, \varkappa_2) > 0$ and $[u_{,111}(0)] > 0$. Then

$$\int\limits_{\mathring{\gamma}_{ob}} [t^\nu]\zeta \geqslant [u_{,111}(0)]\zeta(\varkappa_1, \varkappa_2) > 0.$$

Applying the Cauchy–Bunyakowsky inequality, we get

$$\frac{\|\zeta\|_{L^2(\mathring{\gamma}_{ob})}}{\zeta(\varkappa_1, \varkappa_2)} \geqslant \frac{[u_{,111}(0)]}{\|[t^\nu]\|_{L^2(\mathring{\gamma}_{ob})}} > 0.$$

The values $\|\zeta\|_{L^2(\mathring{\gamma}_{ob})}$, $\zeta(\varkappa_1,\varkappa_2)$ do not depend on each other and on the values $[u_{,111}(0)]$, $\|[t^\nu]\|_{L^2(\mathring{\gamma}_{ob})}$. Therefore, the function ζ can chosen in such a way that the obtained chain of inequalities fails. Hence the assumption $[u_{,111}(0)] > 0$ leads to a contradiction, which means that the first condition in (48) is valid. To prove the second condition in (48), we integrate by parts in (39). Then we have the identity

$$\int\limits_{\mathring{\gamma}_{in}} [t^\nu]w + \int\limits_{\mathring{\gamma}_{ob}} [t^\nu]w + \int\limits_{\mathring{\gamma}_{ob}} u_{,1111}u - [u_{,11}(0)]u_{,1}(0) + [u_{,111}(0)]u(0) = 0.$$

Applying the above-proved conditions (44)–(46), we obtain the identity

$$a_0 \int\limits_{\mathring{\gamma}_{in}} [t^\nu] + a_1 \int\limits_{\mathring{\gamma}_{in}} [t^\nu]x_1 + a_2 \int\limits_{\mathring{\gamma}_{in}} [t^\nu]x_2 + [u_{,111}(0)]u(0) = 0.$$

By (47), from the last identity it follows that

$$-[u_{,111}(0)](a_0 + a_1\varkappa_1 + a_2\varkappa_2 - u(0)) = 0.$$

By (38), the obtained equality means that the second relation in (48) is valid. Thus, we have proved that a sufficiently smooth solution to the problem (36)–(41) satisfies (42)–(49).

Now, let functions w and u be smooth solutions to the problem (42)–(49). We show that they satisfy (36)–(41). In fact, it suffices to prove (39) and (40) since the remaining relations are contained in (42)–(49). Let $(\overline{w}, \overline{u}) \in K_r$ be arbitrary. We multiply both sides of the first relation in (45) by $\overline{u}$ and integrate over $\mathring{\gamma}_{ob}$. Integrating by parts and taking into account (46), we arrive at the equality

$$\int\limits_{\mathring{\gamma}_{ob}} [t^\nu]\overline{u} = -\int\limits_{\gamma_{ob}} u_{,11}\overline{u}_{,11} + [u_{,111}(0)]\overline{u}(0)$$

which implies

$$\int\limits_{\mathring{\gamma}} [t^\nu]\overline{w} + \int\limits_{\gamma_{ob}} u_{,11}\overline{u}_{,11} = \int\limits_{\mathring{\gamma}_{in}} [t^\nu]\overline{w} + \int\limits_{\mathring{\gamma}_{ob}} [t^\nu](\overline{w} - \overline{u}) + [u_{,111}(0)]\overline{u}(0).$$

By the condition $\overline{w}\big|_{\gamma_{in}} \in L(\gamma_{in})$ and Equation (47), we get

$$\int\limits_{\mathring{\gamma}} [t^\nu]\overline{w} + \int\limits_{\gamma_{ob}} u_{,11}\overline{u}_{,11} = \int\limits_{\mathring{\gamma}_{ob}} [t^\nu](\overline{w} - \overline{u}) - [u_{,111}(0)](\overline{w}(\varkappa_1,\varkappa_2) - \overline{u}(0)).$$

The right-hand side of the obtained identity is nonnegative because of the second condition in (45), the first condition in (48), and the inequalities $\overline{w} - \overline{u} \geqslant 0$ on $\mathring{\gamma}_{ob}$ and $\overline{w}(\varkappa_1,\varkappa_2) - \overline{u}(0) \geqslant 0$. The relation (40) is proved. On the other hand, taking $(\overline{w}, \overline{u}) = (w, u)$, we have

$$\int\limits_{\mathring{\gamma}} [t^\nu]w + \int\limits_{\gamma_{ob}} u_{,11}u_{,11} = \int\limits_{\mathring{\gamma}_{ob}} [t^\nu](w - u) + [u_{,111}(0)](w(\varkappa_1,\varkappa_2) - u(0))$$

which implies (39) in view of the last conditions in (45) and (48). Thus, the conditions (39) and (40) are proved and, consequently, a smooth solution to the problem (42)–(49) satisfies (36)–(41). Thus, we have proved the equivalence of the problems (36)–(41) and (42)–(49) on the class of sufficiently smooth solutions.

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References

1. P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston etc. (1985).
2. A. M. Khludnev and V. A. Kovtunenko, *Analysis of Cracks in Solids*, WIT Press, Southampton etc. (2000).
3. A. M. Khludnev, *Elasticity Problems in Nonsmooth Domains* [in Russian], Fizmatlit, Moscow (2010).
4. A. L. Bessoud, F. Krasucki, and M. Serpilli, "Plate–like and shell–like inclusions with high rigidity," *C. R., Math., Acad. Sci. Paris* **346**, No. 11-12, 697–702 (2008).
5. N. V. Neustroeva, "A rigid inclusion in the contact problem for elastic plates," *J. Appl. Ind. Math.* **4**, No. 4, 526–538 (2010).
6. A. M. Khludnev, "On bending an elastic plate with a delaminated thin rigid inclusion," *J. Appl. Ind. Math.* **5**, No. 4, 582–594 (2011).
7. E. M. Rudoy, "The Griffith formula and Cherepanov-Rice integral for a plate with a rigid inclusion and a crack," *J. Math. Sci., New York* **186**, No. 3, 511–529 (2012).
8. A. M. Khludnev and M. Negri, "Optimal rigid inclusion shapes in elastic bodies with cracks," *Z. Angew. Math. Phys.* **64**, No. 1, 179–191 (2013).
9. N. P. Lazarev, "Fictitious domain method in the equilibrium problem for a Timoshenko type plate contacting with a rigid obstacle," *J. Math. Sci., New York* **203**, No. 4, 527–539 (2014).
10. V. V. Shcherbakov, "Existence of an optimal shape of the thin rigid inclusions in the Kirchhoff–Love plate," *J. Appl. Ind. Math.* **8**, No. 1, 97–105 (2014).
11. V. V. Shcherbakov, "Choosing an optimal shape of thin rigid inclusions in elastic bodies," *J. Appl. Mech. Tech. Phys.* **56**, No. 2, 321–329 (2015).
12. L. A. Caffarelli and A. Friedman, "The obstacle problem for the biharmonic operator," *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.* **6**, No. 1, 151–184 (1979).
13. G. Dal Maso and G. Paderni, "Variational inequalities for the biharmonic operator with variable obstacles," *Ann. Mat. Pura Appl., IV. Ser.* **153**, No. 1, 203–227 (1988).
14. A. M. Khludnev, K.- H. Hoffmann, and N. D. Botkin, "The variational contact problem for elastic objects of different dimensions," *Sib. Math. J.* **47**, No. 3, 584–593 (2006).

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