DOI 10.1007/s10958-019-04172-6
Journal of Mathematical Sciences, Vol. 237, No. 3, March, 2019

A STUDY OF ELASTIC-PLASTIC BOUNDARY PROPAGATION IN A TUBE OF ELASTIC-PERFECTLY PLASTIC MATERIAL UNDER DYNAMIC LOADINGS OF DIFFERENT TYPES

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UDC 539.375

Abstract. The dynamics of distribution of the border between areas of elasticity and plasticity for a hollow thick-walled cylinder under the influence of the internal pressure applied instantly is investigated in this work. Proof of the accuracy of the obtained numerical solution is provided. A more general regime of loading a tube is examined.

1. Introduction

The study of loading a tube with internal pressure is considered to be classical in the mechanics of solids. Elastic and plastic cases were examined, for example, in [9–11]. The case of elastic-plastic material was investigated by V. V. Sokolovsky [12]. Usually, the static case of this problem is studied. The influence of moment stresses on the deformation of metals was studied by B. Hopkinson [5, 6]. The dynamics of propagation of the border between the regions of elasticity and plasticity were studied by E. V. Lomakin [8]. In order to estimate the influence of lagging of fluidity on the spreading of the plasticity region, Lomakin obtained a numerical solution to the problem of dynamical propagation of the border between the elastic and plastic regions of the tube. The aim of this work is to study a more general case of loading and to prove the accuracy of the found numerical solution.

The problem of dynamic deformation of a thick-walled tube under the influence of internal pressure in the conditions of flat deformation is examined. Here, the model of an ideal elastic-plastic body is accepted.

Let us look at the equations of the relation between displacement and strain and at the differential equations of motion in polar coordinates:

$$\begin{aligned} \varepsilon_r &= \frac{\partial u_r}{\partial r}, \\ \varepsilon_\theta &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{\theta}, \\ \varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r}\right), \end{aligned} \tag{1}$$

$$\begin{aligned} &\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = \rho \frac{\partial^2 u_r}{\partial t^2}, \\ &\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2}{r}\sigma_{r\theta} + F_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2}. \end{aligned} \tag{2}$$

The generalized Hooke's law for isotropic medium is as follows:

$$\sigma_{ij} = \lambda I_1(\varepsilon) g_{ij} + 2\mu \varepsilon_{ij}, \tag{3}$$

where $I_1(\varepsilon) = \varepsilon_{ii}$ is the first invariant of the strain tensor.

Translated from Fundamentalnaya i Prikladnaya Matematika, Vol. 21, No. 3, pp. 201–216, 2016.

2. Formulation of the Problem in the Case of Ideal Elastic-Plastic Body

2.1. The Elastic Case. We will denote the inner radius of the tube by a, and the outer by b, and also assume the tube to be made up of an incompressible material and to be long enough, which means that the axial deformations are equal to zero ($I_1(\varepsilon) = 0$). Because of the Saint-Venant principle, we will conclude that the transversal sections of the tube will remain plain. Stress conditions in them will be equal. Unknown functions depend only on the radius r. Therefore, from Eqs. (1), (2) we will obtain the equation of motion for the tube:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 U}{\partial t^2},$$

where $U = u_r$ are dimensional variables.

The conditions of incompressiveness are

$$\frac{\partial U}{\partial r} + \frac{U}{r} = 0.$$

The conditions of deformation are

$$\varepsilon_r = \frac{\partial U}{\partial r}, \quad \varepsilon_\theta = \frac{U}{r}.$$

Let us denote the nondimensionalized radius by

$$\bar{r} = \frac{r}{a}.$$

Then the equation of motion will be

$$\frac{\partial \sigma_r}{\partial \bar{r}} + \frac{\sigma_r - \sigma_\theta}{\bar{r}} = \rho \frac{\partial^2 U}{\partial t^2} a.$$

The virtual displacement is

$$U = a \frac{C(t)}{\bar{r}}$$

(from the equation of incompressiveness). From Hooke's law (3) in the case of incompressiveness

$$\sigma_r = p + 2\mu\varepsilon_r = 2\mu\varepsilon_r, \quad \sigma_\theta = p + 2\mu\varepsilon_\theta = 2\mu\varepsilon_\theta$$

it follows that

$$\sigma_r - \sigma_\theta = s_r - s_\theta = 2\mu(\varepsilon_r - \varepsilon_\theta),$$

where s_r and s_θ are the deviators of stress. The conditions of incompressiveness with nondimensionalized $\bar{r}$ are the same:

$$\frac{\partial \bar{U}}{\partial \bar{r}} + \frac{\bar{U}}{\bar{r}} = 0,$$

where

$$\bar{U} = \frac{U(r,t)}{a} = \frac{C(t)}{\bar{r}}.$$

The deformations are

$$\varepsilon_r = \frac{\partial U}{\partial r} = \frac{\partial U}{a \partial \bar{r}} = -\frac{C(t)}{\bar{r}^2}, \quad \varepsilon_\theta = \frac{U}{a\bar{r}} = \frac{C(t)}{\bar{r}^2}.$$

The remainder is

$$\sigma_r - \sigma_\theta = 2\mu(\varepsilon_r - \varepsilon_\theta) = -4\mu \frac{C(t)}{\bar{r}^2}. \tag{4}$$

We can use the nondimensionalized time

$$\bar{t} = \frac{t}{t_*},$$

where

$$t_* = \frac{a}{C_0}, \quad C_0 = \sqrt{\frac{\mu}{\rho}},$$

and C_0 is the velocity of the wave of displacement. Then

$$\rho\frac{\partial^2 U}{\partial t^2}a = \rho a^2\frac{C''(\bar t)}{\bar r}\frac{C_0^2}{a^2} = C_0^2\rho\frac{C''(\bar t)}{\bar r} = \mu\frac{C''(\bar t)}{\bar r}.$$

Then the equation of motion can be represented as

$$\frac{\partial\sigma_r}{\partial\bar r} = \mu\left[\frac{C''(\bar t)}{\bar r} + 4\frac{C(\bar t)}{\bar r^3}\right].$$

The border conditions are

$$\bar r = \frac{b}{a} = \bar r_*, \quad \sigma_r = 0, \quad \bar r = 1, \quad \sigma_r = -q,$$

where q is the pressure density on the inner surface of the tube. The initial conditions are

$$U(r,0) = 0 \Longrightarrow C(0) = 0,$$
$$U'(r,0) = 0 \Longrightarrow C'(0) = 0.$$

By integrating the motion equation $\sigma_r = 0$ using conditions

$$\bar r = \bar r_* = \frac{b}{a},$$

we get the distribution of σ_r:

$$\sigma_r = \mu\left[\ln\left(\frac{\bar r}{\bar r_*}\right)C''(\bar t) - 2\left(\frac{1}{\bar r^2} - \frac{1}{\bar r_*^2}\right)C(\bar t)\right].$$

Using the second border condition $\bar r = 1$, $\sigma_r = -q$, we get the differential equation, which can be used to determine the function $C(\bar t)$. The solution under initial conditions $C(0) = 0$, $C'(0) = 0$ is

$$C(t) = \frac{q}{2\mu}\frac{\bar r_*^2}{\bar r_*^2 - 1}(1 - \cos\omega\bar t), \tag{5}$$

here

$$\omega^2 = \frac{2(\bar r_*^2 - 1)}{\bar r_*^2 \ln\bar r_*}.$$

2.2. The Plastic Case. Next, the case of continuous movement of the border between the elastic and plastic domains will be examined. From the Tresca criterion we get condition of plasticity:

$$|\sigma_r - \sigma_\theta| = 2K.$$

The solution (5) is correct provided that $2\mu C(\bar t) < K$.

Let us find the boundary value of the $q = q^*$, under which plastic deformations on the inner radius of the tube will start:

$$q^* = K\frac{\bar r_*^2 - 1}{\bar r_*^2(1 - \cos\omega\bar t)} > \frac{K(\bar r_*^2 - 1)}{2\bar r_*^2}.$$

By examining the static case of the problem, we get

$$q^* > \frac{K(\bar r_*^2 - 1)}{\bar r_*^2}$$

(see [7]). The solution obtained in the elastic case is correct when $q < q^*$. We will study the case where $q > q^*$.

From the border conditions on the border between the plastic and elastic areas and from the incompressiveness of material we get

$$U_{\mathrm{e}}(x) = U_{\mathrm{p}}(x) \Longrightarrow C_{\mathrm{p}} = C_{\mathrm{e}} = C(\bar t),$$

where the indices e and p mean elastic and plastic, respectively.

The motion differential equation for the plastic area is

$$\frac{\partial \sigma_r}{\partial \bar{r}} = \mu \frac{C''(\bar{t})}{\bar{r}} + \frac{2K}{\bar{r}}.$$

Under internal loading, the plasticity area will spread from the inner border. We will use the border conditions:

$$\bar{r} = 1, \quad \sigma_r = -q.$$

Integrating we get

$$\sigma_r = -q + [2K + \mu C''(\bar{t})] \ln \bar{r}.$$

Let us denote the nondimensionalized radius coordinate of the border between elastic and plastic areas by $x(\bar{t})$. On this border, the stress σ_r is considered to be continuous (as a consequence of incompressiveness), so:

$$q - [2K + \mu C''(\bar{t})] \ln x = \mu \left[\left(\ln \frac{\bar{r}_*}{x} \right) C''(\bar{t}) + 2 \left(\frac{1}{x^2} - \frac{1}{\bar{r}_*^2} \right) C(\bar{t}) \right].$$

It can be represented as

$$\frac{\mu \ln \bar{r}_*}{K} C''(\bar{t}) + \ln x^2 + \frac{2\mu}{K} \left(\frac{1}{x^2} - \frac{1}{\bar{r}_*^2} \right) C(\bar{t}) = \frac{q}{K}.$$

On the border between the elastic and plastic areas, $|\sigma_r - \sigma_\theta| = 2K$. Using (4), we get

$$x^2(t) = \frac{2\mu C(\bar{t})}{K}.$$

Let us determine $x^2(\bar{t}) = y(\bar{t})$. Then we get

$$\frac{\ln \bar{r}_*}{2} y'' + \ln y - \frac{y}{\bar{r}_*^2} = \frac{q}{K} - 1. \tag{6}$$

Initial conditions: t_0 is the time of appearing of plastic deformation on the inner radius ($\bar{r} = 1$), and for each value of q/K it is different; t_0 is defined by the solution in elastic case:

$$\frac{K(1 - 1/\bar{r}_*^2)}{q} = \big(1 - \cos(\omega t_0)\big),$$

$$t_0 = \frac{1}{\omega} \arccos \left(1 - \frac{K}{q} \left(1 - \frac{1}{\bar{r}_*^2} \right) \right).$$

The initial conditions can be represented as

$$y(t_0) = \frac{2\mu C(t_0)}{K} = 1, \quad y'(t_0) = \frac{2\mu C'(t_0)}{K}.$$

Equation (6) is an ordinary differential equation with separable variables. Let us integrate it once. We get

$$\frac{\ln \bar{r}_*}{4} y'^2 = \frac{q}{K} y + \frac{y^2}{2\bar{r}_*^2} - y \ln y + P_1, \tag{7}$$

where P_1 is a constant, which is defined by the initial conditions. We have

$$\big(1 - \cos(\omega t_0)\big) = \frac{K(1 - 1/\bar{r}_*^2)}{q} = \frac{K\left(\bar{r}_*^2 - 1\right)}{\bar{r}_*^2 q}, \tag{8}$$

$$\omega^2 = \frac{2(\bar{r}_*^2 - 1)}{\bar{r}_*^2 \ln \bar{r}_*},$$

$$C(t_0) = \frac{q}{2\mu} \frac{\bar{r}_*^2}{\bar{r}_*^2 - 1} \big(1 - \cos(\omega t)\big), \tag{9}$$

$$y'(t_0) = \frac{2\mu C'(t_0)}{K}, \quad y(t_0) = 1. \tag{10}$$

From (9) we get

$$C'(t_0) = \frac{q\bar{r}_*^2}{2\mu(\bar{r}_*^2 - 1)}\sqrt{\frac{2(\bar{r}_*^2 - 1)}{\bar{r}_*^2 \ln \bar{r}_*}} \sin(\omega t_0) = \frac{q}{\mu\omega \ln \bar{r}_*} \sin(\omega t_0).$$

From (9) and (10) we get

$$y'(t_0) = \frac{2q}{K\omega \ln \bar{r}_*} \sin(\omega t_0) = \frac{2q}{K\omega \ln \bar{r}_*} \sqrt{1 - \cos^2(\omega t_0)}. \tag{11}$$

From (8) we get

$$\cos(\omega t_0) = 1 - \frac{K(\bar{r}_*^2 - 1)}{q r_*^2}, \quad \cos^2(\omega t_0) = 1 - \frac{2K(\bar{r}_*^2 - 1)}{q\bar{r}_*^2} + \frac{K^2(\bar{r}_*^2 - 1)^2}{q^2 \bar{r}_*^4}. \tag{12}$$

From (11) and (12) we get

$$y'(t_0) = \frac{2q}{K\omega \ln \bar{r}_*}\sqrt{\frac{2K(\ln \bar{r}_*^2 - 1)}{q\bar{r}_*^2} - \frac{K^2(\bar{r}_*^2 - 1)^2}{q^2\bar{r}_*^4}}.$$

Finally we obtain

$$\begin{aligned}
P_1 &= \frac{\ln \bar{r}_*}{4} y'^2 - \frac{q}{K} - \frac{1}{2\bar{r}_*^2} = \frac{q^2}{K^2\omega^2 \ln \bar{r}_*}\left(\frac{2K(\bar{r}_*^2 - 1)}{q\bar{r}_*^2} - \frac{K^2(\bar{r}_*^2 - 1)^2}{q^2\bar{r}_*^4}\right) - \frac{q}{K} - \frac{1}{2r_*^2} \\
&= \frac{q}{K} - \frac{q}{K} - \frac{\ln \bar{r}_* \omega^2}{4} - \frac{1}{\bar{r}_*^2}, \\
P_1 &= -\frac{\bar{r}_*^2 - 1}{2\bar{r}_*^2} - \frac{1}{2\bar{r}_*^2} = -\frac{1}{2}.
\end{aligned}$$

Solving Eq. (7) of y', we get

$$y' = \frac{2}{\sqrt{\ln \bar{r}_*}}\sqrt{\frac{q}{K}y + \frac{y^2}{2\bar{r}_*^2} - y\ln y - \frac{1}{2}}. \tag{13}$$

Substituting $y = x^2$, we get

$$\begin{cases} x' = \dfrac{1}{\sqrt{\ln \bar{r}_*}}\sqrt{\dfrac{q}{K} + \dfrac{x^2}{2\bar{r}_*^2} - 2\ln x - \dfrac{1}{2x^2}}, \\ x(t_0) = 1. \end{cases}$$

The equation must be solved with the condition that the parameters q/K and $\bar{r}_*$ satisfy the inequality

$$\frac{q}{K} \geq \frac{1}{2}\left(1 - \frac{1}{r_*^2}\right).$$

Now we will calculate under which value of q/K the cylinder will be in plasticity completely:

$$\begin{aligned}
& x' = \frac{1}{\sqrt{\ln \bar{r}_*}}\sqrt{\frac{q}{K} + \frac{x^2}{2\bar{r}_*^2} - 2\ln x - \frac{1}{2x^2}} = 0, \\
& \frac{q}{K} = \xi, \\
& x = \bar{r}_*, \\
& \frac{q}{K} = 2\ln \bar{r}_* + \frac{1}{2\bar{r}_*^2} - \frac{1}{2}.
\end{aligned}$$

When studying the static case, $q/K = 2\ln \bar{r}_*$ [7].

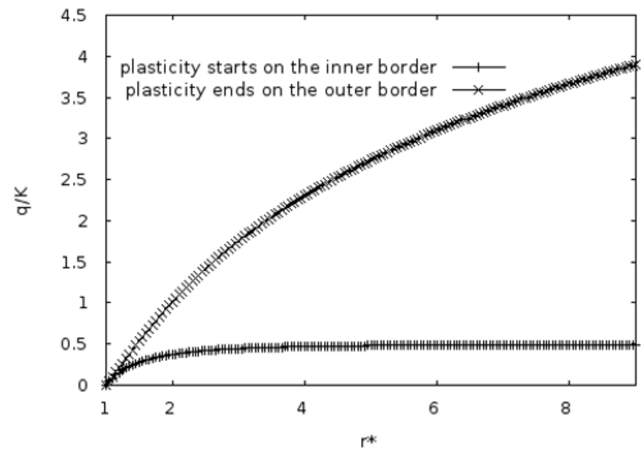

Fig. 1

2.3. Numerical Solution. Let us solve this differential equation numerically using the Runge–Kutta method of the 4th degree with step correction end error estimation [2, 4]. We will use the equations of the Runge–Kutta method with coefficients that are mentioned in [3]. Let us solve the Cauchy problem

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases}$$

The approximate solution in consequent points is obtained from the equation

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ k_2 &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), \\ k_3 &= f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), \\ k_4 &= f(x_n + h, y_n + hk_3), \end{aligned}$$

and h is the step.

In order to estimate the error and choose the step, we will use the method of horizontally estimated step. We will calculate the value of the integral at a point after one "long" step and two "short" steps, taking into account the error of the method. Then we will subtract the found values. We have

$$\Delta = \frac{I_1 - I_{22}}{1 - 1/2^s},$$

where I_1 is the integral value calculated after one step, I_{22} is the integral value calculated after two steps, C is constant, s is the degree of the method, and $\Delta = Ch^{s+1}$ is the main error. We will choose the next step h_{new} so that the error on this step is equal to ε ($Ch_{\text{new}}^{s+1} = \varepsilon$):

$$\left(\frac{h}{h_{\text{new}}}\right)^{s+1} = \frac{\Delta}{\varepsilon} = \chi.$$

We will choose the new step using the equation

$$h_{\text{new}} = \frac{0.95h}{\sqrt[s+1]{\chi}}.$$

The global error can be calculated using the formula

$$\delta_{k+1} = \delta_k e^{\int\limits_{t_k}^{t_{k+1}} \mu\, ds} + r_k,$$

where δ_k is the global error value on the kth step, r_k is the local error on the kth step, μ is the maximum singularity number, which is defined as the maximum eigenvalue of the matrix $(J + J^{\mathrm{T}})/2$, and J is the Jacobian of the system. In our case, the maximum eigenvalue is equal to

$$\mu = \frac{x/\bar{r}_*^2 - 2/x + 1/x^3}{2\sqrt{\bar{r}_*\big(q/K + x^2/(2\bar{r}_*^2) - 2\ln x - 1/(2x^2)\big)}}.$$

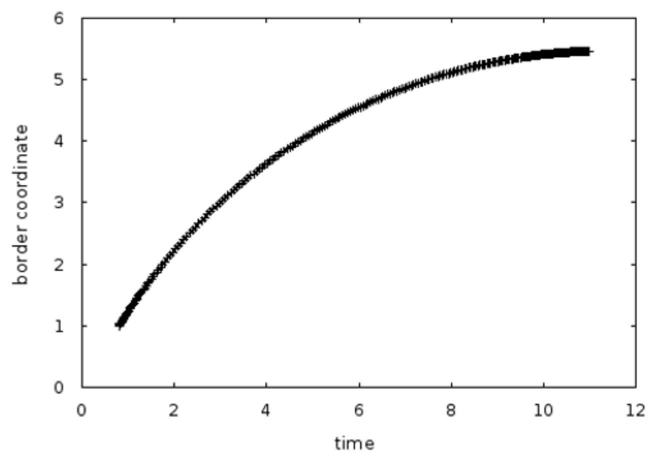

Fig. 2

In order to ensure that our solution is correct, let us calculate the values of the function at three different points ($t = 2, 5, 11$) with three maximum errors on each step.

ε	2	5	11
10^{-7}	2.22802282019651	4.13950553685536	5.46500206374295
10^{-9}	2.22802297613335	4.13950606569059	5.4650026473705
10^{-11}	2.22802298288977	4.1395060800067	5.46500266381324

Global error value at the point $t = 11$ depends on the maximum error on step.

10^{-7}	10^{-9}	10^{-11}
$1.039477084 \cdot 10^{-6}$	$2.9190828 \cdot 10^{-8}$	$6.95469 \cdot 10^{-10}$

So it is possible to solve this problem numerically using this method. A numerical solution concerning $r_* = 6$ is obtained on the net with step 0.01 for different values of q/K from 2 to 3.5.

Now we will study the case where loading q is unloaded after the time t_*. The border conditions on the border of the plastic and elastic areas will remain the same. That is why Eq. (6) is correct. But the initial conditions will change: $y(t_*) = y_*$, $y'(t_*) = y'_*$, where y_* and y'_* are the squared coordinate and

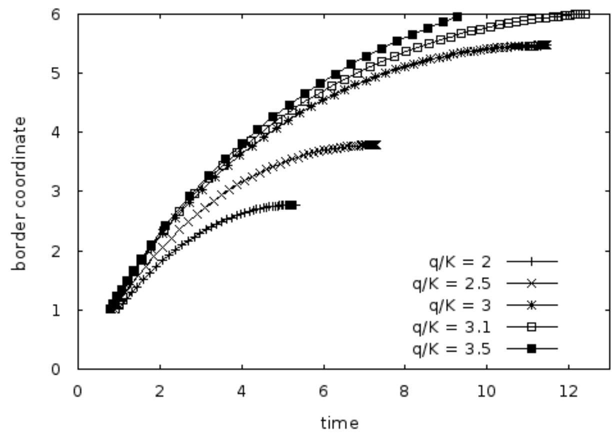

Fig. 3

velocity of the border between areas of elasticity and plasticity, respectively, at the moment of ceasing of the loading. Integrating Eq. (6), using that $q = 0$, we get

$$\frac{\ln \bar{r}_*}{4} y'^2 = \frac{y^2}{2\bar{r}_*^2} - y \ln y + P_2, \tag{14}$$

where P_2 is a constant, which is defined from the new initial conditions. We do not have the equation for $y(t)$ in explicit form, but we have Eq. (13) for $y'(t)$. Let us substitute it in (14), where $t = t_*$:

$$P_2 = \frac{q}{K} y_* - \frac{1}{2}.$$

Finally, we get

$$y'^2 = \frac{4}{\ln \bar{r}_*} \left(\frac{q}{K} y_* + \frac{y^2}{2\bar{r}_*^2} - y \ln y - \frac{1}{2} \right). \tag{15}$$

Substituting $y = x^2$, we have

$$x' = \frac{1}{\sqrt{\ln \bar{r}_*}} \sqrt{\frac{q x_*^2}{K x^2} + \frac{x^2}{2\bar{r}_*^2} - 2 \ln x - \frac{1}{2x^2}}.$$

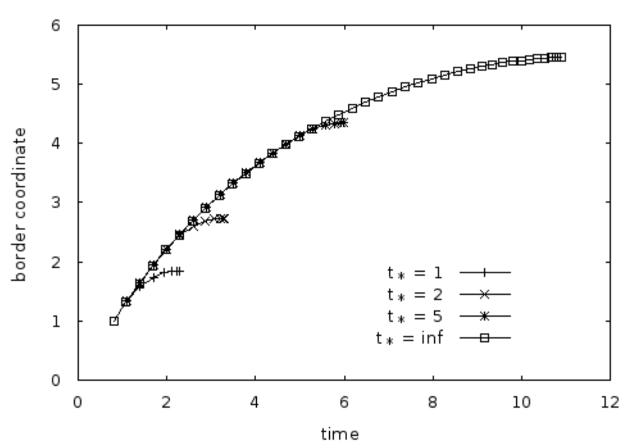

Fig. 4

In Fig. 4, $t_* = \inf$ is mentioned to demonstrate the solution to the problem when loading does not cease.

3. Nonlinear Conditions of Loading

Let us change the scheme of the numerical experiment.

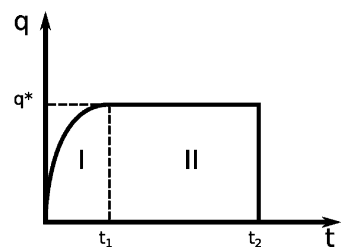

Fig. 5

Loading is increasing by a parabolic trajectory until the value q^* is reached at the moment t_1, then remains the same until the time t_2, when the loading is stopped.

Let us find the solution to the problem in elastic case for the loading area which is marked I:

$$q(t) = \mu\left[(\ln \bar{r}_*)C''(\bar{t}) + 2\left(1 - \frac{1}{\bar{r_*}^2}\right)C(\bar{t})\right].$$

The loading function is

$$q(t) = -\frac{q^*}{t_1^2}t^2 + \frac{2q^*}{t_1}t.$$

The initial conditions are

$$C(0) = 0,$$
$$C'(0) = 0.$$

We get the solution to the ordinary differential equation:

$$C_{oo}(t) = \xi\cos(\omega\bar{t}) + \eta\sin(\omega\bar{t}),$$
$$C_{cn}(t) = \frac{q^*}{t_1\mu(1 - 1/\bar{r}_*^2)}\left(-\frac{t^2}{2t_1} + t + \frac{1}{t1\omega}\right),$$
$$C^{\mathrm{I}}(\bar{t}) = \frac{q^*}{t_1\mu(1 - 1/r_*^2)}\left(-\frac{\cos(\omega\bar{t})}{t_1\omega} - \frac{\sin(\omega\bar{t})}{\omega} - \frac{\bar{t}^2}{2t_1} + t + \frac{1}{t_1\omega}\right).$$

In order to determine the moment of beginning of plastic deformations on the inner border, we need to find the solution to the equation

$$F(t) = \frac{q^*}{t_1\mu(1 - 1/r_*^2)}\left(-\frac{\cos(\omega\bar{t})}{t_1\omega} - \frac{\sin(\omega\bar{t})}{\omega} - \frac{\bar{t}^2}{2t_1} + t + \frac{1}{t_1\omega}\right) - \frac{K}{2\mu}.$$

This can be obtained numerically using the chords method $[1, 2]$. The formula of this method is

$$t_{n+1} = t_n - \frac{(t_n - t_0)}{F(t_n) - F(t_0)}F(t_n).$$

Let us assume the initial approximation for t_0 to be equal to zero. Then on every step we will take for t_0 the value for which $F(t) = F''(t)$ is correct. We are looking for the positive solution to the equation which is the closest to zero. If the found solution is less than t_1, then we solve the three systems of differential equations:

$$\begin{cases} \bar{t} \in [t_0, t_1), \\ \ln \bar{r}_*(x'^2 + xx'') + 2\ln x - \dfrac{x^2}{\bar{r}_*^2} = \dfrac{-(q^*t^2)/t_1^2 + (2q^*t)/t_1}{K} - 1, \\ x(t_0) = 1, \\ x'(t_0) = \dfrac{q^*}{t_1 K(1 - 1/\bar{r}_*^2)} \left(1 - \cos(\omega t_0) + \dfrac{\sin(\omega t_0)}{t_1} - \dfrac{t_0}{t_1}\right). \end{cases}$$

$$\begin{cases} \bar{t} \in [t_1, t_2), \\ \ln r_*(x'^2 + xx'') + 2\ln x - \dfrac{x^2}{\bar{r}_*^2} = \dfrac{q^*}{K} - 1, \\ x(t_1) = x_{1^*}, \\ x'(t_1) = x'_{1^*}, \end{cases}$$

$$\begin{cases} t_2 \le \bar{t}, \\ \ln r_*(x'^2 + xx'') + 2\ln x - \dfrac{x^2}{\bar{r}_*^2} = -1, \\ x(t_2) = x_{2^*}, \\ x'(t_2) = x'_{2^*}. \end{cases}$$

Now we will find the function $C(t)$ for the case of beginning of plastic deformation when the loading area reaches the constant pressure regime:

$$C^{\mathrm{II}}(\bar{t}) = \xi_2 \cos(\omega t) + \eta_2 \sin(\omega t) + \frac{q^* \bar{r}_*^2}{2\mu \bar{r}_*^2 - 1)},$$

$$\begin{cases} C^{\mathrm{II}}(t_1) = C^{\mathrm{I}}(t_1), \\ \left(C^{\mathrm{II}}(t_1)\right)' = \left(C^{\mathrm{I}}(t_1)\right)', \end{cases}$$

$$C^{\mathrm{II}}(\bar{t}) = \frac{q^*}{t_1 \mu(1 - 1/\bar{r}_*^2)} \left(\left(\frac{\cos(\omega t_1)}{\omega t_1} - \frac{1}{\omega t_1} \right) \cos(\omega \bar{t}) + \left(\frac{\sin(\omega t_1)}{\omega t_1} - \frac{1}{\omega} \right) \sin(\omega \bar{t}) + \frac{t_1}{2} \right).$$

Let us find t_0 in the second case:

$$C(t_0) = \frac{K}{2\mu},$$

$$C(t_0) = \xi \cos \omega t_0 + \eta \sin \omega t_0 + \frac{q\bar{r_*}^2}{2\mu(\bar{r_*}^2 - 1)}.$$

We get the square equation of $\cos \omega t_0$. We solve it:

$$\cos \omega t_0 = \frac{A\xi \pm \eta\sqrt{\xi^2 + \eta^2 - A^2}}{\xi^2 + \eta^2},$$

where

$$A = \frac{K}{2\mu} - \frac{qr_*^2}{2\mu(r_*^2 - 1)}.$$

Let us choose the minimal t_0 from the interval between t_1 and t_2. Then

$$x'(t_0) = \frac{\mu}{K} C'(t_0) = \frac{2\mu}{K}\big(-\omega\xi \sin(\omega t_0) + \eta\omega \cos(\omega t_0)\big),$$
$$x(t_0) = 1.$$

In this case, the initial conditions will change ($t_0 \in [t_1, t_2]$):

$$\begin{cases} \bar{t} \in [t_0, t_2), \\ \ln r_*(x'^2 + xx'') + 2\ln x - \dfrac{x^2}{\bar{r}_*^2} = \dfrac{q^*}{K} - 1, \\ x(t_0) = 1, \\ x'(t_0) = \dfrac{q^*}{t_1 K(1 - 1/\bar{r}_*^2)}\left(-\left(\dfrac{\cos(\omega t_1)}{t_1} - \dfrac{1}{t_1}\right)\sin(\omega t_0) + \left(\dfrac{\sin(\omega t_1)}{t_1} - 1\right)\cos(\omega t_0)\right), \end{cases}$$

$$\begin{cases} t_2 \le \bar{t}, \\ \ln r_*(x'^2 + xx'') + 2\ln x - \dfrac{x^2}{\bar{r}_*^2} = -1, \\ x(t_2) = x_{2*}, \\ x'(t_2) = x'_{2*}. \end{cases}$$

We get the numerical solution to the systems of differential equation using the Runge–Kutta method of the fourth degree with automated step estimation. Only border conditions have been changed, so all the conclusions concerning the accuracy of the solution will be correct. For each of the two cases we can get the exact solution to the problem of elasticity. Here we demonstrate the numerical solution when $t_2 = 5$, $q_* = 6$, $K = 3$, $r_* = 6$, $\mu = 1$ for different values of t_1 with maximum local error on step equal to $1e - 5$.

The author thanks E. V. Lomakin, A. N. Sakharov, and A. V. Muravlev.

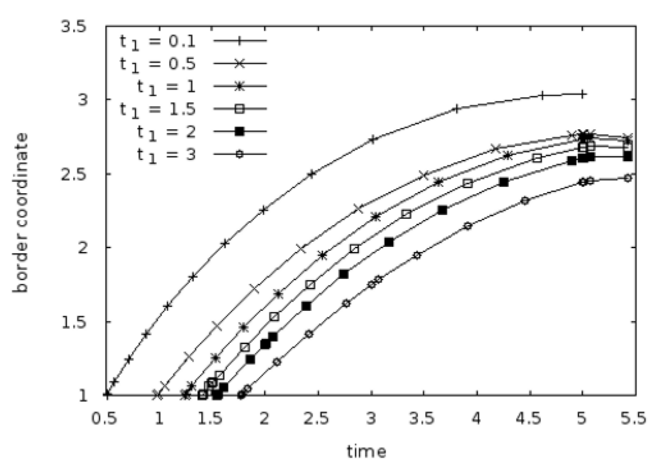

Fig. 6

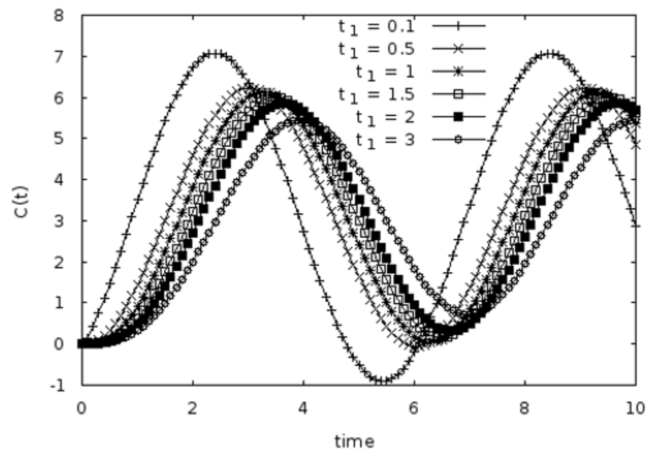

Fig. 7

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