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A BOUND ON THE NUMBER OF LEAVES IN A SPANNING TREE OF A CONNECTED GRAPH OF MINIMUM DEGREE 6

E. N. Simarova* UDC 519.172.1

We prove that a connected graph of minimum degree 6 *has a spanning tree such that at least* $\frac{11}{21}$ *of its vertices are leaves. Bibliography:* 4 *titles.*

1. Introduction

In this paper, we consider a connected graph without loops and multiple edges with minimum vertex degree at least 6. We use the standard notation. For a graph G, we denote the number of its vertices by $v(G)$ and the minimum vertex degree by $\delta(G)$.

Definition 1. Let G be a connected graph. We denote by $n(G)$ the maximum number of leaves in a spanning tree of G. Let

$$t(G) = \frac{n(G)}{v(G)}.$$

Starting from 1981, several papers with lower bounds on $n(G)$ and $t(G)$ were published. In 1981, Linial conjectured that $n(G) \geq \frac{\delta(G)-2}{\delta(G)+1}v(G) + c$ for $\delta(G) \geq 3$, where the constant $c \geq 0$ depends only on $\delta(G)$. Indeed, for any $d \geq 3$, one can construct an infinite series of graphs $G_1, \dots, G_n, \dots$ such that $\delta(G_n) = d$ and $\lim_{n\to\infty} t(G_n) = \frac{d-2}{d+1}$. Hence, if the bound from Linial's conjecture holds for some $\delta(G)$, then it is asymptotically tight.

For $\delta(G) = 3$, this bound was proved in 1991 by Kleitman and West [2]: they proved that $n(G) \geq \frac{1}{4}v(G) + 2$. This bound is tight, it is attained at an infinite series of graphs. The bound for $\delta(G) = 4$ was also proved in [2]: $n(G) \geq \frac{2}{5}v(G) + \frac{8}{5}$. Later, Karpov [4] proved the stronger bound $n(G) \geq \frac{2}{5}v(G) + 2$ for all graphs with $\delta(G) = 4$ except for three exceptional cases. This bound is attained at an infinite series of graphs. In 1992, Griggs and Wu [1] proved the bound $n(G) \geq \frac{1}{2}v(G) + 2$ for $\delta(G) = 5$, which is also tight. Hence, we see that Linial's bound holds for small $\delta(G)$. All these bounds were proved using the *dead vertices* method, which will be described below. For $\delta(G) \geq 6$, Linial's bound is neither proved nor disproved, and no tight bound is known at the moment. It follows from Alon's results [3] that for large $\delta(G)$ Linial's bound is false. However, the case of small $\delta(G)$ remains open: nobody knows where Linial's bound ceases to be true.

In this paper, we prove that $t(G) \geq \frac{11}{21}$ for a connected graph G with $\delta(G) = 6$. This bound is not tight, but it is the best lower bound known at the moment. However, the best known upper bound is Linial's $\frac{4}{7}$.

As we have already mentioned, all known bounds were obtained using the *dead vertices* method. We also use this method. It consists in constructing a spanning tree step by step; at each step, we add new vertices to the tree and "kill" some leaves of the tree constructed earlier.

Definition 2. The tree constructed at the previous steps will be called the *subtree*. We denote by S both the subtree and the set of its vertices. The set of vertices of the original graph G not contained in S will be denoted by T.

*St.Peterburg State University, St.Peterburg, Russia, e-mail: `katerina.1.14@mail.ru`.

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Definition 3. A *dead vertex* is a leaf of the subtree S that is adjacent to no vertex of T.

We successively add vertices to S. If a vertex becomes dead at some step, it remains dead until the end of the construction, and it also remains a leaf of the subtree.

Consider the formula

$$\frac{8}{5}u(S) + \frac{1}{5}b(S) - v(S),$$

where $u(S)$ is the number of leaves, $b(S)$ is the number of dead vertices, and $v(S)$ is the number of vertices in S. We will construct our subtree successively, adding vertices at each step so that the inequality

$$\frac{8}{5}u_1 + \frac{1}{5}b_1 - v_1 \geq 0$$

holds, where u_1, b_1, and v_1 are the increments of the number of leaves, dead vertices, and vertices of S, respectively. If we could construct a spanning tree in this way, then the bound $t(G) \geq \frac{5}{9}$ would hold. Unfortunately, in some cases we cannot perform such a step. Thus, the bound we are proving is smaller.

2. The beginning of the construction. Some general cases of adding vertices

2.1. The initial tree. Take an arbitrary vertex and 6 vertices adjacent to it. We obtain a tree with 7 vertices and 6 leaves. Hence, $\frac{8}{5}u + \frac{1}{5}b - v = \frac{8}{5} \cdot 6 - 7 = \frac{13}{5}$.

2.2. A vertex of T is adjacent to a non-pendant vertex of S. We join this extra vertex with a non-pendant vertex of S and obtain $u_1 = 1$, $b_1 \geq 0$, $v_1 = 1$. Since $\frac{8}{5} - 1 \geq 0$, the desired inequality holds.

2.3. A vertex $a \in S$ has at least three neighbors in T. If a has exactly 3 neighbors, then we join them with a and obtain $u_1 = 2$, $b_1 \geq 0$, $v_1 = 3$. The increment is $\frac{8 \cdot 2}{5} - 3 = \frac{1}{5} \geq 0$.

If a has more than 3 neighbors, then we add three of them and, after that, add the other neighbors of a according to Sec. 2.2.

2.4. A vertex $a \in T$ is adjacent to S and to at least four vertices of T. If a has exactly 5 neighbors in T, then we add a and these neighbors to the subtree S (see Fig. 1). We have $u_1 = 4$, $b_1 \geq 0$, $v_1 = 6$. The increment is $\frac{8}{5} \cdot 4 - 6 = \frac{2}{5} \geq 0$. If a has more than five neighbors, then we add five of them and, after that, add all the others according to Sec. 2.2.

In the remaining case, a has 4 neighbors in T. Since the degree of a is at least 6, it is adjacent to two vertices of S (say, b and c). Clearly, b and c are leaves of S (otherwise, we perform the step described in Sec. 2.2). Assume that b is adjacent in T only to a and its neighbors. We join a with c and join the four neighbors of a with a. Then b becomes dead, and the desired inequality holds: $\frac{8}{5} \cdot 3 + \frac{1}{5} - 5 = 0$.

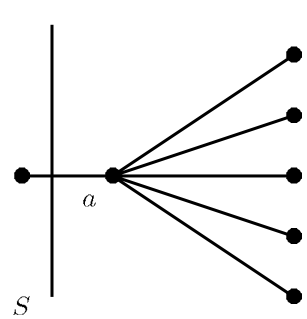

Fig. 1

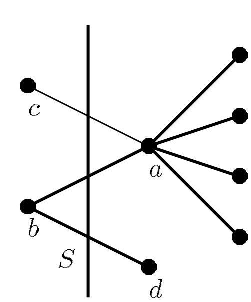

Fig. 2

Now let b has a neighbor $d \in T$ different from a and its neighbors (see Fig. 2). Then we join a and d with b and join the four neighbors of a with a. We obtain $u_1 = 4$, $b_1 \geq 0$, $v_1 = 6$, and $\frac{8 \cdot 4}{5} - 6 = \frac{2}{5} \geq 0$, i.e., the desired inequality holds.

3. There is a vertex in T that is not adjacent to S

Let us say that a vertex x is at level i if the distance between x and S is i (vertices of S are at level 0). Since G is connected, every vertex has a level. In our case, there exists a vertex b of level 2. Clearly, b is adjacent to a vertex a of level 1, and a is adjacent to S.

3.1. The vertex a has a neighbor in T that is not adjacent to b. Then we add a to S, and join with a the vertex b and all neighbors of a that are not adjacent to b. After that, we join the remaining 5 neighbors of b with b. In total, we add 8 vertices to S. One leaf of S becomes non-pendant, and 6 new leaves appear (the five neighbors of b that are different from a and the neighbor of a that is not adjacent to b). We obtain $\frac{8 \cdot 5}{5} - 8 = 0$ (see Fig. 3).

In what follows, all vertices of T that are adjacent to a are also adjacent to b.

3.2. The vertex a has a neighbor $d \in S$ that has two neighbors in T. Then d has a neighbor $e \in T$ different from a. Consider two cases.

3.2.1. *The vertex e is not adjacent to b.* Then we join a and e with d, join b with a, and after that join the five remaining neighbors of b with b. In total, we add 8 vertices to S. One leaf of S becomes non-pendant, and 6 new leaves appear (e and the five neighbors of b that are different from a). We obtain $\frac{8 \cdot 5}{5} - 8 = 0$ (see Fig. 4).

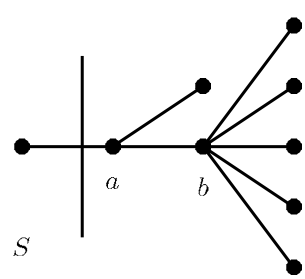

Fig. 3

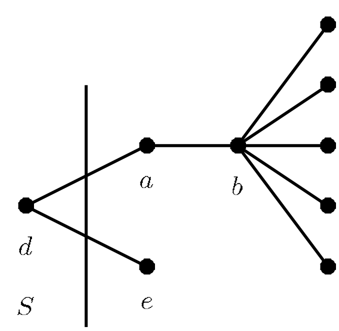

Fig. 4

3.2.2. *The vertex e is adjacent to b.* Then a and e are symmetric. If e has a neighbor in S adjacent to a vertex of T that is not a neighbor of b, then, as in Sec. 3.2.1, we can add 8 vertices to S preserving the desired inequality.

Hence, all vertices of S that are adjacent to a or e can be adjacent in T only to a, e, b and neighbors of b. Then we add a to S, join b with a, and join the five remaining neighbors of b with b. All vertices in S that are adjacent to a or e, except one that is joined with a in the tree, become dead. If there are at least 4 vertices in S adjacent to a or e (see Fig. 5), then we add 7 vertices, one leaf of S becomes non-pendant, 3 leaves become dead, and five new leaves appear (all neighbors of b except a). We obtain $\frac{8 \cdot 4}{5} + \frac{1 \cdot 3}{5} - 7 = 0$.

Assume that at most 3 vertices of S are adjacent to a or e. Since a has a neighbor in S, it must have at least 3 neighbors in S (otherwise, we apply Sec. 2.4), and similarly for e. Hence, there exist exactly 3 vertices in S adjacent to a or e, and all these 3 vertices are adjacent both to a and e and have no other neighbors in T. Then we add a to S, join b with a, and join the 4 remaining neighbors of b with b. As a result, two new dead vertices (the neighbors of a

in S) appear. Moreover, the vertex e also becomes dead, since all its neighbors are added to the subtree. Then, again, we have 3 new dead vertices, and $\frac{8\cdot4}{5}+\frac{1\cdot3}{5}-7=0$ (see Fig. 6).

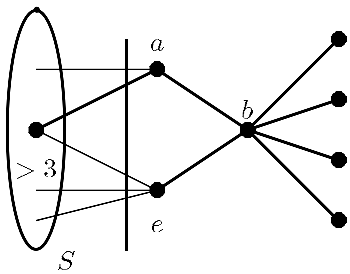

Fig. 5

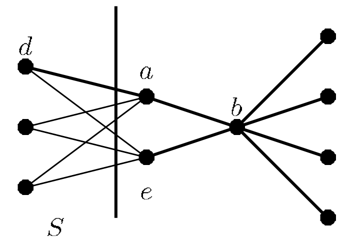

Fig. 6

3.3. All neighbors of a in S are not adjacent to vertices of T different from a. If a has at least 4 neighbors in S, then we add a to S, join b with a, and join all the remaining neighbors of b with b (see Fig. 7). One leaf of S disappears, but 5 new leaves appear (the 5 neighbors of b different from a). Also, three new dead vertices appear (the neighbors of a in S that are not joined with a in the new subtree). We obtain $\frac{8\cdot4}{5}+\frac{1\cdot3}{5}-7=0$.

Thus, the only remaining case is the following one: a is adjacent to exactly 3 vertices of S and to 3 vertices of T (namely, to b and two other vertices, say c and d).

Lemma 1. *The vertices b, c, d are pairwise adjacent and not adjacent to S.*

Proof. Note that c and d are adjacent to b (otherwise, we apply Sec. 3.1). Let us prove that none of the vertices c and d can be adjacent to S. Assume that c is adjacent to a vertex of S (see Fig. 8). This vertex has a unique neighbor in T (namely, c; otherwise, we apply Sec. 3.2). Then we add a to S, join b with a, and join all the other neighbors of b with b. We obtain two dead vertices which are neighbors of a and one extra dead vertex which is a neighbor of c. Thus, $\frac{8\cdot4}{5}+\frac{1\cdot3}{5}-7=0$.

Therefore, neither c nor d is adjacent to S. Hence, c is adjacent to d (otherwise, we apply Sec. 3.1). □

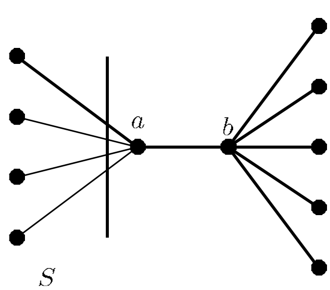

Fig. 7

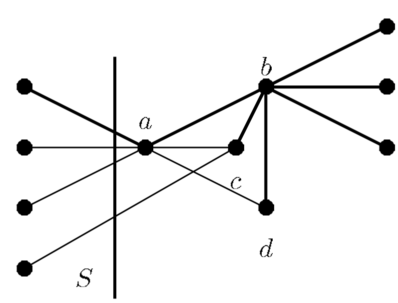

Fig. 8

Lemma 2. *Each of the vertices b, c, and d has exactly 6 neighbors.*

Proof. If one of these vertices (say, b) has at least 7 neighbors, then we add a to S, join b with a, and join the six remaining neighbors of b with b, obtaining $\frac{8\cdot5}{5}+\frac{2}{5}-8\geq0$. □

Lemma 3. *Let e be a neighbor of b, and let $e\neq a$. Then e has no neighbors in S.*

Proof. Assume the converse. By Lemma 1, we have $e \notin \{c, d\}$. Then we add e to S, join b with e, and join all the remaining neighbors of b with b (see Fig. 9). All neighbors of a that belong to S become dead. Therefore, $u_1 = 4$, $b_1 \geq 3$, $v_1 = 7$, and the increment is $\frac{8 \cdot 4}{5} + \frac{3}{5} - 7 \geq 0$. $\square$

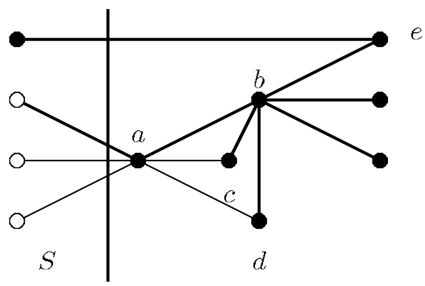

Fig. 9

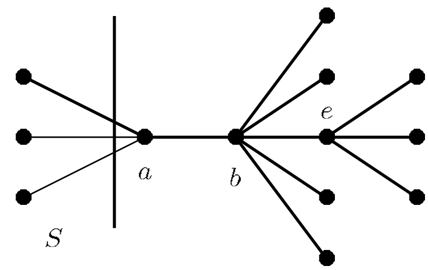

Fig. 10

Lemma 4. *Let e be a neighbor of b, and let $e \neq a$. Then e has at most two neighbors that are not adjacent to b.*

Proof. Assume the converse. Then e has at least 3 neighbors not adjacent to b. By Lemma 3, all these neighbors belong to T. Then we add a to S, join b and all its remaining neighbors with a, and, finally, join three new neighbors with e (see Fig. 10). We obtain $\frac{8 \cdot 6}{5} + \frac{2}{5} - 10 = 0$. $\square$

Corollary 1. *Let e be a neighbor of b. Then b is adjacent to at least one of the vertices c and d.*

Proof. Assume the converse. Then e is adjacent to none of the vertices a, c, d. Hence, e has at least 3 neighbors that are not adjacent to b. This contradicts Lemmas 3 and 4. $\square$

In a similar way, we can replace b with c or d in Lemma 1 and Corollary 4.

Let e, f, g be the three neighbors of b different from a, c, d.

Lemma 5. *The vertices e, f, and g are pairwise adjacent.*

Proof. Assume the converse, and let e and f be nonadjacent. Then both e and f are adjacent to all vertices b, c, d (otherwise, one of the vertices e or f has at least 3 neighbors that are not adjacent to b, a contradiction with Lemma 4). For the same reason, g is adjacent to at least one of the vertices c or d. Assume that g is adjacent to c. Then c already has 6 neighbors. We add a to S, join b with a, and join c, d, e, f, g with b. We obtain $v_1 = 7$, $u_1 = 4$, $b_1 \geq 3$ (two neighbors of a in S and the vertex c become dead, see Fig. 11). We obtain $\frac{8 \cdot 4}{5} + \frac{3}{5} - 7 = 0$. Thus, we may assume that e, f, g are pairwise adjacent. $\square$

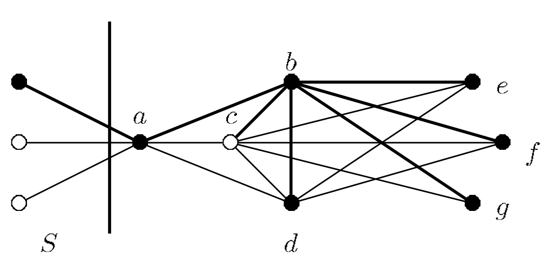

Fig. 11

Now let us consider the neighbors of b and c different from a, b, c, d. Each of the vertices b and c has 3 such neighbors. By Lemma 3, these neighbors belong to T. Assume that b and c have no common neighbor. Then we add a to S, join b, c, d with a, and join all neighbors of b and c with b and c, respectively (see Fig. 12). We have added 10 vertices, the number of leaves has increased by 6 ($= +7 - 1$), and two new dead vertices have appeared (the neighbors of a in S). In total, $\frac{8 \cdot 6}{5} + \frac{2}{5} - 10 = 0$.

Assume that b and c have exactly one common neighbor. Let e, f, g be the neighbors of b and g, h, i be the neighbors of c. By Corollary 1, the vertices e, f, h, i are adjacent to d. Thus, d has 7 neighbors a, b, c, e, f, h, i, a contradiction with Lemma 2.

Assume that b and c have exactly two common neighbors. Namely, let e, f, g be the neighbors of b and f, g, h be the neighbors of c. By Corollary 1, the vertices e and h are adjacent to d. Moreover, all neighbors of d must be adjacent to b or c. Therefore, either f or g is adjacent to d; let it be f. By Lemma 5, the vertices e, f, g are pairwise adjacent (since they are neighbors of b), the vertices e, f, h are also pairwise adjacent (since they are neighbors of d), and, finally, the vertices f, g, h are pairwise adjacent (since they are neighbors of c). Thus, f already has 6 neighbors, namely, b, c, d, e, g, h. Assume that f has no other neighbors. Consider the vertex e. Clearly, e can be adjacent to neither a nor c. Among the vertices considered above, e is adjacent to b, d, f, g, h. The vertex e must have one more neighbor, say i. By Lemma 3, we have $i \in T$. Then we add a to S, join b, c, d with a, join e, f, g with b, and, finally, join h, i with e (see Fig. 13). We add 9 vertices and increase the number of leaves by 5 ($= +6 - 1$). Five new dead vertices appear: the two neighbors of a in S and c, d, f. We obtain $\frac{8 \cdot 5}{5} + \frac{5}{5} - 9 = 0$.

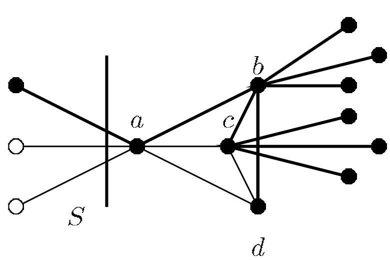

Fig. 12

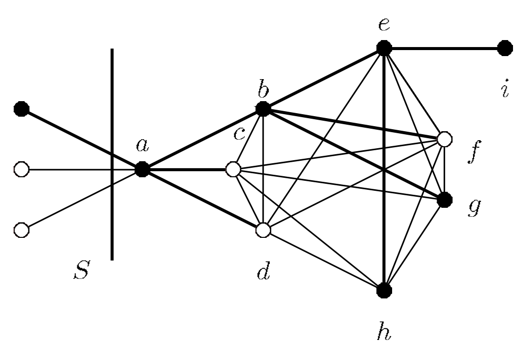

Fig. 13

Now we turn to the last case, where f has one more neighbor, say k. If k is adjacent to at least 3 vertices except those considered above, then we add a to S, join b, c, d with a, join e, f, g with b, join k, h with f, and, finally, join the three new neighbors of k with k. We add 12 vertices and 7 ($= +8 - 1$) leaves. Four new dead vertices appear: the two neighbors of a in S and c, d. We obtain $\frac{8 \cdot 7}{5} + \frac{4}{5} - 12 = 0$.

Assume that k is adjacent to S. Recall that f is adjacent to 7 vertices of T. This is similar to the case where we added a and b (with k instead of a and f instead of b). The only case that is not completely analyzed is where a has exactly 3 neighbors in S and all of them are adjacent in T only to a. By Lemma 2, the vertex b must have exactly 6 neighbors (otherwise, we can increase S preserving our inequality). However, f has at least 7 neighbors, and this case is analyzed.

In what follows, k is not adjacent to S and is adjacent to at most two new vertices of T (i.e., vertices not considered above). Among the vertices considered above, k can be adjacent only to e, f, g, h. Hence, k is adjacent to all these four vertices and to exactly two new vertices of T. The vertex e is adjacent to h and k. By Lemma 4, it can be adjacent only to two vertices of T that are not adjacent to b. Hence, these two vertices are h and k. The same holds for g and h. Then we add a to S, join b, c, d with a, join e, f, g with b, and, finally, join k, h with f. We add 9 vertices, increase the number of leaves by 5 ($= +6-1$), and add 7 new dead vertices (the two neighbors of a in S and c, d, e, g, h). We obtain $\frac{8 \cdot 5}{5} + \frac{7}{5} - 9 \geq 0$ (see Fig. 14).

In the remaining case, b and c have the same three neighbors in T, namely, e, f, g. Let us prove that they are also neighbors of d. By Lemma 2, the vertex d has exactly 3 neighbors in T, and, by Corollary 1, they are adjacent to b or c. Hence, these neighbors are exactly e, f, g (see Fig. 15). Then we add a to S, join b, c, d with a, and join e, f, g with b. We add 7 vertices, increase the number of leaves by 4, and four new dead vertices appear (the two neighbors of a in S and c, d). In total, we have $\frac{8 \cdot 4}{5} + \frac{4}{5} - 7 \geq 0$.

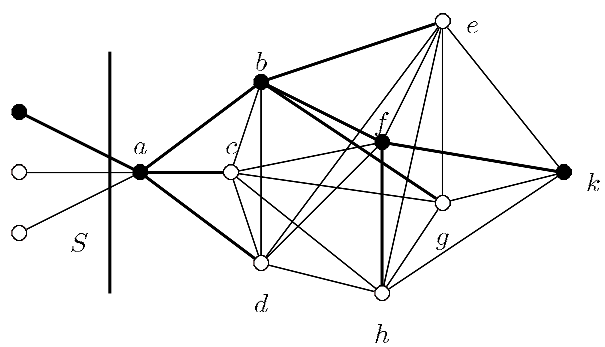

Fig. 14

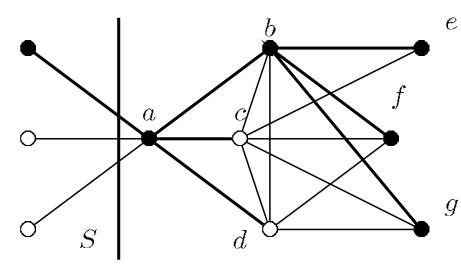

Fig. 15

4. Irreplaceable losses

4.1. New notation and a general concept. Let us add to the subtree all possible constructions for which the inequality $\frac{8}{5}u_1 + \frac{1}{5}b_1 - v_1 \geq 0$ is preserved. If we have built a spanning tree, then we are done. Assume that $T \neq \emptyset$, but we cannot perform any further step of the construction. Then all vertices of T are adjacent to S. Since we cannot perform the step of Sec. 2.4, each vertex of T has at least 3 neighbors in S. Each vertex of S is adjacent to at most 2 vertices of T (otherwise, we can perform the step of Sec. 2.3). We want each vertex of S to be adjacent to at most one vertex of T.

Definition 4. A *tick* consists of three vertices: a vertex of S and two vertices of T adjacent to it.

Choose a maximal set M of ticks such that no two of them have a common vertex. Hence, any vertex of S that is not covered by this set of ticks is adjacent to at most one vertex of T not covered by the ticks of M.

Let us introduce new notation. Denote by A the set of all vertices of S covered by the ticks of M and by B the set of all vertices of T covered by the ticks of M. Let C be the set of all vertices of $S \setminus A$ that are adjacent to T, and let $D = T \setminus B$. Denote by E the set of all vertices of D that are adjacent to B. Let $F = D \setminus E$.

Note that any vertex of A belongs to a tick which contains this vertex and two its neighbors lying in T. As observed above, any vertex of S (in particular, any vertex of A) has at most two neighbors in T. Hence, a vertex of A has exactly two neighbors in T, and these neighbors form a tick together with this vertex. Thus, each vertex of A is adjacent to B and not adjacent to D. Moreover, each vertex of B is adjacent to exactly one vertex of A (namely, to the vertex of the same tick). Since any vertex of T has at least 3 neighbors in S, each vertex of B is adjacent to at least two vertices of C. By construction, each vertex of C has exactly one neighbor in D.

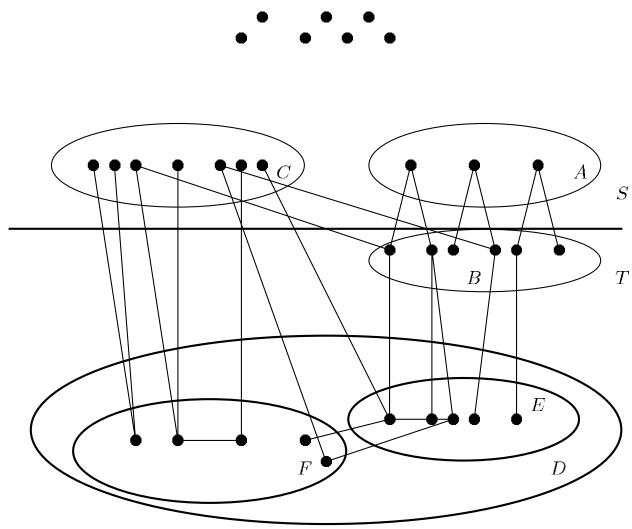

Fig. 16

At the end of this section, we will make a *global addition of vertices*: the vertices from the set B will be added to S via the chosen ticks, i.e., we will join the vertices from B with vertices from A. Hence, if after some adding operation some vertex from C has exactly one neighbor in T and this neighbor belongs to B, then this vertex will become dead after the global addition. We will not take into account the dead vertices from C after the global addition. Hence, we may take them into account after operations preceding the global addition. **This concerns only Sec. 4**. After that, we will return to our standard adding operations.

Remark. Our *local additions* (i.e., all additions of this section except the global one) will change our sets of vertices (A, B, and others). Vertices from B will be added only together with all their neighbors. If a vertex from B is added to S, then all vertices of its tick are also added, with two vertices from B joined to the vertex from A belonging to their tick. In this case, we exclude this tick from M and all its vertices from A and B. The only requirement is that after a local addition, vertices from C that become dead after the global addition should not be counted as dead, but we have agreed about this above.

What happens when we add some vertices from the set D? After a local addition, the set S increases. We add to the set C all vertices of the new set S that are adjacent to the remaining set T. After that, in the new set C, a vertex adjacent to two vertices of the new set D can appear. Let us explain how to perform several more local additions after which any vertex of C will be adjacent to exactly one vertex of the new set D. (The sets C and D can also be modified after an addition.) Assume that C contains a vertex x adjacent to at least two vertices $y_1, \dots, y_m \in D$. Note that each of the vertices $y_1, \dots, y_m$ is adjacent to at least 3 vertices from C (before local additions, each vertex $y_i \in D$ has at least 3 neighbors in C, and x was not in C). After joining $y_1, \dots, y_m$ with x, all neighbors in T of all these $3m$ vertices will belong to B. Let us count these $3m$ vertices as dead now (rather than after the global addition). Having $3m$ dead vertices, we obtain $\frac{8\cdot(m-1)}{5} + \frac{3\cdot m}{5} - m = \frac{6m-8}{5} \geq 0$ for $m \geq 2$. After this operation, we again add to C new vertices and remove vertices that are not adjacent to the new set T. Since T is finite, this process will terminate. Note that no new vertex will be added to A or B.

4.2. Local additions of vertices

4.2.1. *A vertex $x \in B$ is adjacent to at least 2 vertices of E.* Let x belong to a tick $\{u, x, t\}$ where $u \in S$. Let x be adjacent to $y_1, \dots, y_m \in E$. Then we join t, x with u and join $y_1, \dots, y_m$ with x. Since $y_1, \dots, y_m \in D$ (see Fig. 17), each of these vertices has at least 3 neighbors in C. After this addition, all neighbors in T of these $3m$ vertices will belong to B. We will count these $3m$ vertices as dead already and obtain the increment $\frac{8 \cdot m}{5} + \frac{3m}{5} - (m+2) = \frac{6m-10}{5} \geq 0$ for $m \geq 2$.

In what follows, each vertex of B will have at most one neighbor in E.

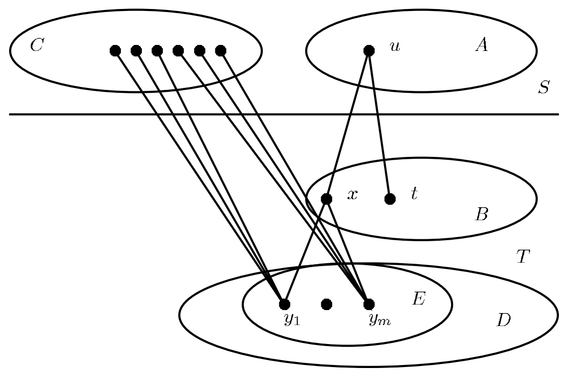

Fig. 17

4.2.2. *A vertex $x \in E$ has at least two neighbors in D.* Let u, v, t be a tick (where $u \in S$) and x be adjacent to v. Let x be adjacent to $y, z \in D$. Before a local addition, each vertex of T had at least 3 neighbors in S. Since the vertices $x, y, z \in D$ had no neighbors in A, all of them had 3 neighbors in C (see Fig. 18). We join v, t with u, join x with v, and join y, z with x. We count the nine neighbors of x, y, z in C as dead vertices now and obtain the increment at least $\frac{8 \cdot 2}{5} + \frac{1 \cdot 9}{5} - 5 = 0$.

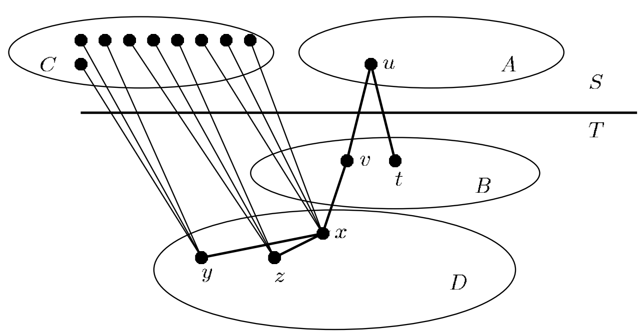

Fig. 18

4.2.3. *Vertices from the set R.* Let R be the set of all vertices from F that are adjacent to E. Let us prove that each vertex from R has one neighbor in E and five neighbors in C. Indeed, assume that a vertex $x \in R$ has two neighbors $y, z \in T$. Clearly, $y, z \in D$. We join x with S via one of the neighbors in C, and join y, z with x. We have $u_1 = 1$, $v_1 = 3$, $b_1 \geq 8$ (since each vertex of D is adjacent to at least 3 vertices of C), and $\frac{8}{5} + \frac{8}{5} - 3 \geq 0$.

If a vertex from the set R has at least 6 neighbors in C, then we join this vertex with one of these neighbors and obtain $\frac{8 \cdot 0}{5} + \frac{5}{5} - 1 = 0$.

4.3. The global addition of vertices. Assume that we can perform no local addition. Then each vertex from B is adjacent in T to at most one vertex from E and, possibly, other vertices from B. Vertices from E can be adjacent only to vertices from the sets B, C, R, and E. Moreover, by Sec. 4.2.2, any vertex from E has at most one neighbor in $R \cup E$. Therefore, other vertices from the set T are not adjacent to A, B, E, R. Then we join the vertices from B with their neighbors in A (in the corresponding ticks), join the vertices from E with their neighbors in B (each vertex is joined with exactly one neighbor), and join the vertices from R with their neighbors in E. The whole this system of vertices is not adjacent to any of the remaining vertices from the set D. Let us study what happens after this global addition of vertices.

Let $|A| = x$; then $|B| = 2x$. By a *branch of a vertex q* we mean the set of all vertices that have been added in the global addition and have q as an ancestor in the subtree S. Then $2x$ new branches with ancestors from the set B have appeared. Since each of these branches contains at least one leaf, at least $2x$ new leaves have appeared. Since vertices that remain in T are not adjacent to A, B, E, R, all these leaves become dead vertices. Hence, the number of leaves increases at least by x, and the number of dead vertices increases at least by $2x$.

Now consider the set E. Let it contain y vertices adjacent to at least two vertices from B and z vertices adjacent to exactly one vertex from B. Each of these vertices is adjacent to at least 3 vertices from C (this holds for all vertices in D). Each of the z vertices adjacent to one vertex from B, by Sec. 4.2.2, has at most one neighbor in D and, therefore, at least 4 neighbors in S. All these neighbors belong to C and become dead after the global addition. Hence, the number of dead vertices increases at least by $3y + 4z$.

Let $|R| = r$. Each of these vertices has 5 neighbors in C. Hence, $5r$ new dead vertices appear. The total increment is at least

$$\frac{8x}{5} + \frac{2x + 3y + 4z + 5r}{5} - 2x - y - r - z = -\frac{2y + z}{5}.$$

Thus, we have decreased our sum. We will return to this later.

Remark. Note that the operation of addition with losses described in this section can be performed at most once. Indeed, assume the converse. Assume that after such an operation a vertex $x \in S$ appears that has exactly two neighbors in T, say y and z. Note that before the first addition with losses, we had $y, z \in T$. Moreover, none of the vertices y, z belonged to B, since all vertices from B had already been added. Therefore, both y and z had 3 neighbors in C at that moment, and all these neighbors are now adjacent in T only to y or z. Hence, if we join y and z with x, then we obtain $u_1 = 1,\ b_1 \geq 6,\ v_1 = 2$, and the inequality $\frac{8}{5} + \frac{6}{5} - 2 \geq 0$ holds.

Thus, we can perform at most one addition with losses and obtain S and T such that each vertex from S has at most one neighbor in T.

5. Each vertex from S has at most one neighbor in T

Note that each vertex has at most 3 neighbors in T, since all the other cases are already analyzed.

5.1. A vertex $x \in T$ has exactly 3 neighbors in T. Let these neighbors be a_1, a_2, a_3. We add x to S and join a_1, a_2, a_3 with x. Each of these 4 vertices has 3 neighbors in S, and all these neighbors except those joined with x become dead after the addition. Therefore, $\frac{8 \cdot 2}{5} + \frac{11}{5} - 4 \geq 0$.

In what follows, each vertex from T has at most 2 neighbors in T and, therefore, at least 4 neighbors in S.

5.2. A vertex $x \in T$ has exactly 2 neighbors in T. Let these neighbors be a_1 and a_2. We add x to S and join a_1, a_2 with x. Each of these 3 vertices has 4 neighbors in S, and all these neighbors except those joined with x become dead after the addition. Therefore, $\frac{8}{5} + \frac{11}{5} - 3 \geq 0$.

In what follows, each vertex of T has at most 1 neighbor in T and, therefore, at least 5 neighbors in S.

5.3. A vertex $x \in T$ has no neighbors in T. Then x has 6 neighbors in S, we add x to S, and obtain 5 new dead vertices. The increment is $\frac{5}{5} - 1 = 0$.

5.4. Each vertex from T has exactly 1 neighbor in T. Consider adjacent vertices $x, y \in T$. We add x to S and join y with x. Each of these vertices is adjacent to at least 5 vertices of S, and 9 of them become dead (all except those joined with x in the subtree). The vertex y also becomes dead, and we obtain $\frac{10}{5} - 2 = 0$.

6. The resulting tree

Consider the moment when S contains all vertices of the graph G. In almost all cases, we add vertices to S so that the inequality

$$\frac{8}{5}\, u \;+\; \frac{1}{5}\, b \;-\; v \;\geq\; 0$$

holds. Problems can appear only in the case where we add vertices of ticks. By the remark at the end of the previous section, we have performed this operation at most once. Let us consider this case in detail.

By Sec. 4.2.2, a vertex from the set E can have at most one neighbor outside $S \cup B$. Let $x(G) = |A|$, let $y(G)$ be the number of vertices in E that have 2 neighbors in B, and let $z(G)$ be the number of vertices in E that have one neighbor in B. Then

$$\frac{8}{5}\, u \;+\; \frac{1}{5}\, b \;-\; v \;\geq\; -\frac{2y(G) + z(G)}{5}.$$

In the unique addition with losses, we have considered $x(G)$ vertices from A and have added $2x(G)$ vertices from B and $y(G) + z(G)$ vertices from E. At least $3y(G) + 4z(G)$ vertices from C have become dead by Sec. 4.3. Thus,

$$\frac{8}{5}u(G) + \frac{1}{5}b(G) - v(G) \geq -\frac{2y(G) + z(G)}{5},$$

and $v(G) \geq 3x(G) + 4y(G) + 5z(G)$. Since all added vertices from E were adjacent to B and it was proved that each vertex from B is adjacent to at most one vertex in E, we obtain $2y(G) + z(G) \leq 2x(G)$. Therefore,

$$\frac{9}{5}u(G) \geq \frac{8}{5}u(G) + \frac{1}{5}b(G) \geq v(G) - \frac{2y(G) + z(G)}{5}.$$

This is equivalent to the inequality

$$\frac{u(G)}{v(G)} \geq \frac{5}{9}\left(1 - \frac{1}{5} \cdot \frac{2y(G) + z(G)}{v(G)}\right).$$

To prove a lower bound on $\frac{u(G)}{v(G)}$, we minimize $1 - \frac{2}{5} \cdot \frac{2y(G)+z(G)}{v(G)}$, i.e., maximize $\frac{2y(G)+z(G)}{v(G)}$.

Note that

$$\frac{2y(G)+z(G)}{v(G)} \le \frac{2y(G)+z(G)}{3x(G)+4y(G)+5z(G)} = \frac{2y(G)+z(G)}{3x(G)+2(2y(G)+z(G))+3z(G)} \le \frac{2x(G)}{7x(G)} = \frac{2}{7}.$$

Therefore,

$$t(G) = \frac{u(G)}{v(G)} \ge \frac{5}{9}\left(1 - \frac{1}{5}\cdot\frac{2}{7}\right) = \frac{5}{9}\cdot\frac{33}{35} = \frac{11}{21}.$$

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