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THE VOLUME FRACTION OF ONE OF THE PHASES IN EQUILIBRIUM TWO-PHASE ELASTIC MEDIUM

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The relationship between the volume fraction of one phase of an equilibrium two-phase medium and other characteristics of the equilibrium state is studied. Bibliography: 9 titles.

1. Introduction

In the quadratic approximation, the energy density of the deformation of each of the phases "±" of a two-phase elastic medium occupying the bounded domain $\Omega \subset R^m$, $m \geq 1$, is given by the functions

$$F^{\pm}(M) = \langle A^{\pm}(e(M) - \zeta^{\pm}), e(M) - \zeta^{\pm} \rangle,$$
$$M \in R^{m\times m}, \quad e(M) = \frac{M + M^*}{2}, \quad \zeta^{\pm} \in R_s^{m\times m}, \tag{1.1}$$

where $R^{m\times m}$ is the space of $m\times m$-matrices, $R_s^{m\times m}$ is the space of $m \times m$-symmetric matrices, the quantity $\langle P, Q\rangle = \operatorname{tr} PQ$, $P, Q \in R_s^{m\times m}$, is the scalar product in $R_s^{m\times m}$, and the linear maps $A^{\pm} : R_s^{m\times m} \to R_s^{m\times m}$ are symmetric and positive definite with respect to the specified scalar product.

The deformation energy functional corresponding to densities (1.1) is defined by

$$I_0[u, \chi, t] = \int_\Omega \{\chi(F^+(\nabla u) + t) + (1 - \chi)F^-(\nabla u)\}\, dx, \tag{1.2}$$

where the m-dimensional vector-valued function $u(x)$ corresponds to the displacement field, $(\nabla u)_{ij} = u^i_{x_j}$, $e(\nabla u)$ is the tensor of deformation, and the matrices $\zeta^{\pm}$ and the parameter $t \in R$ are interpreted as the tensors of residual deformation and the temperature, respectively. The phase distribution in the domain Ω is given by the characteristic function $\chi(x)$, $x \in \Omega$; the phases with index "+" and "−" are located on the support of this function and its complement, respectively. As the domain of definition of functional (1.2), we take the sets

$$u \in \mathbb{H}, \quad \chi \in \mathbb{Z}',$$
$$\mathbb{H} = \mathring{W}^1_2(\Omega, R^m), \quad \mathbb{Z}' \text{ is the set of all measurable characteristic functions.} \tag{1.3}$$

Under the equilibrium state of a two-phase medium for a fixed t, we mean the solution $\widehat{u}_t$, $\widehat{\chi}_t$ of the variational problem

$$I_0[\widehat{u}_t, \widehat{\chi}_t, t] = \inf_{u\in\mathbb{H}, \chi\in\mathbb{Z}'} I_0[u, \chi, t], \qquad \widehat{u}_t \in \mathbb{H}, \quad \widehat{\chi}_t \in \mathbb{Z}'. \tag{1.4}$$

The equilibrium state $\widehat{u}_t$, $\widehat{\chi}_t$ is said to be single-phase if $\widehat{\chi}_t \equiv 0$ or $\widehat{\chi}_t \equiv 1$, and two-phase otherwise. Obviously, for a single-phase equilibrium state $\widehat{u}_t$, $\widehat{\chi}_t$, the equilibrium displacement field equals zero, $\widehat{u}_t \equiv 0$.

The above described approach to determine the equilibrium displacement field $\widehat{u}_t$ and the equilibrium phase distribution $\widehat{\chi}_t$ is traditional, see [4]. An extensive literature is devoted to investigation of problem (1.4) and those close to it (see [1,7] and references therein). Our goal

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is to study some properties of the volume fraction of the phase with the index "+" in the equilibrium state, i.e., the quantity

$$\widehat{Q}(t) = \frac{1}{|\Omega|} \int\limits_{\Omega} \widehat{\chi}_t(x)\, dx \tag{1.5}$$

(here and below, the module of a set in R^m denotes its m-dimensional Lebesgue measure) computed on all the solutions $\widehat{u}_t$, $\widehat{\chi}_t$ of problem (1.4) for a fixed value t. For a better understanding of the nature of quantity (1.5), we make several preliminary remarks.

For problem (1.4), it is known that there exist temperatures $t_\pm$ of phase transitions that are independent of the domain Ω and satisfy the relations

$$t_- \le t^* \le t_+, \quad t^* = -[\langle A\zeta, \zeta\rangle] \tag{1.6}$$

($[\alpha] = \alpha_+ - \alpha_-$ is the jump taking two values $\alpha_\pm$ of the quantity α, in (1.6) both equalities, if they are, hold simultaneously), and are characterized by the following conditions [6]:

$$\begin{aligned}
&\text{if } t < t_-, \text{ then}\\
&\qquad \text{only the single-phase equilibrium with } \widehat{\chi}_t \equiv 1 \text{ is realized,}\\
&\text{if } t > t_+, \text{ then}\\
&\qquad \text{only the single-phase equilibrium with } \widehat{\chi}_t \equiv 0 \text{ is realized,}\\
&\text{if } t = t_\pm, \text{ then}\\
&\qquad \text{there are single-phase equilibriums with } \widehat{\chi}_t \equiv 0 \text{ and } \widehat{\chi}_t \equiv 1, \text{ respectively,}\\
&\text{if } t \in (t_-, t_+), \text{ then there are no single-phase equilibriums.}
\end{aligned} \tag{1.7}$$

For $t \in (t_, t_+)$, solutions (of course, two-phase solutions) may exist or not depending on the parameters of the problem [2, 5]. From what said, it follows that for quantity (1.5),

$$\widehat{Q}(t) = 1 \quad \text{if} \quad t < t_-, \qquad \widehat{Q}(t) = 0 \quad \text{if} \quad t > t_+. \tag{1.8}$$

There is a criterion for the coincidence of the temperatures $t_\pm$ [8],

$$t_\pm = t^* \quad \text{if and only if} \quad [A\zeta] = 0. \tag{1.9}$$

In the case $[A\zeta] = 0$, functional (1.2) is of the form

$$\begin{aligned}
I_0[u, \chi, t] &= |\Omega| \langle A^-\zeta^-, \zeta^-\rangle \\
&+ \int\limits_{\Omega} \Big\{ \chi \langle A^+ e(\nabla u), e(\nabla u)\rangle + (1-\chi)\langle A^- e(\nabla u), e(\nabla u)\rangle + (t - t^*)\chi \Big\}\, dx.
\end{aligned} \tag{1.10}$$

Therefore for $t_+ = t_-$, the set of all the solutions of problem (1.4) is exhausted by the relations

$$\begin{aligned}
&\widehat{u}_t \equiv 0 \quad \text{for all} \quad t \in R,\\
&\widehat{\chi}_t \equiv 1 \quad \text{for} \quad t < t^*,\\
&\widehat{\chi}_t \equiv 0 \quad \text{for} \quad t > t^*,\\
&\widehat{\chi}_{t^*} \ \text{ is an arbitrary element of } \ \mathbb{Z}'.
\end{aligned} \tag{1.11}$$

Consequently in the case $t_+ = t_-$,

$$\begin{aligned}
&\widehat{Q}(t) \equiv 1 \quad \text{for} \quad t < t^*,\\
&\widehat{Q}(t) \equiv 0 \quad \text{for} \quad t > t^*,\\
&\widehat{Q}(t^*) \ \text{ is an arbitrary number from the interval } [0, 1].
\end{aligned} \tag{1.12}$$

From (1.7), (1.8), and (1.12), it follows that function (1.5) does not have to be definite for all values of t, and under certain conditions it can turn out to be many-valued.

2. Formulation of results

We give the formulations of the results to be proved below and comment on them.

(1) *Independence of quantity* (1.5) *from the domain* Ω. Since the phase transition temperatures (1.6) do not depend on the domain Ω, function (1.5) takes values (1.8) for $t \notin (t_-, t_+)$ in any domain. The description (1.12) does not also depend on the domain Ω. Therefore the change of domain is reflected in the function (1.5) (if does) on the interval $(t_, t_+)$ only. Recall that the domain Ω in (1.2) is always considered as bounded.

Theorem 1. (a) *If for $t = t_0$ problem* (1.4) *has a solution in some domain $\Omega = \omega$ with $|\partial\omega| = 0$ and such that $\widehat{Q}(t_0) = Q_0$, then for $t = t_0$ problem* (1.4) *is solvable in an arbitrary domain Ω and has a solution such that $\widehat{Q}(t_0) = Q_0$.*
(b) *If for $t = t_0$ problem* (1.4) *has solutions $\widehat{u}^{(i)}_{t_0}$, $\widehat{\chi}^{(i)}_{t_0}$, $i = 1, 2$, in some domain Ω with $|\partial\Omega| = 0$, and $\widehat{Q}(t_0) = Q_i$, $Q_1 < Q_2$, then in this domain there exists a solution $\widehat{u}_{t_0}$, $\widehat{\chi}_{t_0}$ with any $\widehat{Q}(t_0) \in (Q_1, Q_2)$.*

Statement (a) of the theorem leads to the independence of function (1.5) from the domain Ω. Statement (b) says about the structure of the possible ambiguity of this function, which is confirmed by description (1.12) in the case $t_- = t_+$. For the densities

$$m = 1, \quad F^{\pm}(M) = a_{\pm}(M - c_{\pm})^2, \quad a_{\pm}, c_{\pm} \in R, \quad a_{\pm} > 0, \tag{2.1}$$

$$\begin{gathered} m \geq 2, \quad F^{\pm}(M) = a\,\mathrm{tr}(e(M) - c_{\pm}i)^2 + b_{\pm}\,\mathrm{tr}^2(e(M) - c_{\pm}i), \\ a, b_{\pm}, c_{\pm} \in R, \quad a > 0, \quad b_{\pm} \geq 0, \\ i \text{ is the identity matrix in the space } \quad R^m, \end{gathered} \tag{2.2}$$

function (1.5) was found in explicit form in [7]. It turns out that for $t_- < t_+$, it is single-valued. The situation changes if one takes into account the surface energy of the phase boundary, proportional to its area, in the energy functional, by replacing functional (1.2) with

$$I[u, \chi, t, \sigma] = I_0[u, \chi, t] + \sigma S[\chi], \tag{2.3}$$

where $S[\chi]$ is the area of the phase boundary for $\chi \in \mathbb{Z} = \mathbb{Z}' \cap BV(\Omega)$. For problem (1.4) with functional (2.3), the temperatures of the phase transitions $t_{\pm} = t_{\pm}(\sigma)$ are also introduced. The points $t = t_{\pm}(\sigma)$ are the points of multivaluedness for the function $\widehat{Q}(t, \sigma)$, but (in any case for densities (2.1)) the set of values $\widehat{Q}(t_{\pm}(\sigma), \sigma)$ for each of the signs consists of only two points, and does not fill the interval between them. For details, we refer to [7].

(2) *The connection between the equilibrium displacement field $\widehat{u}_t$ and the equilibrium phase distribution $\widehat{\chi}_t$.* The following theorem discusses the question of the unique determination of one component from the pair $\{\widehat{u}_t, \widehat{\chi}_t\}$ through another.

Theorem 2. (a) *For any t, the function $\widehat{u}_t$ is uniquely determined by the function $\widehat{\chi}_t$.*
(b) *If $t_- = t_+$, but $t \neq t^*$ or $t_- < t_+$, and*

$$[A\zeta] \notin \mathrm{Im}[A], \tag{2.4}$$

or

$$[A\zeta] \in \mathrm{Im}[A] \quad \textit{and either } [A] \geq 0, \textit{ or } [A] \leq 0, \tag{2.5}$$

then $\widehat{\chi}_t$ is uniquely determined by $\widehat{u}_t$.

Since the function $\widehat{u}_t$ is the minimizer of the functional $J[u,t] = I_0[u, \widehat{\chi}_t, t]$, $u \in \mathbb{H}$, the first statement of the theorem follows from the strict convexity of this functional. To explain the second one, we rewrite functional (1.2) in the form

$$I_0[u,\chi,t] = \int_\Omega F^-(\nabla u)\,dx + \int_\Omega \chi(F^+(\nabla u) - F^-(\nabla u) + t)\,dx. \tag{2.6}$$

From (2.6), it follows that

$$\widehat{\chi}_t(x) = \begin{cases} 1 & \text{if} \quad R(x,t) < 0, \\ 0 & \text{if} \quad R(x,t) > 0, \end{cases} \qquad R(x,t) = F^+(\nabla \widehat{u}_t(x)) - F^-(\nabla \widehat{u}_t(x)) + t, \tag{2.7}$$

$$\widehat{\chi}_t(x) \text{ is an arbitrary characteristic function on the set } E_{\widehat{u}_t} = \{x \in \Omega : R(x,t) = 0\}. \tag{2.8}$$

Owing to (1.11), in the case $t_+ = t_-$ the equality $R(x,t) = t - t^*$ holds. Therefore, $|E_{\widehat{u}_t}| = 0$ for $t_+ = t_-$ and $t \neq t^*$, and $E_{\widehat{u}_t} = \Omega$ for $t_+ = t_-$ and $t = t^*$. Hence to prove the theorem it remains to verify that

$$\text{if (2.4) or (2.5) holds and } t_- < t_+, \quad \text{then } |E_{\widehat{u}_t}| = 0 \text{ for any } t. \tag{2.9}$$

Thus if the conditions of the theorem are satisfied, then function (1.5) cannot have two different values for the pair $\{\widehat{u}_t, \widehat{\chi}_t\}$ with fixed first component. By virtue of (1.11) and (1.12), if the condition (b) of the theorem is violated (i.e., for $t_- = t_+$ and $t = t^*$), then $\widehat{u}_{t^*} \equiv 0$, but the values of function (1.5) at the point $t = t^*$ fill the interval $[0,1]$.

(3) *Smooth dependence on the temperature of the equilibrium energy and the point of single-valuedness of function* (1.5). For a fixed domain Ω, we set

$$i(t) = \inf_{u \in \mathbb{H}, \chi \in \mathbb{Z}'} I_0[u,\chi,t]. \tag{2.10}$$

Function (2.10) is called the equilibrium energy of functional (1.2). If for $t = t_0$ problem (1.4) is solvable, then $i(t_0) = I_0[\widehat{u}_{t_0}, \widehat{\chi}_{t_0}, t_0]$ for any of its solutions $\widehat{u}_{t_0}$, $\widehat{\chi}_{t_0}$. Owing to (1.7),

$$\begin{aligned} i(t) &= |\Omega|(t + \langle A^+\zeta^+, \zeta^+\rangle) \quad \text{for} \quad t \le t_-, \\ t(t) &= |\Omega|\langle A^-\zeta^-, \zeta^-\rangle \quad \text{for} \quad t \ge t_+. \end{aligned} \tag{2.11}$$

In the case $t_+ = t_-$, the relation (2.11) can be refined as follows:

$$\begin{aligned} i(t) &= |\Omega|(t + \langle A^+\zeta^+, \zeta^+\rangle) \quad \text{for} \quad t \le t^*, \\ i(t) &= |\Omega|\langle A^-\zeta^-, \zeta^-\rangle \quad \text{for} \quad t \ge t^*. \end{aligned} \tag{2.12}$$

From the definition (1.6) of the number t^*, the continuity of function (2.12) follows.

Theorem 3. (a) *There exists a set of full measure $\mathcal{L} \subset R$ such that function* (2.10) *has a finite classical derivative $i'(t)$ in the points of this set; this derivative is continuous on $\mathcal{L}$ and decreases monotonically. In each point $t \in R \setminus \mathcal{L}$, function* (2.10) *has finite one-sided classical derivatives $i'(t-0) > i'(t+0)$, and also*

$$i'(t-0) = \lim_{\substack{\tau \in \mathcal{L}, \\ \tau < t, \tau \to t}} i'(\tau), \qquad i'(t+0) = \lim_{\substack{\tau \in \mathcal{L}, \\ \tau > t, \tau \to t}} i'(\tau). \tag{2.13}$$

(b) *If problem* (1.4) *is solvable for a given $t = t_0$, then for all its solutions $\widehat{u}_{t_0}$, $\widehat{\chi}_{t_0}$,*

$$\begin{aligned} &|\Omega|\widehat{Q}(t_0) = i'(t_0) \quad \textit{for} \quad t_0 \in \mathcal{L}, \\ &|\Omega|\widehat{Q}(t_0) \in [i'(t_0+0), i'(t_0-0)] \quad \textit{for} \quad t_0 \in E \setminus \mathcal{L}. \end{aligned} \tag{2.14}$$

For arbitrary energy densities in the case $t_- = t_+$, from (2.12) it follows that

$$\begin{gathered}\mathcal{L} = R \setminus \{t^*\}, \quad i'(t) = |\Omega| \quad \text{for} \quad t < t^*, \\ i'(t) = 0 \quad \text{for} \quad t > t^*, \\ i'(t^* - 0) = |\Omega|, \quad i'(t^* + 0) = 0.\end{gathered} \tag{2.15}$$

Relations (1.12) and (2.15) confirm the statement of the theorem. For densities (2.1), (2.2) with $t_- < t_+$, the function $i(t)$ can be written out in explicit form, see [7]. For this function, $\mathcal{L} = R$. The same holds for the density,

$$\begin{gathered}F^{\pm}(M) = a\,\mathrm{tr}(e(M) - c_{\pm}P^{(k)})^2, \\ M \in R^{m\times m}, \quad a, c_{\pm} \in R, \quad a > 0, \quad 1 \le k < m,\end{gathered} \tag{2.16}$$

where $P^{(k)}$ is an orthoprojector in R^m onto a k-dimensional subspace. However if $t \in (t_, t_+)$, then problem (1.4) does not have solutions for these densities [8].

3. Proof of Theorem 1

(a) Given a domain ω, we construct a family of domains

$$\omega_{\xi,\lambda} = \bigl\{x \in R^m : x = \lambda\widetilde{x} + \xi,\ \widetilde{x} \in \omega\bigr\}, \quad \lambda > 0, \quad \xi \in R^m, \tag{3.1}$$

obtained from ω by a stretching in λ times and a subsequent shift to the vector ξ. Let us define sets $\mathbb{H}$, $\mathbb{Z}'$ and consider the domains of the functions u and χ as arguments of the functional I_0. Given $u \in \mathbb{H}(\omega)$ and $\chi \in \mathbb{Z}'(\omega)$, we define the functions

$$u^{\xi,\lambda}(x) = \lambda u(\widetilde{x}), \quad \chi^{\xi,\lambda}(x) = \chi(\widetilde{x}), \quad \widetilde{x} \in \omega, \quad x = \lambda\widetilde{x} + \xi \in \omega_{\xi,\lambda}. \tag{3.2}$$

Obviously, $u^{\xi,\lambda} \in \mathbb{H}(\omega_{\xi,\lambda})$, $\chi^{\xi,\lambda} \in \mathbb{Z}'(\omega_{\xi,\lambda})$, and every function from $\mathbb{H}(\omega_{\xi,\lambda})$ and $\mathbb{Z}'(\omega_{\xi,\lambda})$ is obtained with the help of procedure (3.2) from some function from $\mathbb{H}(\omega)$ and $\mathbb{Z}'(\omega)$, respectively.

After changing coordinates, we have

$$I_0[u^{\xi,\lambda}, \chi^{\xi,\lambda}, t, \omega_{\xi,\lambda}] = \lambda^m I_0[u, \chi, t, \omega]. \tag{3.3}$$

Since $|\omega_{\xi,\lambda}| = \lambda^m|\omega|$, from (3.3) it follows that

$$\frac{1}{|\omega_{\xi,\lambda}|} I_0[u^{\xi,\lambda}, \chi^{\xi,\lambda}, t, \omega_{\xi,\lambda}] = \frac{1}{|\omega|} I_0[u, \chi, t, \omega]. \tag{3.4}$$

The quasi-convex hull $\mathcal{F}(M,t)$ of the function

$$F_{\min}(M,t) = \min\{F^+(M) + t, F^-(M)\}$$

does not depend on the domain ω and is defined by the equality

$$\mathcal{F}(M,t) = \inf_{\substack{u\in\mathbb{H}(\omega),\\ \chi\in\mathbb{Z}'(\omega)}} \frac{1}{|\omega|} \int\limits_{\omega} \Bigl\{\chi(F^+(M+\nabla u)+t) + (1-\chi)F^-(M+\nabla u)\Bigr\}\,dx, \quad M \in R^{m\times m}, \tag{3.5}$$

see [3]. Then $\widehat{u}_{t_0} \in \mathbb{H}(\omega)$, $\widehat{\chi}_{t_0} \in \mathbb{Z}'(\omega)$ is a solution of problem (1.4) for the functional $I_0[u, \chi, t_0, \omega]$ if and only if

$$I_0[\widehat{u}_{t_0}, \widehat{\chi}_{t_0}, t_0, \omega] = |\omega|\mathcal{F}(0, t_0). \tag{3.6}$$

In view of (3.4),

$$I_0[\widehat{u}^{\xi,\lambda}_{t_0}, \widehat{\chi}^{\xi,\lambda}_{t_0}, \omega_{\xi,\lambda}] = |\omega_{\xi,\lambda}|\mathcal{F}(0, t_0). \tag{3.7}$$

Therefore the pair $\widehat{u}_{t_0}^{\xi,\lambda} \in \mathbb{H}(\omega_{\xi,\lambda})$, $\widehat{\chi}_{t_0}^{\xi,\lambda} \in \mathbb{Z}'(\omega_{\xi,\lambda})$ is a solution of problem (1.4) for the functional $I_0[u,\chi,t_0,\omega_{\xi,\lambda}]$. Taking into account (3.1) and (3.2), we have

$$\begin{gathered}\frac{1}{|\omega|}\int\limits_{\omega} \widehat{\chi}_{t_0}(\widetilde{x})\,d\widetilde{x} = \frac{1}{|\omega_{\xi,\lambda}|}\int\limits_{\omega_{\xi,\lambda}} \widehat{\chi}_{t_0}^{\xi,\lambda}(x)\,dx,\\ \int\limits_{\omega_{\xi,\lambda}} |\nabla \widehat{u}_{t_0}^{\xi,\lambda}(x)|^2\,dx = \lambda^m \int\limits_{\omega} |\nabla \widehat{u}_{t_0}(\widetilde{x})|^2\,d\widetilde{x} = \frac{|\omega_{\xi,\lambda}|}{|\omega|}\int\limits_{\omega} |\nabla \widehat{u}_{t_0}(\widetilde{x})|^2\,d\widetilde{x}.\end{gathered} \tag{3.8}$$

From the first relation of (3.8), it follows that quantity (1.5) for the solutions $\widehat{u}_{t_0} \in \mathbb{H}(\omega)$, $\widehat{\chi}_{t_0} \in \mathbb{Z}'(\omega)$ and $\widehat{u}_{t_0}^{\xi,\lambda} \in \mathbb{H}(\omega_{\xi,\lambda})$, $\widehat{\chi}_{t_0}^{\xi,\lambda} \in \mathbb{Z}'(\omega_{\xi,\lambda})$ is the same.

From definition (3.1) of the domains $\omega_{\xi,\lambda}$, it follows that the sets $\bar{\omega}_{\xi,\lambda}$ satisfy all the requirements in [9, Chap. 4, Sec. 3] for constructing the Vitali cover of an arbitrary domain $\Omega \subset R^m$: namely, there exist $\lambda = \lambda^i$, $\xi = \xi^i$, $i = 1, 2, \ldots$, for which $E^i = \bar{\omega}^i$ and $\omega^i = \omega_{\xi^i,\lambda^i}$ are such that

$$E^i \subset \Omega, \qquad E^i \cap E^j = \varnothing \quad \text{for} \quad i \neq j, \qquad |\Omega \setminus \cup_i E^i| = 0. \tag{3.9}$$

Since $|\partial\omega^i| = 0$, we have $|E^i| = |\omega^i|$ for all i. Therefore,

$$|\Omega| = \Sigma_i |E^i| = \Sigma_i |\omega^i|. \tag{3.10}$$

Set

$$u^{(i)}(x) = \widehat{u}_{t_0}^{\xi^i,\lambda^i}(x), \quad \chi^{(i)}(x) = \widehat{\chi}_{t_0}^{\xi^i,\lambda^i}(x), \qquad x \in \omega^i.$$

Denote by $\bar{u}^{(1)}$, $\bar{\chi}^{(i)}$ the extension of these functions by zero to the domain Ω. Obviously, $\bar{u}^{(i)} \in \mathbb{H}(\Omega)$ and $\bar{\chi}^{(i)} \in \mathbb{Z}'(\Omega)$. From (3.8) and (3.10), it follows that the series

$$\bar{u} = \Sigma_i \bar{u}^{(i)}, \quad \bar{\chi} = \Sigma_i \bar{\chi}^{(i)}$$

converge in the spaces $\mathbb{H}(\Omega)$ and $L_1(\Omega)$, respectively. Consequently, $\bar{u} \in \mathbb{H}(\Omega)$ and $\bar{\chi} \in \mathbb{Z}'(\Omega)$. In view of (3.10),

$$\frac{1}{|\Omega|}\int\limits_{\Omega} \bar{\chi}(x)\,dx = \frac{1}{|\omega|}\int_{\omega} \widehat{\chi}_{t_0}(\widetilde{x})\,d\widetilde{x}. \tag{3.11}$$

Taking into account (3.7) and (3.10), we obtain

$$I_0[\bar{u},\bar{\chi},t_0,\Omega] = \Sigma_i I_0[u^{(i)},\chi^{(i)},t_0,\omega^i] = (\Sigma_i|\omega^i|)\mathcal{F}(0,t_0) = |\Omega|\mathcal{F}(0,t_0). \tag{3.12}$$

Therefore, the pair $\bar{u}$, $\bar{\chi}$ is a solution of problem (1.4) for the functional $I_0[u,\chi,t_0,\Omega]$. By (3.11), quantities (1.5) for this and initial functionals $\widehat{u}_{t_0}$ and $\widehat{\chi}_{t_0}$, in the domains Ω and ω, respectively, coincide.

(b) For each $\nu \in R$, we divide the domain Ω by the hyperplane

$$T_{e,\nu} = \{x \in R^m : x \cdot e = \nu\}, \quad e \in R^m, \quad |e| = 1$$

into the two parts

$$\Omega_+^\nu = \{x \in \Omega : x \cdot e > \nu\}, \quad \Omega_-^\nu = \{x \in \Omega : x \cdot e < \nu\}.$$

Obviously, the $\Omega_\pm^\nu$ are open sets, $|\Omega_+^\nu|$ depends on ν continuously, and $|\Omega| = |\Omega_+^\nu| + |\Omega_-^\nu|$. Then for any $\mu \in [0,1]$, there exists ν such that $|\Omega_+^\nu| = \mu|\Omega|$ and $|\Omega_-^\nu| = (1-\mu)||\Omega|$.

In the sequel, we use the technique used in the proof of the part (a) of the theorem and the notation from the statement of the part (*b*).

For each connected component ω_+^i of the set Ω_+^ν, we construct a solution $\bar{u}_i^+$, $\bar{\chi}_i^+$ of problem (1.4) for the functional $I_0[u,\chi,t_0,\omega_+^i]$ with quantity (1.5) equal to Q_1. For each connected component ω_-^i of the set Ω_-^ν, let $\bar{u}_i^-$, $\bar{\chi}_i^-$ be a solution of problem (1.4) for the functional

$I_0[u,\chi,t_0,\omega_-^i]$ with quantity (1.5) equal to Q_2. Analogously to (3.12), we arrive at the conclusion that the pair $\widehat{u}_{t_0}^{(3)}$, $\widehat{\chi}_{t_0}^{(3)}$ such that

$$\widehat{u}_{t_0}^{(3)}(x) = \bar{u}_i^+(x), \quad \widehat{\chi}_{t_0}^{(3)}(x) == \bar{\chi}_i^+(x) \quad \text{for} \quad x \in \omega_+^i,$$
$$\widehat{u}_{t_0}^{(3)}(x) = \bar{u}_i^-(x), \quad \widehat{\chi}_{t_0}^{(3)}(x) == \bar{\chi}_i^-(x) \quad \text{for} \quad x \in \omega_-^i$$

is a solution of problem (1.4) for the functional $I_0[u,\chi,t_0,\Omega]$ for which

$$\begin{aligned}|\Omega|Q_3 = \int_\Omega \widehat{\chi}_{t_0}^{(3)}\,dx = \Sigma_i \int_{\omega_+^i} \bar{\chi}_i^+\,dx + \Sigma_i \int_{\omega_-^i} \bar{\chi}_i^-\,dx = Q_1\Sigma_i|\omega_+^i| + Q_2\Sigma_i|\omega_-^i| \\ = Q_1|\Omega_+^\nu| + Q_2|\Omega_-^\nu| = |\Omega|(\mu Q_1 + (1-\mu)Q_2).\end{aligned}$$

4. Proof of Theorem 2

As has already been established, only statement (2.9) needs justification. We divide the proof into a number of steps.
(1) *For almost all* $x \in E_{\widehat{u}_t}$,

$$[A]e(\nabla\widehat{u}_t(x)) = [A\zeta]. \tag{4.1}$$

Let the pair $\widehat{u}_t$, $\widehat{\chi}_t$ minimize functional (1.2) rewritten in the form

$$\begin{aligned}I_0[u,\chi,t] = \int_\Omega F^-(\nabla u)\,dx + \int_{\Omega\setminus E_{\widehat{u}_t}} \chi(F^+(\nabla u) - F^-(\nabla u) + t)\,dx \\ + \int_{E_{\widehat{u}_t}} \chi(F^+(\nabla u) - F^-(\nabla u) + t)\,dx.\end{aligned} \tag{4.2}$$

Then the pair $\widehat{u}_t$, $\widehat{\chi}_t'$,

$$\widehat{\chi}_t'(x) = \widehat{\chi}_t(x) \quad \text{for} \quad x \in \Omega\setminus E_{\widehat{u}_t}, \qquad \widehat{\chi}_t'(x) = \psi(x) \quad \text{for} \quad x \in E_{\widehat{u}_t},$$

with any measurable function ψ characteristic on $E_{\widehat{u}_t}$ also minimizes this functional. Varying functional (4.2) over u at the point $\widehat{u}_t$, $\widehat{\chi}_t'$, we arrive at the conclusion that for all $h \in \mathbb{H}$ (the subscript of $F^\pm$ means the derivative with respect to the matrix argument M),

$$\begin{aligned}\int_\Omega F_M^-(\nabla\widehat{u}_t)\nabla h\,dx + \int_{\Omega\setminus E_{\widehat{u}_t}} \widehat{\chi}_t(F_M^+(\nabla\widehat{u}_t) - F_M^-(\nabla\widehat{u}_t))\nabla h\,dx \\ = -\int_{E_{\widehat{u}_t}} \psi(F_M^+(\nabla\widehat{u}_t) - F_M^-(\nabla\widehat{u}_t))\nabla h\,dx.\end{aligned} \tag{4.3}$$

Taking the function $\psi = 0$ in (4.3), we see that the left-hand side of this relation is zero. Therefore for all ψ,

$$\int_\Omega \chi_{E_{\widehat{u}_t}}\psi(F_M^+(\nabla\widehat{u}_t) - F_M^-(\nabla\widehat{u}_t))\nabla h\,dx = 0, \tag{4.4}$$

where $\chi_{E_{\widehat{u}_t}}$ is the characteristic function of $E_{\widehat{u}_t}$.

Fix $x^0 \in \Omega$ and set

$$h(x) = \phi(x)Bx,$$

$\psi(x)$ to be the characteristic function of $E_{\widehat{u}_t} \cap B_r(x^0)$,

$$\phi \in C_0^\infty(\Omega), \quad \phi(x) \equiv 1 \quad \text{in} \quad B_\rho(x^0), \quad B \in R_s^{m\times m}, \quad r \in (0,\rho).$$

Since the matrix B is arbitrary, formula (4.4) implies that

$$\int\limits_{B_r(x^0)} \chi_{E_{\widehat{u}_t}} (F_M^+(\nabla \widehat{u}_t) - F_M^-(\nabla \widehat{u}_t))\, dx = 0 \quad \text{for all} \quad r \in (0,\rho).$$

Therefore the integrand vanishes at each of its Lebesgue points x^0. Consequently,

$$F_M^+(\nabla \widehat{u}_t(x)) - F_M^-(\nabla \widehat{u}_t(x)) = 0$$

almost everywhere in $E_{\widehat{u}_t}$, which coincides with (4.1).

(2) *Proof of statement* (2.9) *under condition* (2.4). If condition (2.4) is satisfied, then equality (4.1) is satisfied almost everywhere on $E_{\widehat{u}_t}$ if and only if $|E_{\widehat{u}_t}| = 0$.

(3) *Determination of the value t for which equality* (4.1) *is possible in the case of* $[A\zeta] \in \mathrm{Im}[A]$ *and* $|E_{\widehat{u}_t}| > 0$. From the quadraticity of the energy densities $F^\pm(M)$, it follows that

$$\langle A^\pm \zeta^\pm, \zeta^\pm \rangle = F^\pm(0) = F^\pm(M - M) = F^\pm(M) - F_M^\pm(M)M + \frac{1}{2} F_{MM}^\pm(M, M).$$

Then

$$t - t^* = (F^+(M) - F^-(M) + t) - [F_M(M)]M + \frac{1}{2}[F_{MM}](M, M).$$

Set $M = \nabla \widehat{u}_t$. Taking into account definition (2.8) of the set $E_{\widehat{u}_t}$ and equality (4.1), we obtain

$$\begin{aligned} t - t^* &= \frac{1}{2}[F_{MM}](e(\nabla \widehat{u}_t), e(\nabla \widehat{u}_t)) \\ &= \langle [A] e(\nabla \widehat{u}_t), e(\nabla \widehat{u}_t) \rangle = \langle [A\zeta], e(\nabla \widehat{u}_t) \rangle \text{ almost everywhere on } E_{\widehat{u}_t}. \end{aligned} \tag{4.5}$$

In some cases [2], the energy functional (1.2) can be simplified if the equality $\zeta^+ = \zeta^-$ holds true. When implementing this statement, we make use of the scheme proposed in [8].

Under our assumptions, there exists a solution $\xi \in R_s^{m\times m}$ of the linear equation

$$[A]\xi = [A\zeta]. \tag{4.6}$$

The presence of this solution makes it possible to represent the functional (1.2) in the following way (the tensors of residual deformation are temporarily considered as its arguments):

$$\begin{aligned} I_0[u, \chi, t, \zeta^\pm] &= I_0[u, \chi, t', \xi] + |\Omega| (\langle A^- \zeta^-, \zeta^- \langle - \rangle A^- \xi, \xi \rangle), \\ t' &= t + [\langle A\zeta, \zeta \rangle] - \langle [A]\xi, \xi \rangle. \end{aligned} \tag{4.7}$$

Obviously, the set of minimizers $\widehat{u}_t$, $\widehat{\chi}_t$ of the functional $I_0[u, \chi, t', \xi]$ coincides with the set of minimizers $\widehat{u}_{t'}$, $\widehat{\chi}_{t'}$ of the functional $I_0[u, \chi, t', \xi]$, and

$$t^* + [\langle A\zeta, \zeta \rangle] - \langle [A]\xi, \xi \rangle = t'^* = -\langle [A]\xi, \xi \rangle, \quad t - t^* = t' - t'^*, \quad E_{\widehat{u}_t} = E_{\widehat{u}_{t'}}. \tag{4.8}$$

Using (4.1) and (4.6), we obtain

$$\langle [A\zeta], e(\nabla \widehat{u}_t) \rangle = \langle [A]\xi, \xi \rangle \quad \text{almost everywhere on} \quad E_{\widehat{u}_t}. \tag{4.9}$$

Integrating both sides of equalities (4.5) and (4.9) over the set $E_{\widehat{u}_t}$, taking into account the positivity of its measure and the second relation of (4.8), we come to the conclusion that $t' - t'^* = \langle [A]\xi, \xi \rangle$. Then by the second equality, from the first relation of (4.8), we have $t' = 0$.

Thus,

$$\text{for } [A\zeta] \in \operatorname{Im}[A] \text{ the inequality } |E_{\widehat{u}_{t'}}| > 0 \text{ can be true only for } t' = 0. \tag{4.10}$$

(4) *Calculation of the minimizers of the functional* $I_0[u,\chi,0,\xi]$. We write $I_0[u,\chi,0,\xi]$ in two different ways

$$\begin{aligned} I_0[u,\chi,0,\xi] &= \int_\Omega \Big\{F^-(\nabla u) + \chi(F^+(\nabla u) - F^-(\nabla u)\Big\}\,dx \\ &= \int_\Omega \Big\{F^+(\nabla u) - (1-\chi)(F^+(\nabla u) - F^-(\nabla u)\Big\}\,dx. \end{aligned}$$

Then

$$\begin{aligned} &I_0[u,\chi,0,\xi] - |\Omega|\langle A^-\xi,\xi\rangle \\ &= \int_\Omega \langle A^- e(\nabla u), e(\nabla u)\rangle + \int_\Omega \chi\{\langle [A]e(\nabla u), e(\nabla u)\rangle - 2\langle [A]\xi, e(\nabla u)\rangle + \langle [A]\xi,\xi\rangle\}\,dx, \\ &I_0[u,\chi,0,\xi] - |\Omega|\langle A^+\xi,\xi\rangle \\ &= \int_\Omega \langle A^+ e(\nabla u), e(\nabla u)\rangle - \int_\Omega (1-\chi)\{\langle [A]e(\nabla u), e(\nabla u)\rangle - 2\langle [A]\xi, e(\nabla u)\rangle + \langle [A]\xi,\xi\rangle\}\,dx. \end{aligned}$$

Therefore,

$$\begin{aligned} &I_0[u,\chi,0,\xi] - |\Omega|\langle A^-\xi,\xi\rangle \\ &= \int_\Omega \{\langle A^- e(\nabla u), e(\nabla u)\rangle + \chi|[A]^{1/2}(e(\nabla u) - \xi)|^2\}\,dx \text{ for } [A] \geq 0, \\ &I_0[u,\chi,0,\xi] - |\Omega|\langle A^+\xi,\xi\rangle \\ &= \int_\Omega \{\langle A^+ e(\nabla u), e(\nabla u)\rangle + (1-\chi)|[-A]^{1/2}(e(\nabla u) - \xi)|^2\}\,dx \text{ for } [A] \leq 0. \end{aligned}$$

Consequently, the minimizers of the functional $I_0[u,\chi,0,\xi]$, i.e., the functions $\widehat{u}_0$, $\widehat{\chi}_0$, have the form

$$\begin{aligned} &\widehat{u}_0 \equiv 0,\ \widehat{\chi}_0 \equiv 0 \quad \text{for} \quad [A] \geq 0 \quad \text{and} \quad [A]^{1/2}\xi \neq 0, \\ &\widehat{u}_0 \equiv 0,\ \widehat{\chi}_0 \equiv 1 \quad \text{for} \quad [A] \leq 0 \quad \text{and} \quad [-A]^{1/2}\xi \neq 0. \end{aligned} \tag{4.11}$$

(5) *Proof of statement* (2.9) *under condition* (2.5). Since $t_- < t_+$, from (1.9) it follows that the matrix $[A]\xi$ is nonzero on the solution of problem (4.6). By symmetry and assumptions on the sign of the mappings $[A]$, the quantity $\langle [A]\xi,\xi\rangle$ is nonzero, which proves the inequalities in (4.11). Then the functional $I_0[u,\chi,0,\xi]$ has a unique (one for each of the signs of mapping $[A]$) minimizer (4.11). Consequently, the function $R(x,0) = \langle [A]\xi,\xi\rangle$, defined in (2.7), is nonzero, which makes the realization of (4.10) impossible.

5. Proof of Theorem 3

(a) For fixed u and χ, the function $I_0[u,\chi,t]$ is linear in $t \in R$. Consequently, function (2.10), as the infimum of the family of concave functions, is concave. From the concavity and (2.11), it follows that it is uniform Lipschitz. Therefore, $i(.) \in W^1_{\infty,\mathrm{loc}}(R)$ and is locally absolutely continuous. It has the Sobolev derivative $Di(t)$, and for almost all $t \in R$, the classical derivative $i'(t)$, and also $i'(t) = Di(t)$ almost everywhere on R.

Further arguments are traditional and are based only on the properties of concave functions. For the sake of completeness, we discuss them briefly.

We fix a representative of the function $Di(t)$ with a uniformly bounded module. Denote by $\mathcal{L}'$ the set of all Lebesgue points of this representative, i.e., the set of points $t \in R$ for which

$$\frac{1}{2h} \int\limits_{t-h}^{t+h} |Di(\xi) - Di(t)| \, d\xi \to 0 \quad \text{as} \quad h \to 0.$$

Averaging preserves the concavity property. Therefore, $i_\rho(t)$ is a smooth concave function. Hence, $(i_\rho)'(t_2) \leq (i_\rho)'(t_1)$ as $t_1 < t_2$. Since $D(i_\rho) = (Di)_\rho$ and $(Di)_\rho(t) \to Di(t)$ as $\rho \to 0$ and $t \in \mathcal{L}'$, we arrive at the monotonicity of the Sobolev derivative,

$$Di(t_2) \leq Di(t_1), \quad \text{for} \quad t_1 < t_2, \quad t_1, t_2 \in \mathcal{L}'. \tag{5.1}$$

In view of the absolute continuity of the function $i(t)$,

$$\frac{i(t+h) - i(t)}{h} = \frac{1}{h} \int\limits_{t}^{t+h} Di(\xi) \, d\xi.$$

Then as $h \to 0$, for any $t \in \mathcal{L}'$ we have

$$\left| \frac{i(t+h) - i(t)}{h} - Di(t) \right| \leq 2 \frac{1}{2|h|} \int\limits_{|t-\xi|<|h|} |Di(\xi) - Di(t)| \, dx \to 0.$$

Consequently, at each point $t \in \mathcal{L}'$ there exists a finite classical derivative and the equality $i'(t) = Di(t)$ is fulfilled.

Let $t \in R$ and $\tau \in \mathcal{L}'$. From (5.1), it follows that the limits below exist and are finite:

$$\lim_{\tau \to t, \tau < t} Di(\tau) = \alpha_-, \qquad \lim_{\tau \to t, \tau > t} Di(\tau) = \alpha_+, \quad \alpha_- \geq \alpha_+. \tag{5.2}$$

Since for $\zeta \in R$, $\zeta \neq t$,

$$\begin{aligned} \frac{i(t) - i(\zeta)}{t - \zeta} - \alpha_- &= \frac{1}{t - \zeta} \int\limits_{\zeta}^{t} (Di(\xi) - \alpha_-) \, d\xi \quad \text{for} \quad \zeta < t, \\ \alpha_+ - \frac{i(t) - i(\zeta)}{t - \zeta} &= \frac{1}{\zeta - t} \int\limits_{t}^{\zeta} (\alpha_+ - Di(\xi)) \, d\xi \quad \text{for} \quad \zeta > t, \end{aligned} \tag{5.3}$$

relations (5.2) imply the existence of the limits

$$\begin{aligned} i'(t-0) &= \lim_{\zeta \to t, \zeta < t} \frac{i(t) - i(\zeta)}{t - \zeta} = \alpha_-, \\ i'(t+0) &= \lim_{\zeta \to t, \zeta > t} \frac{i(t) - i(\zeta)}{t - \zeta} = \alpha_+. \end{aligned} \tag{5.4}$$

If $\alpha_- = \alpha_+ = \alpha$, then at the point t there exists a finite classical derivative $i'(t) = \alpha$. We redefine the function $Di(t)$ at these points by setting $Di(t) = i'(t)$. Obviously, the set $\mathcal{L}'$ contains only Lebesgue points of the redefined function. By virtue of the sign-definiteness almost everywhere on the integration intervals of the integrands in (5.3), the points t, for which $\alpha_\pm = \alpha$, are also Lebesgue points of the redefined function. Denote by $\mathcal{L}$ the union of $\mathcal{L}'$ with these points. From (5.3), it follows that the points of $R \setminus \mathcal{L}$ are not Lebesgue points for it.

For $\alpha_+ = \alpha_-$, relations (5.2) mean the continuity of the function $i'(t)$ on the set $\mathcal{L}$, and for $\alpha_+ < \alpha_-$, they express the validity of (2.13).

Thus, the set $\mathcal{L}$ from Theorem 3(a) is the set of all Lebesgue points of a special representative of the function $Di(t)$.

(b) For an arbitrary $t \in R$ and any solution $\widehat{u}_{t_0}$, $\widehat{\chi}_{t_0}$ of problem (1.4), for the functional $I[u, \chi, t_0]$ we have

$$i(t) \leq I_0[\widehat{u}_{t_0}, \widehat{\chi}_{t_0}, t] = I_0[\widehat{u}_{t_0}, \widehat{\chi}_{t_0}, t_0] + (t - t_0)|\Omega|\widehat{Q}(t_0) = i(t_0) + (t - t_0)|\Omega|\widehat{Q}(t_0).$$

Consequently,

$$\frac{t(t) - i(t_0)}{t - t_0} \leq |\Omega|\widehat{Q}(t_0) \quad \text{for} \quad t > t_0,$$
$$\frac{i(t) - i(t_0)}{t - t_0} \geq |\Omega|\widehat{Q}(t_0)| \quad \text{for} \quad t < t_0.$$

For $t_0 \in \mathcal{L}$, the left-hand sides of the last inequalities have the same limit $i'(t_0)$. For $t_0 \in R \setminus \mathcal{L}$, these limits coincide with $i'(t_0 \pm 0)$, respectively.

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