DOI 10.1007/s10958-018-4108-2
Journal of Mathematical Sciences, Vol. 236, No. 3, January, 2019

ON THE ASYMPTOTIC PROPERTIES OF SOLUTIONS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH LINEARLY TRANSFORMED ARGUMENT

D. V. Bel'skii and G. P. Pelyukh UDC 517.929

We establish new properties the of solutions of functional-differential equation with linearly transformed argument

In the present paper, we consider an equation

$$x'(t) = ax(t) + bx(qt) + cx'(qt), \tag{1}$$

where $\{a, b, c\} \subset R$ and $0 < q < 1$. Special cases of this equation were studied by numerous mathematicians. Thus, the asymptotic properties of solutions of the equation $y'(x) = ay(\lambda x) + by(x)$ were investigated in [1], new properties of solutions of the equation $y'(x) = ay(\lambda x)$ were obtained in [2], the conditions for the existence of analytic almost periodic solutions of the equation $y'(x) = ay(\lambda x) + by(x)$ were established in [3], a representation of the general solution of Eq. (1) for $|c| > 1$ was constructed in [4], a series of new results on the existence of bounded and finite solutions of equations with linearly transformed argument was obtained in [5], the behavior of solutions of Eq. (1) in a neighborhood of the point $t = 0$ was studied in [6], the existence of solutions of the equation $x'(t) = F(x(2t))$ with periodic modulus was proved in [7], and Eq. (1) was investigated for $a = 0$ in [11] and for $a < 0$ in [12]. Nevertheless, despite these results and extensive applications of the analyzed equations in various fields of science and engineering (see [8] and the references therein), numerous problems of the theory of the functional-differential equation (1) are studied quite poorly. First of all, this is true for the asymptotic properties of solutions of this equation as $t \to +\infty$.

In what follows, we need the following particular solutions:

Example 1. If $\left|\dfrac{b}{a}\right| < 1$, then one of the solutions of Eq. (1) has the form

$$x(t) = \sum_{n=0}^{+\infty} x_n e^{aq^n t},$$

where $x_0 = 1$ and

$$x_n = \frac{b + acq^{n-1}}{a\,(q^n - 1)}\, x_{n-1}, \quad n \geq 1,$$

or, in the expanded form,

$$x(t) = e^{at}\left\{1 + \sum_{n=1}^{+\infty} (-1)^n \frac{(b + ac)(b + acq)\ldots\left(b + acq^{n-1}\right)}{a^n(1 - q)\left(1 - q^2\right)\ldots(1 - q^n)} e^{-a(1-q^n)t}\right\}.$$

Institute of Mathematics, Ukrainian National Academy of Sciences, Tereshchenkivs'ka Str., 3, Kyiv, 01004, Ukraine.

Translated from Neliniini Kolyvannya, Vol. 20, No. 3, pp. 291–302, July–September, 2017. Original article submitted August 29, 2015; revision submitted March 26, 2017.

Example 2. One more particular solution of Eq. (1) convergent for $t > 0$ is given by a series

$$x(t) = \sum_{n=0}^{+\infty} x_n t^{v_2+n},$$

where the quantity v_2 is determined from the equality $q^{v_2} = \frac{q}{c}$ and satisfies the condition $v_2 \neq -n \ \forall n \in \mathrm{N}$, $x_0 = 1$, and

$$x_{n+1} = \frac{a + bq^{v_2+n}}{\left(1 - cq^{v_2+n}\right)(v_2 + n + 1)} x_n, \quad n \geq 0.$$

In the expanded form, we can write

$$x(t) = t^{v_2} \left\{ 1 + \sum_{n=1}^{+\infty} \frac{\left(a + \frac{b}{c} q\right)\left(a + \frac{b}{c} q^2\right) \ldots \left(a + \frac{b}{c} q^n\right)}{(1-q)\left(1-q^2\right) \ldots (1-q^n)(v_2+1)(v_2+2) \ldots (v_2+n)} t^n \right\}.$$

By using methods proposed in [1], we prove the following theorem:

Theorem. *Suppose that the following conditions are satisfied:*

(i) $a > 0$, $bc \neq 0$;

(ii) $a + bq^n \neq 0 \ \forall n \in \mathrm{N} \bigcup \{0\}$ *or* $c > 0$, $1 + \frac{\ln c}{\ln q^{-1}} \neq l \ \forall l \in \mathrm{Z}$;

(iii) the quantity $v_1 \in \mathrm{C}$ *is determined from the equality* $a + bq^{v_1} = 0$;

(iv) for the parameters $\{j, m\} \subset \mathrm{N} \bigcup \{0\}$, *the inequalities*

$$v_0 \stackrel{\mathrm{df}}{=} \frac{\ln\left(\frac{|b|}{a}\right)}{\ln q^{-1}} = \operatorname{Re} v_1 \geq v_{\min} \stackrel{\mathrm{df}}{=} \frac{\ln\left(\left|cq^j\right| q^{-1} + \frac{\left|bq^j + acq^j q^{-1}\right|}{a}\right)}{\ln q^{-1}},$$

$$q^{-\operatorname{Re} v_1 + m}\left(\left|\frac{q}{c}\right| + \left|\frac{a}{b} + \frac{q}{c}\right|\right) < 1 \quad \text{and} \quad \left(\left|c^{-1}\right| + 2\left|ac^{-1} + qbc^{-2}\right|\right) q^{-\operatorname{Re} v_1 + m} < 1,$$

are true.

Then any continuously differentiable solution of Eq. (1) possesses the property $x(t)e^{-at} \to L$ *as* $t \to \infty$, *where* L *is a constant and, for any number* L, *there exists a solution with the indicated property and, in addition, for* $bc < 0$, *the following assertions are true:*

(i) for any $m + 1$ *times continuously differentiable periodic function* $f_0(u)$ *with period* 1, *there exists a continuously differentiable solution of Eq. (1)*

$$x_f(t) = t^{v_1} f_0\left(\frac{\ln t}{\ln q^{-1}}\right) + t^{v_1-1} f_1\left(\frac{\ln t}{\ln q^{-1}}\right) + \ldots + t^{v_1-m} f_m\left(\frac{\ln t}{\ln q^{-1}}\right) + \sum_{n=1}^{+\infty} z_n(t), \quad t > 0,$$

where $f_p(u)$, $1 \le p \le m$, *are periodic functions with period* 1 *given by the recurrence formula*

$$f_{p+1}(u) = \frac{\left(bq^{p+1} + ac\right)}{ba\left(q^{p+1} - 1\right)} \left((v_1 - p) f_p(u) + \frac{1}{\ln q^{-1}} f_p'(u)\right), \quad 0 \le p \le m-1,$$

$$z_1(t) = \left(bc^{-2}q^{-v_1+m+1} - bc^{-1}\right) \times e^{-bc^{-1}t} \int\limits_t^{+\infty} \left[u^{v_1-m} f_m\left(\frac{\ln u}{\ln q^{-1}}\right) - t^{v_1-m} f_m\left(\frac{\ln t}{\ln q^{-1}}\right)\right] e^{bc^{-1}u}\, du,$$

$$z_{n+1}(t) = c^{-1}qz_n\left(q^{-1}t\right) + \left(ac^{-1} + qbc^{-2}\right) e^{-bc^{-1}t} \int\limits_t^{+\infty} z_n\left(q^{-1}u\right) e^{bc^{-1}u}\, du, \quad n = 1, 2, 3, \ldots,$$

the functional series $\sum_{n=1}^{+\infty} z_n(t)$ *is continuously differentiable and has the asymptotic property*

$$\sum_{n=1}^{+\infty} z_n(t) = O\left(t^{v_1-m-1}\right)$$

as $t \to +\infty$;

(ii) *every* $m + j + 4$ *times continuously differentiable solution* $x(t)$ *of Eq. (1) is identically equal to the sum* $x(t) = Lx_1(t) + x_f(t)$, *where* L *is a constant,* $x_1(t)$ *is a solution of Eq. (1) with the property* $x_1(t)e^{-at} \to 1$ *as* $t \to \infty$, *and* $x_f(t)$ *is the solution from the previous item constructed on the basis of a certain* $m + 1$ *times continuously differentiable periodic function* $f_0(u)$ *with period* 1;

for $bc > 0$, *the following assertions are true:*

(i) *for any* $m + 1$ *times continuously differentiable periodic function* $f_0(u)$ *with period* 1, *there exists a continuously differentiable solution of Eq. (1)*

$$x_f(t) = t^{v_1} f_0\left(\frac{\ln t}{\ln q^{-1}}\right) + t^{v_1-1} f_1\left(\frac{\ln t}{\ln q^{-1}}\right) + \ldots + t^{v_1-m} f_m\left(\frac{\ln t}{\ln q^{-1}}\right) + \sum_{n=1}^{+\infty} z_n(t) + \gamma\, x_*(t), \qquad t \ge \rho > 0,$$

where ρ *is a sufficiently large constant independent of the function* $f_0(u)$, $f_p(u)$, $1 \le p \le m$, *is a periodic function with period 1 given by the recurrence formula*

$$f_{p+1}(u) = \frac{\left(bq^{p+1} + ac\right)}{ba\left(q^{p+1} - 1\right)} \left((v_1 - p) f_p(u) + \frac{1}{\ln q^{-1}} f_p'(u)\right), \quad 0 \le p \le m-1,$$

$$z_1(t) = \left(c^{-1}q^{-v_1+m+1} - 1\right)\left[e^{-bc^{-1}(t-\rho)}t^{v_1-m}f_m\left(\frac{\ln t}{\ln q^{-1}}\right)\right.$$

$$\left.-bc^{-1}\int\limits_{\rho}^{t} e^{-bc^{-1}(t-u)}\left\{u^{v_1-m}f_m\left(\frac{\ln u}{\ln q^{-1}}\right) - t^{v_1-m}f_m\left(\frac{\ln t}{\ln q^{-1}}\right)\right\}du\right],$$

$$z_{n+1}(t) = c^{-1}qz_n\left(q^{-1}t\right)$$

$$-\left(qbc^{-2} + ac^{-1}\right)\int\limits_{\rho}^{t} e^{-bc^{-1}(t-u)}z_n\left(q^{-1}u\right)du, \quad n = 1, 2, 3, \ldots,$$

the functional series $\sum_{n=1}^{+\infty} z_n(t)$ *is continuously differentiable and has the asymptotic property*

$$\sum_{n=1}^{+\infty} z_n(t) = O\left(t^{v_1-m-1}\right), \quad t \to +\infty,$$

and the function $x_*(t)$ *is a particular solution of Eq. (1) given by the formula*

$$x_*(t) = \sum_{n=0}^{+\infty} x_n e^{-\frac{b}{c}q^{-n}t},$$

where

$$x_n = \frac{ac + bq^{-n+1}}{bc\left(q^{-n} - 1\right)}x_{n-1}, \qquad n \geq 1, \qquad x_0 = 1,$$

and γ *is an arbitrary constant;*

(ii) *every* $m + j + 4$ *times continuously differentiable solution* $x(t)$ *of Eq. (1) is identically equal to the sum* $x(t) = Lx_1(t) + x_f(t)$, *where* L *is a constant and* $x_f(t)$ *is the solution from the previous item constructed on the basis of a certain* $m + 1$ *times continuously differentiable periodic function* $f_0(u)$ *with period* 1 *and a certain constant* γ.

Proof. We rewrite Eq. (1) in the form

$$\frac{d}{dt}\left\{e^{-at}x(t)\right\} = be^{-at}x(qt) + ce^{-at}x'(qt)$$

and integrate it:

$$e^{-at}x(t) = e^{-aq^{-n}}x\left(q^{-n}\right) + cq^{-1}\left\{e^{-a(1-q)t}e^{-aqt}x(qt) - e^{-aq^{-n}(1-q)}e^{-aq^{-(n-1)}}x\left(q^{-(n-1)}\right)\right\}$$

$$+\left(b + acq^{-1}\right)\int\limits_{q^{-n}}^{t} e^{-a(1-q)s}e^{-aqs}x(qs)\,ds.$$

We define

$$\sup_{t\in[q^{-n+1},q^{-n}]}\left|e^{-at}x(t)\right| \stackrel{\text{df}}{=} M_n.$$

Let $q^{-n} \le t \le q^{-n-1}$. Thus, we get

$$\begin{aligned}\left|e^{-at}x(t)\right| &\le \left|e^{-aq^{-n}}x\left(q^{-n}\right)\right| + \left|\frac{c}{q}\right| \\ &\quad\times\left\{e^{-a(1-q)t}\left|e^{-aqt}x(qt)\right| + e^{-aq^{-n}(1-q)}\left|e^{-aq^{-(n-1)}}x\left(q^{-(n-1)}\right)\right|\right\} \\ &\quad+\left|b+acq^{-1}\right|\int\limits_{q^{-n}}^{t} e^{-a(1-q)s}\left|e^{-aqs}x(qs)\right|\,ds \\ &\le M_n + 2\left|\frac{c}{q}\right|e^{-a(1-q)q^{-n}}M_n + \left|b+acq^{-1}\right|M_n\frac{e^{-a(1-q)q^{-n}}}{a(1-q)} \\ &= M_n\left\{1+\left(2\left|\frac{c}{q}\right| + \frac{\left|b+acq^{-1}\right|}{a(1-q)}\right)e^{-a(1-q)q^{-n}}\right\}.\end{aligned}$$

This yields the inequality

$$M_{n+1} \le M_n\left\{1+\left(2\left|\frac{c}{q}\right| + \frac{\left|b+acq^{-1}\right|}{a(1-q)}\right)e^{-a(1-q)q^{-n}}\right\}$$

and the estimate $x(t) = O\left(e^{at}\right)$ as $t\to\infty$. By using the identity

$$\begin{aligned}e^{-at_2}x(t_2) - e^{-at_1}x(t_1) &= cq^{-1}\left\{e^{-a(1-q)t_2}e^{-aqt_2}x(qt_2) - e^{-a(1-q)t_1}e^{-aqt_1}x(qt_1)\right\} \\ &\quad+\left(b+acq^{-1}\right)\int\limits_{t_1}^{t_2} e^{-a(1-q)s}e^{-aqs}x(qs)\,ds,\end{aligned}$$

for some constant M such that

$$\left|e^{-at}x(t)\right| \le M, \qquad t \ge q^{-n+1},$$

we arrive at the inequality

$$\left|e^{-at_2}x(t_2) - e^{-at_1}x(t_1)\right| \le \Bigg(|c|q^{-1}\left\{e^{-a(1-q)t_2} + e^{-a(1-q)t_1}\right\}$$

$$+ \left|b + acq^{-1}\right| \frac{e^{-a(1-q)t_1} - e^{-a(1-q)t_2}}{a(1-q)}\Bigg) M.$$

By using the Cauchy principle, we conclude that the limit $\lim_{t\to\infty} e^{-at}x(t) \in \mathrm{C}$ exists.

The particular solution of the first example exists for

$$\left|\frac{b}{a}\right| < 1.$$

We differentiate Eq. (1) p times to guarantee that the inequality $|bq^p| < a$ is true. As a result we obtain

$$x^{(p+1)}(t) = ax^{(p)}(t) + bq^p x^{(p)}(qt) + cq^p x^{(p+1)}(qt).$$

By $y_p(t)$ we denote the solution of the equation

$$y_p'(t) = ay_p(t) + bq^p y_p(qt) + cq^p y_p'(qt) \tag{2}$$

with the property $y_p(t)e^{-at} \to a^p$, $t \to \infty$. This is a solution of Eq. (2) for from first example multiplied by a^p. We define a function

$$y_{p-1}(t) = \int_1^t y_p(u)\,du + h_p$$

and integrate Eq. (2) over the segment $[1, t]$:

$$y_{p-1}'(t) = ay_{p-1}(t) + bq^{p-1} y_{p-1}(qt) + cq^{p-1} y_{p-1}'(qt) - h_p\left(a + bq^{p-1}\right)$$

$$+ bq^{p-1} \int_q^1 y_p(u)\,du - cq^{p-1} y_p(q) + y_p(1).$$

If $a + bq^{p-1} \neq 0$, then, selecting the corresponding h_p, we obtain

$$y_{p-1}'(t) = ay_{p-1}(t) + bq^{p-1} y_{p-1}(qt) + cq^{p-1} y_{p-1}'(qt).$$

It is easy to see that $y_{p-1}(t)e^{-at} \to a^{p-1}$ as $t \to \infty$. Repeating these arguments several times, we get

$$x(t)e^{-at} = y_0(t)e^{-at} \to 1, \quad t \to \infty.$$

Assume that $a + bq^n = 0$, $n \in \mathrm{N}\bigcup\{0\}$, but $c > 0$ and

$$1 + \frac{\ln c}{\ln q^{-1}} \neq l \quad \forall l \in \mathrm{Z}.$$

Thus, as a result of replacement of the coefficients b and c by the quantities bq^n and cq^n, the solution from the second example becomes the unbounded infinitely differentiable solution $x_2(t)$ of the equation

$$x^{(n+1)}(t) = ax^{(n)}(t) + bq^n x^{(n)}(qt) + cq^n x^{(n+1)}(qt).$$

In what follows, we show that the assumption $x^{(n)}(t) = o\left(e^{at}\right)$ as $t \to \infty$ for a sufficiently smooth solution implies the estimate $x^{(n)}(t) = O(1)$ as $t \to \infty$. Hence, $x_2(t)e^{-at} \to h \neq 0$ as $t \to \infty$. Multiplying the last expression by the corresponding quantity, we arrive at a solution with the property $y_n(t)e^{-at} \to a^n$ as $t \to \infty$. The subsequent reasoning is similar to the previous arguments. The first part of the theorem is proved.

Assume that $x(t) = o\left(e^{at}\right)$ as $t \to \infty$. In the identity

$$e^{-at_1}x(t_1) - e^{-at}x(t) = cq^{-1}\left\{e^{-a(1-q)t_1}e^{-aqt_1}x(qt_1) - e^{-a(1-q)t}e^{-aqt}x(qt)\right\}$$

$$+\left(b + acq^{-1}\right)\int_t^{t_1} e^{-a(1-q)s}e^{-aqs}x(qs)\,ds$$

we pass to the limit as the argument t_1 tends to ∞. This yields

$$x(t) = cq^{-1}x(qt) - \left(b + acq^{-1}\right)e^{at}\int_t^{+\infty} e^{-a(1-q)s}e^{-aqs}x(qs)\,ds.$$

If

$$|x(t)| \le Me^{at}, \quad t \ge U,$$

where M and U are constants, then, for $t \ge q^{-1}U$, the inequality

$$|x(t)| \le |c|q^{-1}|x(qt)| + \left|b + acq^{-1}\right|e^{at}\int_t^{+\infty} e^{-a(1-q)s}e^{-aqs}|x(qs)|ds$$

$$\le |c|q^{-1}Me^{aqt} + \left|b + acq^{-1}\right|e^{at}\int_t^{+\infty} e^{-a(1-q)s}M\,ds$$

$$= M\left\{|c|q^{-1} + \frac{\left|b + acq^{-1}\right|}{a(1-q)}\right\}e^{aqt}$$

is true. Repeating the process, for $t \ge q^{-n}U$, we obtain

$$|x(t)| \le M\left\{|c|q^{-1} + \frac{\left|b + acq^{-1}\right|}{a(1-q)}\right\}\left\{|c|q^{-1} + \frac{\left|b + acq^{-1}\right|}{a\left(1-q^2\right)}\right\}\cdots\left\{|c|q^{-1} + \frac{\left|b + acq^{-1}\right|}{a\left(1-q^n\right)}\right\}e^{aq^nt}.$$

Thus, in the intermediate segment $q^{-n}U \le t \le q^{-n-1}U$, we get

$$|x(t)| \le M\left\{|c|q^{-1} + \frac{\left|b + acq^{-1}\right|}{a(1-q)}\right\}\left\{|c|q^{-1} + \frac{\left|b + acq^{-1}\right|}{a\left(1-q^2\right)}\right\}$$

$$\dots\left\{|c|q^{-1} + \frac{\left|b + acq^{-1}\right|}{a\left(1-q^n\right)}\right\} e^{aq^{-1}U}$$

$$\le Me^{aq^{-1}U}\left(|c|q^{-1} + \frac{\left|b + acq^{-1}\right|}{a}\right)^n \prod_{k=1}^{n}\left(1 + Lq^k\right)$$

$$\le Me^{aq^{-1}U}\prod_{k=1}^{+\infty}\left(1 + Lq^k\right)\left(|c|q^{-1} + \frac{\left|b + acq^{-1}\right|}{a}\right)^n,$$

where L is a constant. It follows from the condition $q^{-n}U \le t \le q^{-n-1}U$ that

$$\frac{\ln t}{\ln q^{-1}} - 1 - \frac{\ln U}{\ln q^{-1}} \le n \le \frac{\ln t}{\ln q^{-1}} - \frac{\ln U}{\ln q^{-1}}.$$

Then the estimate for $|x(t)|$ can be continued as follows:

$$|x(t)| \le L_1\left(|c|q^{-1} + \frac{\left|b + acq^{-1}\right|}{a}\right)^{\frac{\ln t}{\ln q^{-1}}} = L_1 t^{\frac{\ln\left(|c|q^{-1} + \frac{\left|b+acq^{-1}\right|}{a}\right)}{\ln q^{-1}}}$$

for a constant L_1. The function on the right-hand side of the last inequality is independent of n.

For the sake of brevity, we define

$$\frac{\ln\left(|c|q^{-1} + \frac{\left|b + acq^{-1}\right|}{a}\right)}{\ln q^{-1}} \overset{\text{df}}{=} v,$$

and establish an approximate (qualitative) estimate of the derivative $x'(t)$. We rewrite Eq. (1) in the following form:

$$x'(t) = cx'(qt) + ax(t) + bx(qt) \overset{\text{df}}{=} cx'(qt) + f(t).$$

For the inhomogeneity, the following equality is true: $f(t) = O\left(t^v\right)$ as $t \to \infty$. We perform the change of variables $x'(t) = t^{v_3}y(t)$, $v_3 > v$:

$$y(t) = cq^{v_3}y(qt) + t^{-v_3}f(t).$$

We estimate the coefficient $c_1 \overset{\text{df}}{=} cq^{v_3}$ and the inhomogeneity $t^{-v_3}f(t) \overset{\text{df}}{=} g(t)$ as follows:

$$|c_1| = |c|q^{v_3} < |c|q^v < q < 1, \quad |g(t)| < M < +\infty$$

for a constant M. Then

$$y(t) = c_1 y(qt) + g(t)$$

and, for $q^{-n-1} \le t \le T$, we obtain

$$\begin{aligned}
|y(t)| &\le |c_1|\,|y(qt)| + M \le |c_1| \sup_{q^{-n-1}\le t\le T} |y(qt)| + M = |c_1| \sup_{q^{-n}\le t\le qT} |y(t)| + M \\
&\le |c_1| \sup_{q^{-n}\le t\le T} |y(t)| + M = |c_1| \max\left\{ \sup_{q^{-n}\le t\le q^{-n-1}} |y(t)|, \sup_{q^{-n-1}\le t\le T} |y(t)| \right\} + M \\
&\le |c_1| \sup_{q^{-n}\le t\le q^{-n-1}} |y(t)| + |c_1| \sup_{q^{-n-1}\le t\le T} |y(t)| + M,
\end{aligned}$$

whence it follows that

$$\sup_{q^{-n-1}\le t\le T} |y(t)| \le |c_1| \sup_{q^{-n}\le t\le q^{-n-1}} |y(t)| + |c_1| \sup_{q^{-n-1}\le t\le T} |y(t)| + M,$$

$$\sup_{q^{-n-1}\le t\le T} |y(t)| \le (1-|c_1|)^{-1} \left(|c_1| \sup_{q^{-n}\le t\le q^{-n-1}} |y(t)| + M \right).$$

Since T is an arbitrary number, we get

$$|y(t)| \le (1-|c_1|)^{-1} \left(|c_1| \sup_{q^{-n}\le t\le q^{-n-1}} |y(t)| + M \right) \quad \forall t \ge q^{-n-1},$$

i.e.,

$$x'(t) = t^{v_3} y(t) = O\left(t^{v_3}\right), \quad t \to \infty$$

Differentiating Eq. (1) and successively using the above-mentioned reasoning, we conclude that

$$x^{(m)}(t) = O\left(t^{v_{m+2}}\right), \quad t \to \infty,$$

where $v < v_3 < \ldots < v_{m+1} < v_{m+2}$, i.e., all derivatives are $o\left(e^{at}\right)$ as $t \to \infty$.

Differentiating Eq. (1) j times, we get

$$x^{(j+1)}(t) = ax^{(j)}(t) + bq^j x^{(j)}(qt) + cq^j x^{(j+1)}(qt).$$

As in the case of the function $x(t)$, we arrive at the estimate $x^{(j)}(t) = O\left(t^{v_{\min}}\right)$ as $t \to \infty$, from the condition $x^{(j)}(t) = o\left(e^{at}\right)$, $t \to \infty$. In the equation

$$x^{(j)}(t) = ax^{(j-1)}(t) + bq^{j-1} x^{(j-1)}(qt) + cq^{j-1} x^{(j)}(qt),$$

$$x^{(j-1)}(t) = -\frac{b}{a} q^{j-1} x^{(j-1)}(qt) - \frac{c}{a} q^{j-1} x^{(j)}(qt) + \frac{1}{a} x^{(j)}(t) \stackrel{\text{df}}{=} -\frac{b}{a} q^{j-1} x^{(j-1)}(qt) + f(t),$$

we perform the change of variables $x^{(j-1)}(t) = t^{v_*} y(t)$, where $v_* \geq v_{\min}$ and

$$v_* > \frac{\ln\left(\dfrac{|bq^{j-1}|}{a}\right)}{\ln q^{-1}} = v_0 - (j-1).$$

This yields

$$y(t) = -\frac{b}{a} q^{j-1} q^{v_*} y(qt) + t^{-v_*} f(t).$$

We redefine the auxiliary coefficient

$$c_1 \stackrel{\text{df}}{=} -\frac{b}{a} q^{j-1} q^{v_*}$$

and the inhomogeneity $g(t) \stackrel{\text{df}}{=} t^{-v_*} f(t)$ and estimate them in view of the choice of v_*:

$$|g(t)| = O\left(t^{v_{\min} - v_*}\right) < M < +\infty, \quad t \to \infty,$$

for a constant M,

$$|c_1| = \exp\left\{\left(\frac{\ln\left|\dfrac{bq^{j-1}}{a}\right|}{\ln q^{-1}} - v_*\right)\ln q^{-1}\right\} < 1.$$

By applying the previous reasoning to the equation $y(t) = c_1 y(qt) + g(t)$, we establish the boundedness of $|y(t)|$ and the property

$$x^{(j-1)}(t) = O\left(t^{v_*}\right) = O\left(t^{\max\{v_{\min}, v_0 - (j-1) + \varepsilon\}}\right) \quad \text{as} \quad t \to \infty,$$

where $\varepsilon > 0$ is an arbitrary number. Repeating this process, we find

$$x^{(j-2)}(t) = O\left(t^{\max\{v_0 - (j-2) + \varepsilon; \max\{v_{\min}, v_0 - (j-1) + \varepsilon\}\}}\right) = O\left(t^{\max\{v_0 - (j-2) + \varepsilon, v_{\min}\}}\right), \quad t \to .$$

Further, after several steps (by the condition of the theorem, $v_0 \geq v_{\min}$), we get

$$x(t) = O\left(t^{\max\{v_0 + \varepsilon, v_{\min}\}}\right) = O\left(t^{v_0 + \varepsilon}\right), \quad t \to \infty.$$

Similarly, the derivative admits the following estimate: $x'(t) = O\left(t^{v_0 - 1 + \varepsilon}\right)$, $t \to \infty$.

We rewrite Eq. (1) in the form

$$x(t) = -\frac{b}{a} x(qt) + \frac{1}{a} x'(t) - \frac{c}{a} x'(qt) \stackrel{\text{df}}{=} -\frac{b}{a} x(qt) + f(t)$$

and perform the change of variables $x(t) = t^{v_1-\varepsilon} y(t)$:

$$y(t) = -\frac{b}{a} q^{v_1-\varepsilon} y(qt) + t^{-(v_1-\varepsilon)} f(t) = q^{-\varepsilon} y(qt) + t^{-(v_1-\varepsilon)} f(t).$$

We now estimate the inhomogeneity $g(t) \overset{\text{df}}{=} t^{-(v_1-\varepsilon)} f(t)$:

$$|g(t)| = t^{-(v_1-\varepsilon)} O\left(t^{v_1-1+\varepsilon}\right) = O\left(t^{-1+2\varepsilon}\right) = O(1), \quad t \to \infty,$$

for small ε and $|g(t)| < M < +\infty,\ t \geq 1$, for some constant M. We now rewrite the identity in the form

$$\begin{aligned} y(t) &= q^{-\varepsilon} y(qt) + g(t) \\ &= \ldots = q^{-n\varepsilon} y\left(q^n t\right) + q^{-(n-1)\varepsilon} g\left(q^{n-1} t\right) \\ &\quad + \ldots + q^{-2\varepsilon} g\left(q^2 t\right) + q^{-\varepsilon} g(qt) + g(t) \end{aligned}$$

and select n such that the inequality $qt_0 \leq q^n t \leq t_0$ is true. Thus, we get

$$\begin{aligned} |y(t)| &\leq q^{-n\varepsilon} \left\{ \left| y\left(q^n t\right)\right| + q^{\varepsilon} M + \ldots + q^{(n-2)\varepsilon} M + q^{(n-1)\varepsilon} M + q^{n\varepsilon} M \right\} \\ &\leq q^{-n\varepsilon} \left\{ \sup_{qt_0 \leq u \leq t_0} |y(u)| + \frac{M}{q^{-\varepsilon} - 1} \right\}. \end{aligned}$$

The condition $qt_0 \leq q^n t \leq t_0$ implies the inequality

$$n \leq \frac{\ln t}{\ln q^{-1}} + 1 + \frac{\ln t_0}{\ln q}.$$

Hence, extending the estimate for $|y(t)|$, we get

$$|y(t)| \leq t^{\varepsilon} \left(q^{-\varepsilon}\right)^{1+\frac{\ln t_0}{\ln q}} \left\{ \sup_{qt_0 \leq u \leq t_0} |y(u)| + \frac{M}{q^{-\varepsilon} - 1} \right\},$$

$$x(t) = t^{v_1-\varepsilon} y(t) = t^{v_1-\varepsilon} O\left(t^{\varepsilon}\right) = O\left(t^{v_1}\right), \quad t \to \infty.$$

Repeating these reasoning for the derivative, we obtain $x'(t) = O\left(t^{v_1-1}\right)$ as $t \to \infty$.

Performing the change

$$x(t) = t^{v_1} y\left(\frac{\ln t}{\ln q^{-1}}\right)$$

in Eq. (1) and using the estimate deduced for the derivative $x'(t)$, we get

$$y\left(\frac{\ln t}{\ln q^{-1}}\right) - y\left(\frac{\ln t}{\ln q^{-1}} - 1\right) = O\left(t^{-1}\right) = O\left(e^{-\ln q^{-1} \frac{\ln t}{\ln q^{-1}}}\right), \quad t \to \infty.$$

Denote $s \overset{\text{df}}{=} \dfrac{\ln t}{\ln q^{-1}}$ and $l \overset{\text{df}}{=} \ln q^{-1} > 0$,

$$y(s) - y(s+1) = O\left(e^{-ls}\right), \quad s \to \infty.$$

This implies that the sequence $y(s+n)$ is fundamental and, hence, convergent. We denote its limit by $g(s)$. This is a periodic function with period 1 satisfying the equality

$$y(s) - g(s) = O\left(e^{-ls}\right), \quad s \to \infty.$$

In view of the uniform convergence of continuous functions to $g(s)$, this function is continuous. Returning to the required function, we get

$$x(t) = t^{v_1}\left\{g(s) + O\left(t^{-1}\right)\right\}, \quad t \to \infty.$$

For the $m + j + 4$ times continuous differentiable solution $x(t) = o\left(e^{at}\right)$, $t \to \infty$, of Eq. (1), we repeat this process for the derivatives and arrive at the equalities

$$x^{(k)}(t) = t^{v_1 - k}\left\{f_{k,0}(s) + O\left(t^{-1}\right)\right\}, \quad t \to \infty,$$

where $0 \le k \le m+1$, $f_{k,0}(s)$ are continuous periodic functions with period 1. Further, by using the same reasoning as in the proof of Theorem 5 in [9] (Sec. 2) or in [10], we obtain the representation

$$\begin{aligned} x(t) = {} & t^{v_1} f_0\left(\frac{\ln t}{\ln q^{-1}}\right) + t^{v_1-1} f_1\left(\frac{\ln t}{\ln q^{-1}}\right) + t^{v_1-2} f_2\left(\frac{\ln t}{\ln q^{-1}}\right) \\ & + \ldots + t^{v_1-m+1} f_{m-1}\left(\frac{\ln t}{\ln q^{-1}}\right) + t^{v_1-m} f_m\left(\frac{\ln t}{\ln q^{-1}}\right) \\ & + t^{v_1-m-1} d_{m+1}\left(\frac{\ln t}{\ln q^{-1}}\right), \quad t \ge 1, \end{aligned} \tag{3}$$

where $f_p(u)$, $0 \le p \le m$, are periodic functions with period 1 such that $f_0(u) \in C^{m+1}(R)$,

$$f_{p+1}(u) = \frac{bq^{p+1} + ac}{ba\left(q^{p+1} - 1\right)}\left((v_1 - p) f_p(u) + \frac{1}{\ln q^{-1}} f_p'(u)\right), \quad 0 \le p \le m-1;$$

and $d_{m+1}(u)$ is a continuously differentiable bounded function. Thus, rewriting Eq. (1) as an advance equation

$$x'(t) = -bc^{-1}x(t) - ac^{-1}x\left(q^{-1}t\right) + c^{-1}x'\left(q^{-1}t\right)$$

and applying the reasoning used in the proof of the theorem in [12] to this equation, we obtain the equalities $x(t) = x_f(t)$, where the functions $x_f(t)$ are given in the condition of the theorem.

Since any solution has the property $x(t)e^{-at} \to L \in \mathrm{C}$, $t \to \infty$, the difference $x(t) - Lx_1(t) = o\left(e^{at}\right)$ as $t \to \infty$, where the function $x_1(t)$ is defined in the condition of the theorem. Thus, for a sufficiently smooth solution $x(t)$, we obtain the identities $x(t) - Lx_1(t) = x_f(t)$.

The theorem is proved.

In the proof of the theorem, the construction of the solution $x_1(t)$ for $a + bq^n = 0$, $n \in \mathrm{N}\bigcup\{0\}$, was based on the solution from the second example. If it does not exist, then we can try to construct, as in [1], or again find an unbounded particular solution of the equation

$$x^{(n+1)}(t) = ax^{(n)}(t) + bq^n x^{(n)}(qt) + cq^n x^{(n+1)}(qt).$$

This solution, e.g., $y(t)$, has the property $y(t)e^{-at} \to h \neq 0$, $t \to \infty$, and can be regarded as a starting point in the construction of the solution $x_1(t)$.

For sufficiently smooth solutions with the property $x(t) = o\left(e^{at}\right)$ as $t \to \infty$, representation (3) was obtained from the formal solution

$$x_\phi(t) = t^{v_1} f_0\left(\frac{\ln t}{\ln q^{-1}}\right) + t^{v_1-1} f_1\left(\frac{\ln t}{\ln q^{-1}}\right) + t^{v_1-2} f_2\left(\frac{\ln t}{\ln q^{-1}}\right) + \ldots,$$

where $f_0(u)$ is an arbitrary periodic function with period 1,

$$f_{p+1}(u) = \frac{bq^{p+1} + ac}{ba\left(q^{p+1} - 1\right)}\left((v_1 - p)\, f_p(u) + \frac{1}{\ln q^{-1}}\, f_p'(u)\right), \quad p \geq 0.$$

This solution is a divergent power series for $f_0(u) \equiv \text{const} \neq 0$.

REFERENCES

1. T. Kato and J. B. McLeod, "The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$," *Bull. Amer. Math. Soc.,* **77**, 891–937 (1971).
2. N. G. de Bruijn, "The difference-differential equation $F'(x) = e^{\alpha x+\beta} F(x-1)$ I, II," *Ned. Akad. Wetensch. Proc. Ser. A 56-Indag. Math.,* **15**, 449–464 (1953).
3. P. O. Frederickson, "Series solutions for certain functional-differential equations," in: *Lecture Notes in Mathematics,* **243** (1971), pp. 249–254.
4. G. P. Pelyukh and A. N. Sharkovskii, *Introduction to the Theory of Functional Equations* [in Russian], Naukova Dumka, Kiev (1974).
5. G. A. Derfel', "Probabilistic method for a class of functional-differential equations," *Ukr. Mat. Zh.,* **41**, No. 10, 1483–1491 (1989); ***English translation:*** *Ukr. Math. J.,* **41**, No. 10, 1137–1141 (1989).
6. V. M. Polishchuk and A. N. Sharkovskii, "Representation of solutions of linear difference-differential equations of neutral type," *Differents. Uravn.,* **9**, No. 9, 1627–1645 (1973).
7. P. O. Frederickson, "Global solutions to certain nonlinear functional differential equations," *J. Math. Anal. Appl.,* **33**, 355–358 (1971).
8. I. Gumovski and C. Mira, *Recurrences and Discrete Dynamic Systems,* Springer, Berlin (1980).
9. G. P. Pelyukh and D. V. Bel'skii, *On the Asymptotic Properties of Solutions of Functional and Functional-Differential Equations with Linearly Transformed Argument* [in Russian], Preprint, Institute of Mathematics, Ukrainian National Academy of Sciences, Kiev (2011).
10. G. P. Pelyukh and D. V. Bel'skii, "On the asymptotic properties of solutions of some functional-differential equations," *Nelin. Kolyv.,* **19**, No. 3, 311–348 (2016); ***English translation:*** *J. Math. Sci.,* **226**, No. 3, 197–239 (2017).
11. D. V. Bel'skii and G. P. Pelyukh, "On the asymptotic properties of solutions of one functional-differential equation with linearly transformed argument," *Nelin. Kolyv.,* **16**, No. 3, 291–313 (2013); ***English translation:*** *J. Math. Sci.,* **201**, No. 3, 263-287 (2014).
12. G. P. Pelyukh and D. V. Bel'skii, "On the asymptotic properties of solutions of a linear functional-differential equation of neutral type with constant coefficients and linearly transformed argument," *Nelin. Kolyv.,* **15**, No. 4, 466–493 (2012); ***English translation:*** *J. Math. Sci.,* **194**, No. 4, 374–403 (2013).