DOI 10.1007/s10958-018-4100-x

Journal of Mathematical Sciences, Vol. 236, No. 1, January, 2019

COMBINATION OF THE LAGUERRE TRANSFORM WITH THE BOUNDARY-ELEMENT METHOD FOR THE SOLUTION OF INTEGRAL EQUATIONS WITH RETARDED KERNEL

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We apply the Laguerre transform with respect to time to a time-dependent boundary-value integral equation encountered in the solution of three-dimensional Dirichlet initial-boundary-value problems for the homogeneous wave equation with homogeneous initial conditions by using the retarded potential of single layer. The obtained system of boundary integral equations is reduced to a sequence of Fredholm integral equations of the first kind that differ solely by the recursively dependent right-hand sides. To find their numerical solution, we use the boundary-element method. We establish an asymptotic estimate of the error of numerical solution and present the results of numerical simulations aimed at finding the solutions of retarded-potential integral equations for model examples.

The retarded-potential integral equations (RPIE) are encountered in modeling acoustic and electromagnetic fields with the use of the Kirchhoff formula or one of its components, namely, the retarded potentials of single (or double) layer. Both these potentials are solutions of a homogeneous wave equation and satisfy homogeneous initial conditions for any densities obeying relatively simple restrictions [7, 18]. The specific form of the equation used to determine the unknown density depends on the limit properties of the applied potential and the boundary conditions.

For the investigation of an RPIE, it is reasonable to use energy spaces of functions depending on the time variable and taking values in required Hilbert spaces (see, e.g., [16, Chap. XVIII]). In particular, in [10, 11], the existence of the unique solution of integral equations equivalent to the Dirichlet and Neumann problems for the wave equation was proved for the first time in these spaces, and the Galerkin method for their numerical solutions was justified. At the same time, the complexity of realization of this method for nontrivial boundary surfaces caused by the specific dependence of the densities of potentials on the so-called "retardation" $t-|x|/c$, where t, x, and c denote, respectively, the time, point in the space, and the velocity of propagation of vibrations in the environment, was indicated in [20].

The combination of discretization with respect to the space variables (e.g., with the help of boundary elements) with the solution of some intermediate problems aimed at taking into account the dependence of unknown functions on time is one of the approaches used for the solution of the indicated problem. In particular, this can be a family of convolution quadrature methods [24] based on the application of stable methods for the solution of ordinary differential equations. These methods are used in various applied problems and their survey can be found, e.g., in [12, 19].

In the present work, we determine the time dependence of the solution of RPIE with the help of the Laguerre transform. The properties of this transformation in the above-mentioned functional spaces were investigated in [3], where, in addition, the requirements to the boundary conditions guaranteeing the possibility of

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Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 59, No. 3, pp. 89–101, July–September, 2016. Original article submitted June 17, 2016.

 1072-3374/19/2361–0098

application of the Laguerre transform to RPIE were established. Moreover, this transform was used to reduce the original equation to an infinite sequence of boundary integral equations (BIE). The obtained BIE differ only in the recursively dependent right-hand sides. This circumstance makes it possible to construct efficient algorithms for the successive determination of their solutions by the boundary-element method (BEM). In the present work, by an example of equation equivalent to the Dirichlet problem for the wave equation, we consider the computational aspects of this combined approach.

Note that approaches based on the use of the Laguerre transform with respect to time are applicable (in some classes of functions) to arbitrary evolutionary problems linear in time. Furthermore, different methods can be used for the solution of problems obtained in the space of transforms. Thus, in [1, 9] (see also the bibliography therein) devoted to the analysis of complex problems of mechanics, the Laguerre transform with respect to time is combined with other integral transformations with respect to space variables. For the solution of these problems, it is necessary to perform the inverse transformation in a domain where we seek the unknown quantities. From the viewpoint of numerical analysis, with the exception of separate cases, this is a quite complicated procedure. This is why, in domains of the general form, the Laguerre transform is combined with some methods from the family of boundary-element methods. A survey of these combined methods is presented in [4, 25], where the boundary-value problems obtained in the space of transforms are justified, and their solutions are constructed with the help of potentials of single and double layers.

The structure of the work is as follows: In Section 1, we first introduce required functional spaces and define Laguerre transforms with respect to their elements. Then, by using this transform, we reduce an RPIE corresponding to the Dirichlet problem for the wave equation to an infinite triangular system of BIE. Moreover, we establish the conditions under which it is possible to find the solution of the RPIE by applying the inverse Laguerre transform to the solution of the triangular system of BIE and show how to reduce the obtained system to a sequence of equations that differ solely by the recursively dependent right-hand sides. In Section 2, we deduce basic relations of the BEM for the solution of the obtained sequence of BIE and establish asymptotic estimates for the error of the numerical solution of the RPIE that depends on the discretization parameter of the boundary surface. In Section 3, we present the results of a series of numerical experiments on the solution of the model RPIE.

1. Reduction of the Retarded-Potential Integral Equation to a Sequence of Boundary Integral Equations

Consider a Lipschitz surface Γ bounding a domain $\Omega \in \mathbb{R}^3$, $\mathbb{R}_+ := (0,\infty)$, and $\Sigma := \Gamma \times \mathbb{R}_+$. If we seek the solution of the Dirichlet problem for the wave equation in the form of the retarded potential of single layer

$$(\mathcal{S}\mu)(x,t) := \frac{1}{4\pi}\int_\Gamma \frac{\mu(y,t-|x-y|)}{|x-y|}\,d\Gamma_y, \qquad (x,t) \in \bar{Q},$$

with unknown density

$$\mu : \Gamma \times \mathbb{R} \to \mathbb{R},$$

then we get the following RPIE for its determination:

$$\frac{1}{4\pi}\int_\Gamma \frac{\mu(y,t-|x-y|)}{|x-y|}\,d\Gamma_y = g(x,t), \qquad (x,t) \in \Sigma, \tag{1}$$

where $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a given function.

We now introduce functional spaces required in what follows. Let $L^2(\Omega)$ be the Lebesgue space of square-integrable functions $v:\Omega\to\mathbb{R}$ with the inner product

$$(v,w)_{L^2(\Omega)} := \int_\Omega vw\,dx, \qquad v,w\in L^2(\Omega),$$

and the norm

$$\|v\|_{L^2(\Omega)} := \sqrt{(v,v)_{L^2(\Omega)}},$$

and let $H^1(\Omega)$ be a Sobolev space of functions $v\in L^2(\Omega)$ with generalized derivatives v_{x_1}, v_{x_2}, and v_{x_3} from $L^2(\Omega)$ with the inner product

$$(v,w)_{H^1(\Omega)} := \int_\Omega (\nabla v\nabla w + vw)\,dx, \qquad v,w\in H^1(\Omega),$$

and the norm

$$\|v\|_{H^1(\Omega)} := \sqrt{(v,v)_{H^1(\Omega)}}, \qquad v\in H^1(\Omega).$$

By $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma) := (H^{1/2}(\Gamma))'$ we denote, respectively, the space of traces of the elements $H^1(\Omega)$ on the surface Γ and the dual space; the duality relation on $H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$ is denoted by $\langle\cdot,\cdot\rangle_\Gamma$.

Let X be a Hilbert space with the inner product $(\cdot,\cdot)_X$ and the norm $\|\cdot\|_X$ generated by this product. Let $\sigma>0$ be an arbitrary number. Denote by $L^2_\sigma(\mathbb{R}_+;X)$ a weighted Lebesgue space [16] with a weight

$$\rho_\sigma(t) = e^{-\sigma t}, \; t\in\mathbb{R}_+,$$

whose elements are measurable functions $v:\mathbb{R}_+\to X$ such that

$$\int_{\mathbb{R}_+} \|v(t)\|_X^2\, e^{-\sigma t}\,dt < \infty,$$

with the inner product

$$(v,w)_{L^2_\sigma(\mathbb{R}_+;X)} = \int_{\mathbb{R}_+} (v(t),w(t))_X\, e^{-\sigma t}\,dt, \qquad v,w\in L^2_\sigma(\mathbb{R}_+;X),$$

and the norm

$$\|v\|_{L^2_\sigma(\mathbb{R}_+;X)} = \sqrt{(v,v)_{L^2_\sigma(\mathbb{R}_+;X)}}, \qquad v\in L^2_\sigma(\mathbb{R}_+;X).$$

Note that the space $L^2_\sigma(\mathbb{R}_+;X)$ is complete [8]. Assume that the elements of the space $L^2_\sigma(\mathbb{R}_+;X)$ are extended by zero to nonpositive values of the argument.

For any $m \in \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers), we define a weighted Sobolev space

$$H_\sigma^m(\mathbb{R}_+;X) := \{v \in L_\sigma^2(\mathbb{R}_+;X) \,|\, v^{(k)} \in L_\sigma^2(\mathbb{R}_+;X),\ k = 1,\ldots,m\}$$

with the norm

$$\|v\|_{H_\sigma^m(\mathbb{R}_+;X)} = \left(\sum_{k=0}^{m} \left\|v^{(k)}\right\|_{L_\sigma^2(\mathbb{R}_+;X)}^2\right)^{1/2}.$$

Here, the derivatives $v^{(k)}$, $k \in \mathbb{N}$, are understood in a sense of the space $\mathcal{D}'(\mathbb{R}_+;X)$ whose elements are distributions with values in the space X.

We study the RPIE (1) in the weighted Lebesgue and Sobolev spaces. Note that the existence and uniqueness of solutions of these equations was proved for broader spaces in [10]. Moreover, in [3], additional conditions were established for the function g to guarantee that the solution of Eq. (1) belongs to the weighted Sobolev space. As a consequence of the formulated assertions, we get the following proposition:

Proposition 1 [3]. *Let* $g \in H_{\sigma_0}^{m+2}(\mathbb{R}_+;H^{1/2}(\Gamma))$ *for some* $\sigma_0 > 0$ *and* $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. *Then there exists a unique solution of the RPIE (1) from the space* $H_{\sigma_0}^m(\mathbb{R}_+;H^{-1/2}(\Gamma))$. *Moreover, for any* $\sigma \geq \sigma_0$, *the following estimate is true:*

$$\|\mu\|_{H_\sigma^m(\mathbb{R}_+;H^{-1/2}(\Gamma))} \leq C\|g\|_{H_\sigma^{m+2}(\mathbb{R}_+;H^{1/2}(\Gamma))},$$

where $C > 0$ *is a constant independent of* g.

In what follows, we represent the infinite sequences of elements of any set X in the form of a column vector $\mathbf{v} := (v_0, v_1, \ldots)^\top$. Let X be a Hilbert space with the inner product $(\cdot,\cdot)_X$ and the norm $\|\cdot\|_X$ generated by this product. Consider a Hilbert space

$$\ell^2(X) := \left\{\mathbf{v} \in X^\infty \,\Big|\, \sum_{j=0}^{\infty} \left\|v_j\right\|_X^2 < +\infty\right\}$$

with the inner product $(\mathbf{v},\mathbf{w}) = \sum_{j=0}^{\infty}(v_j, w_j)_X$, $\mathbf{v},\mathbf{w} \in \ell^2(X)$, and the norm

$$\|\mathbf{v}\|_{\ell^2(X)} := \left(\sum_{j=0}^{\infty} \left\|v_j\right\|_X^2\right)^{1/2}, \qquad \mathbf{v} \in \ell^2(X).$$

If $X = \mathbb{R}$, then we write $\ell^2 := \ell^2(\mathbb{R})$.

We now consider in more detail the relationship between the spaces $L_\sigma^2(\mathbb{R}_+;X)$ and $\ell^2(X)$. In the case where $X = \mathbb{R}$, the first of these spaces is the space $L_\sigma^2(\mathbb{R}_+)$ in which the Laguerre polynomials $\{L_k(\sigma\,\cdot)\}_{k\in\mathbb{N}_0}$

form an orthogonal basis [23], i.e., an arbitrary function $f \in L^2_\sigma(\mathbb{R}_+)$ admits the expansion

$$f(t) = \sum_{k=0}^{\infty} f_k L_k(\sigma t), \qquad t \in \mathbb{R}_+,$$

where the coefficients $f_0, f_1, \ldots, f_k, \ldots$ are expressed in the same way by the formula

$$f_k := \sigma \int_{\mathbb{R}_+} f(t) L_k(\sigma t) e^{-\sigma t}\, dt, \qquad k \in \mathbb{N}_0. \tag{2}$$

A mapping $\mathcal{L}: L^2_\sigma(\mathbb{R}_+) \to \ell^2$ that associates a sequence $\mathbf{f} = (f_0, f_1, \ldots, f_k, \ldots)^\top$ [whose components are given by relation (2)] with a function f is called the discrete Laguerre integral transform [2, 23]. The Laguerre transform $\mathcal{L}$ is a bijective mapping and the inverse transform $\mathcal{L}^{-1}: \ell^2 \to L^2_\sigma(\mathbb{R}_+)$ is given, for any $\mathbf{h} \in \ell^2$, by the formula

$$(\mathcal{L}^{-1}\mathbf{h})(t) := \sum_{k=0}^{\infty} h_k L_k(\sigma t), \qquad t \in \mathbb{R}_+. \tag{3}$$

It is clear that, for any function $f \in L^2_\sigma(\mathbb{R}_+)$, the following equality is true:

$$\mathcal{L}^{-1}\mathcal{L} f = f. \tag{4}$$

In [3], the notion of Laguerre transform was generalized to the case of vector valued functions from $L^2_\sigma(\mathbb{R}_+; X)$, i.e., the authors considered a mapping $\mathcal{L}: L^2_\sigma(\mathbb{R}_+; X) \to X^\infty$ acting according to rule (2). If X is a Hilbert space with the inner product $(\cdot,\cdot)_X$ and the norm $\|\cdot\|_X$ generated by this product, then the following proposition is true:

Proposition 2 [3]. *A mapping $\mathcal{L}: L^2_\sigma(\mathbb{R}_+; X) \to X^\infty$ that associates a sequence $\mathbf{f} = (f_0, f_1, \ldots, f_k, \ldots)^\top$ with a function f by relation (2) is injective, the space $\ell^2(X)$ is its image and, in addition,*

$$\|f\|^2_{L^2_\sigma(\mathbb{R}_+; X)} = \frac{1}{\sigma} \sum_{k=0}^{\infty} \|f_k\|^2_X. \tag{5}$$

Moreover, equality (4) is true for any function $f \in L^2_\sigma(\mathbb{R}_+; X)$, where $\mathcal{L}^{-1}: \ell^2(X) \to L^2_\sigma(\mathbb{R}_+; X)$ is a mapping inverse to $\mathcal{L}$ that associates the function h with an arbitrary sequence $\mathbf{h} = (h_0, h_1, \ldots, h_k, \ldots)^\top$ by relation (3).

Definition 1 [3]. *Let $\sigma > 0$ and let X be a Hilbert space. The mappings*

$$\mathcal{L}: L^2_\sigma(\mathbb{R}_+; X) \to \ell^2(X) \qquad \text{and} \qquad \mathcal{L}^{-1}: \ell^2(X) \to L^2_\sigma(\mathbb{R}_+; X),$$

from Proposition 2 are called the direct and inverse Laguerre transforms, respectively, and relation (5) is called the Parseval equality.

Consider a sequence $\mathbf{V}$ whose components are given by the formula

$$(V_k\xi)(x) := \int_\Gamma \xi(y) e_k(x-y)\, d\Gamma_y, \qquad x \in \Gamma,$$

where ξ is an arbitrary function measurable on Γ and the functions e_k for $z \in \mathbb{R}^3$ are defined as follows:

$$e_k(z) := (4\pi|z|)^{-1}\zeta_k(|z|), \qquad k \in \mathbb{N}_0,$$

$$\zeta_0(z) := e^{-\sigma|z|}, \qquad \zeta_k(z) := e^{-\sigma|z|}(L_k(\sigma|z|) - L_{k-1}(\sigma|z|)), \qquad k \in \mathbb{N}. \tag{6}$$

Note that, at the point $z = 0$, the function e_0 has an integrable singularity and, for any $k \in \mathbb{N}$, the function e_k (6) has a removable singularity. This means [14] that $\mathbf{V}$ can be interpreted as a sequence of boundary operators $V_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $k \in \mathbb{N}_0$, such that

$$(V_k\xi)(x) := \langle \xi, e_k(x-\cdot)\rangle_\Gamma.$$

Assume that $\boldsymbol{\mu} = \mathcal{L}\mu$ for any $\mu \in L^2_{\sigma_0}(\mathbb{R}_+; H^{-1/2}(\Gamma))$ and some $\sigma_0 > 0$. We construct a sequence $\mathbf{w}$ according to the rule

$$w_k = \sum_{i=0}^{k} V_{k-i}\mu_i, \qquad k \in \mathbb{N}_0.$$

The operation over sequences defined as indicated above is called q-convolution [4], and we write

$$\mathbf{w} := \mathbf{V} \underset{H^{1/2}(\Gamma)}{\circ} \boldsymbol{\mu}.$$

In [3], it was shown that, as a result of the application of the Laguerre transform to the RPIE (1), we get the following infinite triangular system of BIE:

$$\mathbf{V} \underset{H^{1/2}(\Gamma)}{\circ} \boldsymbol{\mu} = \mathbf{g} \quad \text{in} \quad \ell^2(H^{1/2}(\Gamma)), \tag{7}$$

where $\mathbf{g} := \mathcal{L}g$.

Proposition 3 [3]. *Let* $g \in H^2_{\sigma_0}(\mathbb{R}_+; H^{1/2}(\Gamma))$ *for some* $\sigma_0 > 0$. *Then there exists a unique solution* $\boldsymbol{\mu} \in \ell^2(H^{-1/2}(\Gamma))$ *of the system of BIE (7), and the solution of the RPIE (1) is determined by the inverse Laguerre transform* $\mu := \mathcal{L}^{-1}\boldsymbol{\mu}$.

It is easy to see that system (7) can be represented in the form of a sequence of equations as follows:

$$\begin{aligned} (V_0\mu_0)(x) &= g_0(x), \\ (V_0\mu_1)(x) &= \tilde{g}_1(x), \\ &\dots\dots\dots\dots, \\ (V_0\mu_k)(x) &= \tilde{g}_k(x), \quad k \in \mathbb{N}, \quad x \in \Gamma, \\ &\dots\dots\dots\dots, \end{aligned} \tag{8}$$

where

$$\tilde{g}_k(x) := g_k(x) - \sum_{i=0}^{k-1} (V_{k-i}\mu_i)(x), \quad k \in \mathbb{N}.$$

We choose a value of the parameter N from certain considerations and, as a result of solution of the BIE (8), successively determine the components $\mu_0, \mu_1, \dots, \mu_N$. Thus, according to Lemma 1 in [3], we can interpret a partial sum

$$\tilde{\mu}^N(y, t - |x-y|) = \sum_{k=0}^{N} \left(\sum_{i=0}^{k} \zeta_{k-i}(\sigma(x-y))\mu_i(y) \right) L_k(\sigma t), \quad x, y \in \Gamma, \quad t \in \mathbb{R}_+, \tag{9}$$

as an approximate solution of the RPIE (1).

2. Solution of the System of BIE by the Boundary Element Method

We now consider the sequence of BIE (8) in more detail. For any $k \in \mathbb{N}_0$, the left-hand side of the kth equation is specified by the boundary operator V_0, while the right-hand side depends on the corresponding element of the sequence $\mathbf{g}$ and on the solutions of equations with the previous numbers $i = 0, \dots, k-1$. It is known [14, 22] that the integral operator $V_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is elliptic and bounded

$$\langle V_0\eta, \eta \rangle_\Gamma \geq c_1 \|\eta\|^2_{H^{-1/2}(\Gamma)},$$

$$\|V_0\eta\|_{H^{1/2}(\Gamma)} \leq c_2 \|\eta\|_{H^{-1/2}(\Gamma)} \quad \forall \eta \in H^{-1/2}(\Gamma).$$

These properties are used not only to prove the existence and uniqueness of the solution of BIE

$$V_0\eta = f \quad \text{in} \quad H^{1/2}(\Gamma), \tag{10}$$

but also to find the approximate solution of this equation by the Bubnov–Galerkin method or its specific realization by the boundary-element method [21]. We now deduce the main relations of this method for the investigated BIE.

Let $X^M \subset H^{-1/2}(\Gamma)$, $M \in \mathbb{N}$, be a sequence of finite-dimensional half spaces that are linear spans of the functions $\{\phi_i\}_{i=1}^M$ which form a basis in X^M. In each of these spaces, we approximate the solution of Eq. (10) by a linear combination

$$\eta^M := \sum_{i=1}^{M} \eta_i \phi_i \in X^M,$$

obtained as a solution of the following variational problem:

$$\left\langle V_0 \eta^M, \eta \right\rangle_\Gamma = \langle f, \eta \rangle_\Gamma \qquad \forall \eta \in X^M. \tag{11}$$

Thus, if we choose the basis functions ϕ_j as test functions, then, for the unknown coefficients

$$\boldsymbol{\eta}^{[M]} := \{\eta_i\}_{i=1}^M \in \mathbb{R}^M,$$

we get the following system of linear algebraic equations (SLAE):

$$\mathbf{V}_0^{[M]} \mathbf{h}^{[M]} = \mathbf{f}^{[M]}, \tag{12}$$

where

$$V_0^{[M]}[j,i] := \left\langle V_0 \phi_i, \phi_j \right\rangle_\Gamma, \quad f_j^{[M]} := \left\langle f, \phi_j \right\rangle_\Gamma, \quad i, j = 1, \ldots, M.$$

Note that, in view of the $H^{-1/2}(\Gamma)$-ellipticity of the operator V_0, the matrix of the obtained system is positive definite. This is why, for any right-hand side, system (12) possesses a unique solution, i.e., $\forall M \in \mathbb{N}$, we obtain an approximate solution of Eq. (10). According to the Céa lemma (see, e.g., [27]), this solution satisfies the inequality

$$\left\| \eta^M \right\|_{H^{-1/2}(\Gamma)} \le c_1 \| f \|_{H^{1/2}(\Gamma)}$$

and the value of its error can be estimated as follows:

$$\left\| \eta - \eta^M \right\|_{H^{-1/2}(\Gamma)} \le \frac{c_2}{c_1} \inf_{\xi \in X^M} \| \eta - \xi \|_{H^{-1/2}(\Gamma)}.$$

This implies the convergence of the approximate solution

$$\eta^M \to \eta \in H^{-1/2}(\Gamma) \quad \text{as} \quad M \to \infty \quad \text{in} \quad H^{-1/2}(\Gamma),$$

where η is the solution of the corresponding BIE in sequence (8).

We now specify the SLAE (12) by using the notation presented in [15, 27]. Let

$$\Gamma_{\tilde{M}} = \bigcup_{\ell=1}^{\tilde{M}} \overline{\tau}_\ell$$

be an approximation of the surface Γ formed by $\tilde{M}$ triangular boundary elements $\{\tau_\ell\}_{\ell=1}^{\tilde{M}}$ with vertices $\{x^{[\ell_1]}, x^{[\ell_2]}, x^{[\ell_3]}\}$. The quantity

$$h := \max_{\ell=1,\ldots,\tilde{M}} \left\{ \int_{\tau_\ell} ds \right\}^{1/2}$$

is regarded as a parameter of approximation.

We now construct a set of piecewise-constant functions linearly independent on $\Gamma_{\tilde{M}}$ $\{\varphi_\ell^0\}_{\ell=1}^{M}$, $M = \tilde{M}$:

$$\varphi_\ell^0(x) = \begin{cases} 1, & x \in \tau_\ell, \\ 0, & x \notin \tau_\ell. \end{cases}$$

In finite-dimensional functional spaces $X^M = S_h^0(\Gamma) := \operatorname{span}\{\varphi_\ell^0\}_{\ell=1}^{M}$, we consider Eq. (10) and assume that it corresponds to the kth equation in sequence (8). Its numerical solution μ_k^h is sought in the form of a linear combination of the piecewise-constant functions

$$\mu_k^h = \sum_{\ell=1}^{M} \mu_{k,\ell}^h \varphi_\ell^0 \in S_h^0(\Gamma), \qquad k \in \mathbb{N}_0. \tag{13}$$

Here, $\mathbb{R}^M \ni \{\mu_{k,\ell}^h\}_{\ell=1}^{M} =: \boldsymbol{\mu}_k^h$ is the vector of unknown coefficients. This vector is found from the SLAE

$$\mathbf{V}_0^h \boldsymbol{\mu}_k^h = \tilde{\mathbf{g}}_k^h, \qquad k \in \mathbb{N}_0. \tag{14}$$

The matrix $\mathbf{V}_0^h$ is a specific realization of the matrix of system (12), and the integrals

$$V_0^h[i,\ell] = \int_{\tau_i} \int_{\tau_\ell} e_0(x-y)\, ds_y\, ds_x, \qquad i, \ell = 1, \ldots, M,$$

are its elements. The components of the vector on the right-hand side in system (14) have the form

$$\tilde{g}_k^h[i] = \int_{\tau_i} \left\{ g_k(x) - \sum_{j=0}^{k-1} (V_{k-j}\mu_j^h)(x) \right\} ds_x, \qquad j = 1, \ldots, M.$$

Thus, if we specify the values of the parameter N, then we can find the approximate values of the densities μ_k^h on the cylinder Σ by using relation (13) as a result of the solution of system (14) for the successive values $k = 0,\ldots,N$ and finding the vectors μ_k^h. The sequence $\mu^{N,h} := (\mu_0^h, \mu_1^h, \ldots, \mu_N^h, 0, 0, \ldots)^\top$ is an approximate solution of the system of BIE (7) and the sum

$$\tilde{\mu}^{N,h}(y, t - |x-y|) = \sum_{k=0}^{N} \left(\sum_{i=0}^{k} \zeta_{k-i}(\sigma|x-y|)\mu_i^h(y) \right) L_k(\sigma t), \qquad x, y \in \Gamma, \qquad t \in \mathbb{R}_+, \tag{15}$$

is regarded as an approximate (numerical) solution of the RPIE (1).

We now deduce an *a priori* estimate for the error of the numerical solution $\tilde{u}^{N,h}$ relative to the approximate solution $\tilde{u}^N$ given by relation (9). Assume that the surface Γ can be represented in the form of a union

$$\Gamma = \bigcup_{i=1}^{\tilde{N}} \overline{\Gamma}_i$$

of surfaces Γ_i (such that $\Gamma_i \cap \Gamma_j = \varnothing,\ i \neq j$) each of which admits a sufficiently smooth parametrization

$$\Gamma_i := \{ x \in \mathbb{R}^3 : x = \tilde{\chi}_i(\xi), \xi \in \tilde{\tau}_i \subset \mathbb{R}^2 \}.$$

By using a set of nonnegative functions $\phi_i \in C_0^\infty(\mathbb{R}^3)$ such that

$$\sum_{i=1}^{\tilde{N}} \phi_i(x) = 1 \qquad \forall x \in \Gamma, \qquad \phi_i(x) = 0 \qquad \forall x \in \Gamma \setminus \Gamma_i,$$

we can represent an arbitrary function v given on the surface Γ in the form

$$v(x) = \sum_{i=1}^{\tilde{N}} \phi_i(x) v(x) = \sum_{i=1}^{\tilde{N}} v_i(x) \qquad \forall x \in \Gamma,$$

where $v_i(x) := \phi_i(x) v(x)$ $\forall x \in \Gamma_i$. For $s \in (0,1]$, we consider the spaces of the piecewise-smooth functions [27]

$$H^s_{\text{pw}}(\Gamma) := \{ v \in L^2(\Gamma) : v|_{\Gamma_i} \in H^s(\Gamma_i), i = 1,\ldots,\tilde{N} \}$$

with the following norm and seminorm:

$$\|v\|_{H^s_{\text{pw}}(\Gamma)} := (\|v\|^2_{L^2(\Gamma)} + |v|^2_{H^s_{\text{pw}}(\Gamma)})^{1/2}, \qquad |v|_{H^s_{\text{pw}}(\Gamma)} := \left(\sum_{i=1}^{\tilde{N}} \big| v|_{\Gamma_i} \big|^2_{H^s(\Gamma_i)} \right)^{1/2},$$

where the seminorms on the parts of Γ are specified with regard for the parametrization

$$v_i(\tilde{\chi}_i(\xi)) =: \tilde{v}_i(\xi) \qquad \text{for} \quad \xi \in \tilde{\tau}_i,$$

$$|\tilde{v}_i|_{H^1(\tilde{\tau}_i)} := \left(\int_{\tilde{\tau}_i} |\nabla_\xi \tilde{v}_i(\xi)|^2 \, ds_\xi \right)^{1/2},$$

$$|\tilde{v}_i|_{H^s(\tilde{\tau}_i)} := \left(\int_{\tilde{\tau}_i} \int_{\tilde{\tau}_i} \frac{(\tilde{v}_i(\xi) - \tilde{v}_i(\eta))^2}{|\xi - \eta|^{2+2s}} ds_\xi \, ds_\eta \right)^{1/2}, \quad s \in (0,1).$$

Lemma 1. *Let* $\mu \in (H^s_{\text{pw}}(\Gamma))^\infty$ *be a solution of the system of BIE (7) for some* $s \in (0,1]$ *and let the following inequality be true:*

$$\sum_{j=0}^{\infty} |\mu_j|_{H^s_{\text{pw}}(\Gamma)} < +\infty. \tag{16}$$

Then, for any value of the parameter $N \in \mathbb{N}_0$, *the numerical solution* $\tilde{\mu}^{N,h}$ *of the RPIE (1) admits the following asymptotic estimate:*

$$\left\| \tilde{\mu}^N(\cdot,t) - \tilde{\mu}^{N,h}(\cdot,t) \right\|_{H^{-1/2}(\Gamma)} \le \tilde{C}_{N,T} h^{s+1/2} \sum_{k=0}^{N} |\mu_k|_{H^s_{\text{pw}}(\Gamma)}, \quad t \in (0,T), \tag{17}$$

where $T \in \mathbb{R}_+$ *is an arbitrary fixed number and* $C_{N,T}$ *and* $\tilde{C}_k$ *are quantities independent of the parameter* h.

Proof. For arbitrary fixed $N \in \mathbb{N}_0$, $T \in \mathbb{R}_+$, and $t \in (0,T)$, we consider the quantity

$$\delta_{N,T} := \left\| \tilde{\mu}^N(\cdot,t) - \tilde{\mu}^{N,h}(\cdot,t) \right\|_{H^{-1/2}(\Gamma)} = \left\| \sum_{k=0}^{N} (\mu_k(\cdot) - \mu_k^h(\cdot)) L_k(\sigma t) \right\|_{H^{-1/2}(\Gamma)}.$$

Denote

$$C_{N,T} := \max_{t \in [0,T], k=0,\ldots,N} |L_k(\sigma t)|.$$

Thus, we can write

$$\delta_{N,T} \le C_{N,T} \sum_{k=0}^{N} \left\| \mu_k - \mu_k^h \right\|_{H^{-1/2}(\Gamma)}. \tag{18}$$

Note that, in the case where the solution of the system of BIE (7) satisfies inequality (16), the following estimate is true [27]:

$$\left\| \mu_k - \mu_k^h \right\|_{H^{-1/2}(\Gamma)} \le \tilde{C}_k h^{s+1/2} |\mu_k|_{H^s_{\text{pw}}(\Gamma)}, \quad k \in \mathbb{N}_0,$$

where $\tilde{C}_k$ are quantities that do not depend on the parameter h. By using this estimate in inequality (18) and introducing the notation

$$\tilde{C}_{N,T} := C_{N,T} \max_{k=0,\ldots,N} \{\tilde{C}_k\},$$

we get (17).

3. Results of the Numerical Experiments

We now illustrate the numerical solution of the RPIE (1) by the considered combined method. Assume that Γ is the surface of the cube

$$\Omega := (-1,1)\times(-1,1)\times(-1,1)$$

and that the right-hand side of Eq. (1) has the form

$$g(x,t) := \frac{f_3(t-|x|+1)}{|x|}, \qquad (x,t)\in\Sigma,$$

where f_3 is the so-called cubic beta-spline [6]. As the value of the parameter of Laguerre transform, we take $\sigma = 2$.

We first present the results obtained for the components of the numerical solution $\mu^{N,h}$ of the system of BIE (7) for different values of the parameter N. For comparison, we also consider the numerical solution

$$\hat{\mu}^{N,h} := (\hat{\mu}_0^h, \hat{\mu}_1^h, \ldots, \hat{\mu}_N^h, 0, 0, \ldots)^\top$$

of the same system obtained by a different method, namely, by the collocation method [5, 21, 26]. Despite the fact that, for a given class of integral equations, this method does not have appropriate justification (unlike the BEM constructed on the basis of the Bubnov–Galerkin method), it is used in practice for a long time. Note that, in the case of application of the same finite-dimensional functional spaces $S_h^0(\Gamma)$ with piecewise-constant basis functions $\{\varphi_\ell^0\}_{\ell=1}^M$ in the collocation method, the elements of the matrices $\mathbf{V}_k^h$ and the vector of the right-hand side in the SLAE (14) have the form

$$V_k^h[i,\ell] = \int_{\tau_\ell} e_k(x_i - y)\,ds_y,$$

$$\tilde{g}_k^h[i] = g_k(x_i) - \sum_{j=0}^{k-1} (V_{k-j}\mu_j^h)(x_i), \qquad i,\ell = 1,\ldots,M,$$

where every collocation point x_i is taken at the center-of-mass of the corresponding triangle τ_i.

We apply both mentioned methods to the first BIE in sequence (8). Since, in this stage, the parameter of discretization h is the main parameter, we consider different partitions $\Gamma_{\tilde{M}}$ of the boundary surface by decreasing

Table 1

x_2 \ $\tilde{M}$	300	588	972	1452	2700
0	$3.09064\cdot10^{-1}$ $2.96648\cdot10^{-1}$	$3.05025\cdot10^{-1}$ $2.98582\cdot10^{-1}$	$3.02762\cdot10^{-1}$ $2.98911\cdot10^{-1}$	$3.01466\cdot10^{-1}$ $2.98932\cdot10^{-1}$	$3.00211\cdot10^{-1}$ $2.98837\cdot10^{-1}$
0.2	$3.09064\cdot10^{-1}$ $2.96648\cdot10^{-1}$	$2.69065\cdot10^{-1}$ $2.64675\cdot10^{-1}$	$2.80764\cdot10^{-1}$ $2.77803\cdot10^{-1}$	$2.70912\cdot10^{-1}$ $2.68793\cdot10^{-1}$	$2.83511\cdot10^{-1}$ $2.82207\cdot10^{-1}$
0.5	$1.83769\cdot10^{-1}$ $1.79747\cdot10^{-1}$	$1.62174\cdot10^{-1}$ $1.61857\cdot10^{-1}$	$1.75864\cdot10^{-1}$ $1.74803\cdot10^{-1}$	$1.64615\cdot10^{-1}$ $1.64473\cdot10^{-1}$	$1.65704\cdot10^{-1}$ $1.65664\cdot10^{-1}$

Table 2

k \ $\tilde{M}$	300	588	972	1452	2700
0	$3.09064\cdot10^{-1}$ $2.96648\cdot10^{-1}$	$3.05025\cdot10^{-1}$ $2.98582\cdot10^{-1}$	$3.02762\cdot10^{-1}$ $2.98911\cdot10^{-1}$	$3.01466\cdot10^{-1}$ $2.98932\cdot10^{-1}$	$3.00211\cdot10^{-1}$ $2.98837\cdot10^{-1}$
10	$-7.27805\cdot10^{-2}$ $-6.93435\cdot10^{-2}$	$-7.12230\cdot10^{-2}$ $-6.94413\cdot10^{-2}$	$-7.04907\cdot10^{-2}$ $-6.94286\cdot10^{-2}$	$-7.02050\cdot10^{-2}$ $-6.94037\cdot10^{-2}$	$-6.98376\cdot10^{-2}$ $-6.93674\cdot10^{-2}$
15	$3.41835\cdot10^{-2}$ $3.20066\cdot10^{-2}$	$3.28953\cdot10^{-2}$ $3.19480\cdot10^{-2}$	$3.24581\cdot10^{-2}$ $3.18885\cdot10^{-2}$	$3.21571\cdot10^{-2}$ $3.18604\cdot10^{-2}$	$3.21783\cdot10^{-2}$ $3.18375\cdot10^{-2}$
20	$-1.23603\cdot10^{-2}$ $-1.22545\cdot10^{-2}$	$-1.22948\cdot10^{-2}$ $-1.21227\cdot10^{-2}$	$-1.22314\cdot10^{-2}$ $-1.21263\cdot10^{-2}$	$-1.19848\cdot10^{-2}$ $-1.21325\cdot10^{-2}$	$-1.21693\cdot10^{-2}$ $-1.21354\cdot10^{-2}$

the value of h. In Table 1, we present the values of the numerical solutions $\mu_0^h(x)$ and $\hat{\mu}_0^h(x)$ (the upper and lower rows, respectively) of the BIE (8) for $k=0$ and $\sigma=2$ on the sequence of partitions $\Gamma_{\tilde{M}}$ at the point $x=(1,x_2,0)$ on one of the lateral faces of the cube. It is easy to see that the values of the solutions obtained by different numerical methods are close at all points of observation with different locations on the faces of the cube.

The values of the numerical solutions $\mu_k^h(x)$ and $\hat{\mu}_0^h(x)$ (the upper and lower rows, respectively) of the BIE (8) for a fixed observation point $x=(1,0,0)$ and various values of the parameter k and for $\sigma=2$ on the same sequence of partitions $\Gamma_{\tilde{M}}$ are presented in Table 2. It is easy to see that the difference between the values of the solutions obtained by different methods at different points is insignificant, and the behaviors of these solutions as the value of the parameter h decreases for different numbers k are identical.

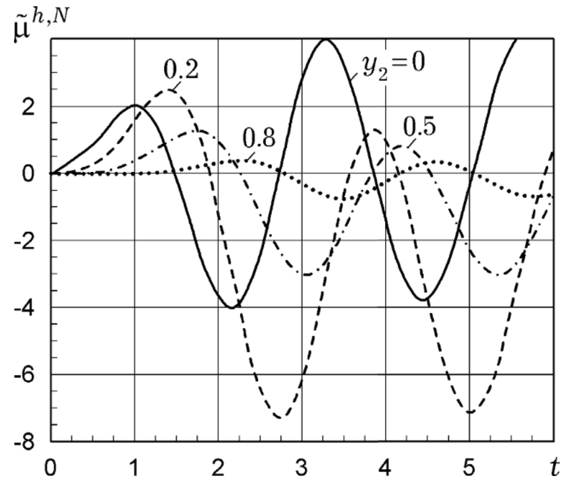

Fig. 1

At the same time, we note that the results obtained by both methods are sensitive to the errors of evaluation of elements of the matrices $\mathbf{V}_k^h$, $k = 0,\ldots,N$, in formation of the SLAE (14), which have the form of double (collocation method) or quadruple (BEM) integrals. This is especially important for the matrix $\mathbf{V}_0^h$ because (as indicated above) the function $e_0(x-y)$ has an integrable singularity in the case where the points x and y coincide. In finding numerical solutions of the analyzed model problems, we used the additive subtraction of these singularities (for similar problems, this procedure was described in [5]). With an aim to simplify the algorithms of realization, these methods were applied only to the cases where the points x and y belong to the same boundary element, i.e., only on the principal diagonal of the matrix. For too fine boundary elements, it is necessary to use a special procedure of integration also in the cases where the analyzed points are located in neighboring boundary elements [5, 13, 17]. Otherwise, one may observe an instability of the behavior of numerical solutions, which can be observed for some values of the solutions and for $\tilde{M} = 2700$ in the tables presented above.

Having a collection of the components μ_k^h, $k = 0,\ldots,N$, we can compute the values of the numerical solution (15) of the RPIE (1) on Σ. In Fig. 1, we present the plots of the solutions $\tilde{\mu}^{h,N}$ at the observation point $x = (1,0,0)$ for $y = (1, y_2, 0)$, $N = 40$ and $\tilde{M} = 972$. If we consider $\tilde{\mu}^{h,N}(y, t-|x-y|)$ on a sequence of the values of the parameter N, then we can observe its point-by-point convergence in time on a certain interval $[0,T]$ that becomes larger as N increases. We mention the following clear reproduction of the effect of retardation: The value of the solution $\tilde{\mu}^{h,N}(y, t-|x-y|)$ that characterizes the influence of the source located at a point y on the perturbation at a point x is close to zero for $t \in [0,|x-y|]$. Note that the plots of the numerical solution of Eq. (1) determined by substituting the coefficients $\hat{\mu}_k^h(x)$, $k = 0,\ldots,N$, obtained by the collocation method in expansion (15) exhibit no visual difference with the previous plots in Fig. 1.

We now consider a sequence

$$\mathbf{v} := (v_0, v_1, \ldots, v_k, \ldots)^\top, \qquad v_k(x) = \frac{e^{\sigma}\zeta_k(x)}{|x|}, \qquad k \in \mathbb{N}_0. \tag{19}$$

To within the coefficients, this sequence coincides with the fundamental solution of the infinite triangular system

Table 3

x_1 \ k		0	10	20
1.5	$u_k^h(x)$	$2.45961 \cdot 10^{-1}$	$8.91296 \cdot 10^{-2}$	$-7.76775 \cdot 10^{-2}$
	$\hat{u}_k^h(x)$	$2.45157 \cdot 10^{-1}$	$8.88135 \cdot 10^{-2}$	$-7.65805 \cdot 10^{-2}$
	$v_k(x)$	$2.45253 \cdot 10^{-1}$	$8.85703 \cdot 10^{-2}$	$-7.66715 \cdot 10^{-2}$
2.0	$u_k^h(x)$	$6.78263 \cdot 10^{-2}$	$5.70383 \cdot 10^{-2}$	$4.13326 \cdot 10^{-2}$
	$\hat{u}_k^h(x)$	$6.76731 \cdot 10^{-2}$	$5.65699 \cdot 10^{-2}$	$4.09090 \cdot 10^{-2}$
	$v_k(x)$	$6.76676 \cdot 10^{-2}$	$5.65019 \cdot 10^{-2}$	$4.07837 \cdot 10^{-2}$
2.5	$u_k^h(x)$	$1.99574 \cdot 10^{-2}$	$-1.86551 \cdot 10^{-2}$	$-3.81745 \cdot 10^{-3}$
	$\hat{u}_k^h(x)$	$1.99229 \cdot 10^{-2}$	$-1.86508 \cdot 10^{-2}$	$-3.84610 \cdot 10^{-3}$
	$v_k(x)$	$1.99148 \cdot 10^{-2}$	$-1.86297 \cdot 10^{-2}$	$-3.80306 \cdot 10^{-3}$
3.0	$u_k^h(x)$	$6.11747 \cdot 10^{-3}$	$-1.97795 \cdot 10^{-2}$	$-1.06752 \cdot 10^{-2}$
	$\hat{u}_k^h(x)$	$6.10901 \cdot 10^{-3}$	$-1.97095 \cdot 10^{-2}$	$-1.05927 \cdot 10^{-2}$
	$v_k(x)$	$6.10521 \cdot 10^{-3}$	$-1.96762 \cdot 10^{-2}$	$-1.05491 \cdot 10^{-2}$
3.5	$u_k^h(x)$	$1.92886 \cdot 10^{-3}$	$-2.61027 \cdot 10^{-3}$	$3.34234 \cdot 10^{-3}$
	$\hat{u}_k^h(x)$	$1.92660 \cdot 10^{-3}$	$-2.59210 \cdot 10^{-3}$	$3.33455 \cdot 10^{-3}$
	$v_k(x)$	$1.92513 \cdot 10^{-3}$	$-2.57739 \cdot 10^{-3}$	$3.33181 \cdot 10^{-3}$

of elliptic equations obtained as a result of the application of the Laguerre transform to the wave equation [3]. This is why sequence (19) can be regarded as an analytic solution of the Dirichlet problem for this infinite system with the boundary value $g_k = v_k$, $k \in \mathbb{N}_0$ on Γ. Moreover, it is known [25] that the sequence of BIE (8) is equivalent, in a certain sense, to the analyzed boundary-value problem. In particular, this means that the sequence

$$\mathbf{u}^h := \mathbf{V} \underset{H^{1/2}(\Gamma)}{\circ} \boldsymbol{\mu}^h$$

constructed on the basis of the numerical solution of the BIE (8)

$$\boldsymbol{\mu}^h := (\mu_0^h, \mu_1^h, \ldots, \mu_k^h, \ldots)^\top$$

is a solution of this boundary-value problem if its trace coincides with the sequence $\mathbf{v} := (v_0, v_1, \ldots, v_k, \ldots)^\top$

on Γ. We use this fact to estimate the error of the numerical solution of the sequence of BIE (8) because we do not know its analytic solution.

Hence, we assume that, in the sequence of BIE (8), the right-hand side is specified by using the sequence $\mathbf{v}$ (19). We find (as earlier, by two methods) N components of the numerical solutions μ_k^h and $\hat{\mu}_k^h(x)$, $k = 0,\ldots,N$, and compute, by using these components, the values of the corresponding solutions $u_k^h(x)$ and $\hat{u}_k^h(x)$, $k = 0,\ldots,N$, of the indicated Dirichlet boundary-value problem on the set of points x lying outside the cube. The results of evaluation of the solutions $u_k^h(x)$, $\hat{u}_k^h(x)$, and $v_k(x)$ are presented in Table 3.

It is easy to see that the components of both numerical solutions are close at all points to the corresponding components of the analytic solution. For the analyzed model example, this is an indirect confirmation of the fact that, for a given number of boundary elements $(\tilde{M} = 1452)$, the numerical solutions of the BIE are also found with satisfactory accuracy.

CONCLUSIONS

The combination of the method of Laguerre transform with the boundary-element method makes it possible to efficiently determine the numerical solutions of the retarded-potential integral equations. It is clear that this approach can be extended to other time-dependent integral equations encountered in the solution of the initial-boundary-value problems with different boundary conditions and also in the case where the solution is represented in terms of the retarded potential of double layer or by the Kirchhoff formula.

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