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LIE SUPERALGEBRAS AND CALOGERO–MOSER–SUTHERLAND SYSTEMS

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Abstract. We review recent results obtained at the intersection of the theory of quantum deformed Calogero–Moser–Sutherland systems and the theory of Lie superalgebras. We begin with a definition of admissible deformations of root systems of basic classical Lie superalgebras. For classical series, we prove the existence of Lax pairs. Connections between infinite-dimensional Calogero–Moser–Sutherland systems, deformed quantum CMS systems, and representation theory of Lie superalgebras are discussed.

***Keywords and phrases*:** quantum Calogero–Moser–Sutherland system, Lax pair, Lie superalgebra, symmetric function, Euler character, Grothendieck ring.

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CONTENTS

Introduction

The main goal of this survey is to present the main results obtained in the last 15 years at the intersection of the theory quantum Calogero–Moser–Sutherland systems and the theory of Lie superalgebras. The existence of some connections between these two theories was first observed in [12, 13], where a superanalog of the Calogero–Moser–Sutherland operator was constructed by using the root system of the Lie superalgebra $\mathfrak{gl}(n,m)$. It was also proved in [12] that for the particular case $m = 1$, the constructed superanalog of the CMS operator coincides, up to change of variables and a parameter k, with the operator constructed by A. Veselov, O. Chalykh, and M. Feigin in [3]. Another result obtained in [12] states that for $k = -1$ and $k = -1/2$, the constructed superanalog of CMS operator is the radial part of the Laplace operator for symmetric Lie superalgebras $(\mathfrak{gl} \oplus \mathfrak{gl}, \mathfrak{gl})$ and $(\mathfrak{gl}, \mathfrak{osp})$. These results showed that there should be connections between Lie superalgebras and deformed CMS system similar to the connections discovered by M. A. Olshanetski and A. M. Perelomov in their classical paper [9]. The next important step in an investigation of deformed quantum CMS systems and their connections to Lie superalgebras was taken in [14]. This paper set up a basis for a systematic construction of the theory of the deformed quantum CMS systems. Namely, it was shown that it is possible to construct a deformed CMS operator for any root system of a basic classical Lie superalgebra. The main objects for future investigation were also introduced in the same paper:

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admissible deformations of generalized root systems and the corresponding second-order deformed operator, and generalized invariant algebras and their difference analogs. The case of a generalized root system of the type $A(n,m)$ was considered in the differential case in [15] in more detail, and the corresponding difference case was considered in [17]. As a by-product the paper [6], appeared, in which Jack polynomials for some special parameters was discussed.

These papers showed that there are deep connections between quantum integrable systems and Lie superalgebras, and these connections can be used to develop both these theories. Note that the main difficulty in the theory of the deformed CMS systems is the absence of Dunkl operators, which play a crucial role in the classical case. So we need to develop methods that are independent of the theory of Dunkl operators: quantum Moser matrices and infinite-dimensional analogs of classical systems. In particular, considering infinite-dimensional analogs of quantum integrable systems, it is easy to explain the integrability of the deformed difference Macdonald–Ruijsenaars operator and differential deformed operator of the type BC and to obtain explicit formulas for eigenfunctions (see [16, 17]). These eigenfunctions depend on some parameters that can be specialized to some particular values, and a natural question appears: "What is the meaning of the specializations from the point of view of representation theory? It is well known that in the classical case Jacobi polynomials can be specialized to the characters of irreducible finite-dimensional representations of Lie algebras or spherical functions connected with finite-dimensional representations, and corresponding second-order operators can be interpreted as radial parts of the Laplace operators. To date, the general picture is not clear in the case of the deformed CMS systems. It should be connected with the fact that the representation theory of finite-dimensional simple Lie superalgebras is not semi-simple. But partial results in this direction can be found in [19, 20, 24]. In [19], it was shown that in the case of the deformed CMS operator of the type BC, there exists a natural specialization of eigenfunctions to Euler characters, and in [20], a description of Grothendieck rings of classical simple Lie superalgebras in terms of some groupoid (which is natural from the point of view of integrable systems) is given. In the general case, the spectral decomposition of the algebra of integrals of the deformed quantum CMS problem is not semisimple. An interpretation of this fact in terms of the representation theory of Lie superalgebras is given in [24]. In particular, it was proved in [24] that for $k=-1/2$ the generalized eigenspaces can be naturally described in terms of projective covers of the irreducible finite-dimensional modules.

Connections between Lie superalgebras and deformed quantum Calogero–Moser–Sutherland systems lead to a natural infinite-dimensional generalization of the Jack classical symmetric functions. Moreover, using some analogs of translation functors from representation theory, it is possible to describe the action of the algebra of integrals in the generalized eigenspaces for special values of the parameters (see [21, 23]). In [22], Lax pairs were found for deformed CMS operators of the classical type by means of infinite-dimensional Dunkl operators. It allowed one to obtain a simpler proof of the integrability for the classical deformed CMS systems.

1. Generalized Root Systems and Their Admissible Deformations

1.1. Generalized root systems. We start with the definition of the generalized root systems due to Serganova (see [10]). We mention that there are three slightly different definitions of generalized root systems in [10], and we choose one of them which suits our purpose best for.

Let V be a finite-dimensional complex vector space with a symmetric, nondegenerate bilinear form $\langle\cdot,\cdot\rangle$.

Definition 1.1. A finite set $R\subset V\setminus\{0\}$ is called a *generalized root system* if the following conditions are fulfilled:

(1) R spans V and $R=-R$;

(2) if $\alpha,\beta\in R$ and $\langle\alpha,\alpha\rangle\neq 0$, then

$$\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z},\quad s_\alpha(\beta)=\beta-\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha\in R;$$

(3) if $\alpha \in R$ and $\langle \alpha, \alpha \rangle = 0$, then for any $\beta \in R$ such that $\langle \alpha, \beta \rangle \neq 0$ at least one of the vectors $\beta + \alpha$ or $\beta - \alpha$ belongs to R.

Introduce the following notation:

$$R_{re} = \{\alpha \in R :< \alpha, \alpha >\neq 0\} \quad R_{iso} = \{\alpha \in R :< \alpha, \alpha >= 0\}$$

A generalized root system R is said to be *reducible* if it can be represented as the direct sum of two nonempty generalized root systems R_1 and R_2, i.e., $V = V_1 \oplus V_2$, where $R_1 \subset V_1$, $R_2 \subset V_2$, and $R = R_1 \cup R_2$. Otherwise, the system is called *irreducible.*

Any generalized root system has a partial symmetry described by the finite group W_0 generated by reflections with respect to the real roots. The main result of Serganova's paper [10] is a classification theorem for irreducible generalized root systems. One can introduce the notion of positive and simple root in the same manner as for ordinary root systems. Below, we present a list of all irreducible generalized root systems according to [10].

Classical series.

1. Series $A(n-1, m-1)$, $n \neq m$. Let $V_{n,m} = V_1 \oplus V_2$ be a vector space with a basis $\{e_1, \dots, e_{n+m}\}$ such that $\{e_1, \dots, e_n\}$ is a basis of V_1 and $\{e_{n+1}, \dots, e_{n+m}\}$ is a basis of V_2. Let e^i, $i = 1, \dots, n+m$, denote the corresponding basis in the dual space $V^*_{n,m}$.

Consider the following bilinear symmetric form on $V_{n,m}$:

$$B(u, v) = \sum_{i=1}^{n} u^i v^i - \sum_{j=1}^{m} u^j v^j, \tag{1}$$

where u^i and v^i are the coordinates of the vectors u and v in the basis e_i.

Let us split the set of indices $I = \{1, \dots, n+m\}$ into two groups, $I = I_1 \cup I_2$, where $I_1 = \{1, \dots, n\}$ and $I_2 = \{n+1, \dots, n+m\}$, and rewrite the last formula as follows:

$$B = \sum_{i \in I_1} e^i \otimes e^i - \sum_{j \in I_2} e^j \otimes e^j, \tag{2}$$

where B is now considered as an element of $V^* \otimes V^*$.

A generalized root system of the type $A(n-1, m-1)$, $n \neq m$, is defined as the set

$$R = \Big\{ e_i - e_j,\ i \neq j,\ i, j \in I \Big\},$$

and the corresponding space V is a hyperplane in $V_{n,m}$ generated by this set, with the induced bilinear form. It is easy to see that in this case

$$R_{\mathrm{re}} = A_{n-1} \oplus A_{m-1}, \quad R_{\mathrm{im}} = \Big\{ \pm (e_i - e_j),\ i \in I_1,\ j \in I_2 \Big\}.$$

The corresponding Lie superalgebra is $sl(n, m)$.

2. Series $A(n-1, n-1)$. In the case $m = n$, the restriction of the form B on the corresponding hyperplane V is degenerate. Indeed, the vector

$$v = \sum_{i \in I_1} e_i - \sum_{i \in I_2} e_i$$

belongs to V and is orthogonal to all roots (and, therefore, to the whole space V). In order to obtain a proper generalization of the root system in this case, we consider the quotient $V' = V/\langle v \rangle$ and the corresponding set R', which is the image of R after such a projection. This is the system of type $A(n-1, n-1)$. The corresponding Lie superalgebra is $psl(n|n)$.

3. Series $B(n,m)$**.** In this case, $V = V_{n,m}$ with the same bilinear form B and

$$R = BBB\{\pm e_i \pm e_j,\ i \neq j,\ i,j \in I,\ \pm e_i,\ i \in I,\ \pm 2e_i,\ i \in I_2\Big\}.$$

The real and isotropic roots are

$$R_{\rm re} = B_n \oplus BC_m, \quad R_{\rm im} = \Big\{ \pm e_i \pm e_j,\ i \in I_1,\ j \in I_2\Big\}.$$

This is the root system of the Lie superalgebra $osp(2n+1|2m)$.

4. Series $D(n,m)$**,** $n \geq 2$**.** Here $V = V_{n,m}$ is the same as before, but R is the set

$$R = \Big\{ \pm e_i \pm e_j,\ i \neq j,\ i,j \in I,\ \pm 2e_i,\ i \in I_2\Big\}.$$

We have

$$R_{\rm re} = D_n \oplus C_m, \quad R_{\rm im} = \Big\{ \pm e_i \pm e_j,\ i \in I_1,\ j \in I_2\Big\}.$$

This root system corresponds to the Lie superalgebra $osp(2n|2m)$.

5. Series $C(0,m)$**.** Here $V = V_{1,m}$ and

$$R = \Big\{ \pm e_i \pm e_j,\ i \neq j,\ i,j \in I,\ \pm 2e_i,\ i \in I_2\Big\}.$$

In this case,

$$R_{\rm re} = C_m, \quad R_{\rm im} = \Big\{ \pm e_1 \pm e_j,\ j \in I_2\Big\}.$$

6. Series $C(n,m)$**.** Here $V = V_{n,m}$ and

$$R = \{\pm e_i \pm e_j,\ i \neq j,\ i,j \in I,\ \pm 2e_i,\ i \in I\Big\},$$

so that

$$R_{\rm re} = C_n \oplus C_m, \quad R_{\rm im} = \Big\{ \pm e_i \pm e_j,\ i \in I_1,\ j \in I_2\Big\}.$$

In this and the following cases, there are no related Lie superalgebras, but there are symmetric superspaces with such root systems.

7. Series $BC(n,m)$**.** Here $V = V_{n,m}$ and

$$R = \Big\{ \pm e_i \pm e_j,\ i \neq j,\ i,j \in I,\ \pm e_i,\ \pm 2e_i,\ i \in I\Big\}.$$

In this case,

$$R_{\rm re} = BC_n \oplus BC_m,\ R_{\rm im} = \Big\{ \pm e_i \pm e_j,\ i \in I_1,\ j \in I_2\Big\}.$$

8. Case $AB(1,3)$ **(also known as** $F(4)$**).** Here $V = V_1 \oplus V_2$, where V_1 is a three-dimensional space with basis $\{e_1, e_2, e_3\}$ and V_2 is a one-dimensional space generated by e_4. The bilinear form is

$$B(u,v) = u^1v^1 + u^2v^2 + u^3v^3 - 3u^4v^4.$$

The root system R is the set

$$R = \Big\{\pm e_i \pm e_j,\ i \neq j,\ \pm e_i,\ i,j = 1,2,3,\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\Big\};$$

here

$$R_{\rm re} = B_3 \oplus A_1, \quad R_{\rm im} = \Big\{\frac{1}{2}\big(\pm e_1 \pm e_2 \pm e_3 \pm e_4\big)\Big\}.$$

9. Case $G(1,2)$ (also known as $G(3)$). Here $V = V_1 \oplus V_2$, where V_1 is a two-dimensional space generated by vectors e_1, e_2, and e_3 satisfying the condition $e_1+e_2+e_3 = 0$, and V_2 is a one-dimensional space generated by a vector e_4. The form B is defined as follows:

$$\begin{cases} B(e_i, e_j) = -1, \quad \text{if} \quad i \neq j, \\ B(e_i, e_i) = 2, \\ B(e_i, e_4) = 0, \\ B(e_4, e_4) = -2, \end{cases}$$

where $i, j = 1, 2, 3$. The root system is

$$R = \Big\{ \pm e_i,\ (e_i - e_j),\ \pm e_4,\ \pm 2e_4,\ \pm e_i \pm e_4,\ i \neq j,\ i, j \leq 3 \Big\},$$

$$R_{\text{re}} = G_2 \oplus BC_1, \quad R_{\text{im}} = \Big\{ \pm e_i \pm e_4,\ i = 1, 2, 3 \Big\}.$$

10. Case $D(2,1,\lambda)$. Here $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ are parameters satisfying the relation $\lambda_1 + \lambda_2 + \lambda_3 = 0$. The space V is the direct sum $V_1 \oplus V_2 \oplus V_3$ of three one-dimensional subspaces generated by e_1, e_2, and e_3. The form B is

$$B(u, v) = \lambda_1 u^1 v^1 + \lambda_2 u^2 v^2 + \lambda_3 u^3 v^3;$$

the root system is

$$R = \Big\{ \pm 2e_1,\ \pm 2e_2,\ \pm 2e_3,\ \pm e_1 \pm e_2 \pm e_3 \Big\},$$

$$R_{\text{re}} = A_1 \oplus A_1 \oplus A_1, \quad R_{\text{im}} = \Big\{ \pm e_1 \pm e_2 \pm e_3 \Big\}.$$

1.2. Admissible deformations. In order to construct deformed CMS operators, we need the notion of an admissible deformation introduced in [14]. Consider all possible triples $\mathcal{R} = (R, m, B)$, where R is a generalized root system, m is a complex-valued function on R, and B is a symmetric bilinear form on the space V. We can consider the initial root system as a particular triple of such a form, where m is identically equal to 1 and the form B is the same as in the definition of the generalized root system R. For any triple $\mathcal{R}$, consider two algebras $\mathcal{D}^{\text{rat}}_{\mathcal{R}}$ and $\mathcal{D}^{\text{tr}}_{\mathcal{R}}$ (the rational and trigonometric algebras). Let $\mathcal{R}$ be such a triple. We can consider elements from $\alpha \in R$ as the functions on V. Namely, $\alpha(v) = B(\alpha, v)$, and we will also consider the derivations defined by the rule $\partial_u(v) = B(u, v)$, where $u, v \in V$.

Definition 1.2. Let $\mathbb{C}(V)$ be the algebra of rational functions on the space V and $\text{End}_{\mathbb{C}}(\mathbb{C}(V))$ be the algebra of its linear maps (considered as maps of a vector space). Let us denote by $\mathcal{D}^{\text{rat}}_{\mathcal{R}}$ its subalgebra generated by multiplication operators by the functions $1/\alpha$, $\alpha \in R$, and differentiations ∂_v, $v \in V$. Let us also denote by $D(V)$ the algebra generated by differentiations ∂_v, $v \in V$.

Definition 1.3. A homomorphism $\varphi : \mathcal{D}^{\text{rat}}_{R,B} \to D(V)$ such that $\varphi(1/\alpha) = 0$, $\alpha \in R$, is called a *Harish-Chandra homomorphism.*

Definition 1.4. Let $\mathbb{M}(V)$ be the algebra of meromorphic functions on the space V, and $\text{End}_{\mathbb{C}}(\mathbb{M}(V))$ be its algebra of linear maps (considered as maps of a vector space). Let us denote by $\mathcal{D}^{\text{tr}}_{R,B}$ its subalgebra generated by multiplication operators by the functions $e^{-\alpha}$ and $1/(e^{-\alpha} - 1)$, $\alpha \in R^+$, and differentiations ∂_v, $v \in V$.

Definition 1.5. A homomorphism $\varphi : \mathcal{D}^{\text{tr}}_{R,B} \to D(V)$ such that $\varphi(e^{-\alpha}) = 0$, $\alpha \in R^+$, is called a *Harish-Chandra homomorphism.*

We can identify the algebra $D(V)$ with the algebra of polynomial functions on V by the rule $\partial_v(u) = (v, u)$.

Let us also introduce the following algebras:

$$\Lambda_{\mathcal{R}}^{\mathrm{rat}} = \Big\{p \in S(V) \;\Big|\; p(wu) = p(u),\ w \in W_0,\ \partial_\alpha(p) \in (\alpha),\ \alpha \in R_{\mathrm{iso}}\Big\}, \tag{3}$$

$$\Lambda_{\mathcal{R}}^{\mathrm{tr}} = \Big\{p \in S(V) \;\Big|\; p(wu) = p(u),\ w \in W_0,\ p(u+\alpha/2) - p(u-\alpha/2) \in (\alpha),\ \alpha \in R_{\mathrm{iso}}\Big\}, \tag{4}$$

where (α) means the principal ideal generated by α, and R_{iso} means the set of isotropic roots.

For any triple $\mathcal{R}$, let us define two types of operators, which will be called the *rational* and *trigonometric* Calogero–Moser-Sutherlend operators (CMS):

$$H_2^{\mathrm{rat}} = \Delta_B - \sum_{\alpha\in R^+} \frac{m(\alpha)(m(\alpha)+1)(\alpha,\alpha)}{\alpha^2}, \tag{5}$$

$$H_2^{\mathrm{tr}} = \Delta_B - \sum_{\alpha\in R^+} \frac{m(\alpha)\big(m(\alpha)+m(2\alpha)+1\big)(\alpha,\alpha)}{\big(e^{\alpha/2}-e^{-\alpha/2}\big)^2} \tag{6}$$

or

$$H_2^{\mathrm{tr}} = \Delta_B - \sum_{\alpha\in R^+} \frac{m(\alpha)\big(m(\alpha)+m(2\alpha)+1\big)(\alpha,\alpha)}{\sinh^2(\alpha)}, \tag{7}$$

where Δ_B is the Laplace operator on V defined by the form B. We note that the last two operators differ by the change of variables $v \to 2v$.

Let us denote by $\Psi_{\mathcal{R}}^{\mathrm{rat}}$ the function (called the ground state)

$$\Psi_{\mathcal{R}}^{\mathrm{rat}} = \prod_{\alpha\in R^+} \alpha^{m(\alpha)} \tag{8}$$

in the rational case and

$$\Psi_{\mathcal{R}}^{\mathrm{tr}} = \prod_{\alpha\in R^+} \Big(e^{\alpha/2}-e^{-\alpha/2}\Big)^{m(\alpha)} \tag{9}$$

in the trigonometric case.

Let also $C(H_2)$ be the centralizer of the operator H_2 in the corresponding algebra. The main problem is to describe this centralizer more or less explicitly. It turns out that in the cases under consideration, the restriction of the Harish-Chandra homomorphism to the centralizer is an injection. Therefore, it suffices to describe the image of the centralizer under the Harish-Chandra homomorphism.

Now let us indicate conditions that allow one to choose triples $\mathcal{R}$ corresponding to integrable systems.

Definition 1.6. A triple $\mathcal{R}$ is called an *admissible deformation* if it satisfies the following conditions:
(1) the new form B and the multiplicities are W_0-invariant;
(2) all isotropic roots have multiplicity 1;
(3) the following *fundamental identity* holds:

$$\sum_{\substack{\alpha\not\sim\beta,\\ \alpha,\beta\in R_+}} \frac{m_\alpha m_\beta(\alpha,\beta)}{\alpha\beta} \equiv 0 \tag{10}$$

in the rational case and

$$\sum_{\substack{\alpha\not\sim\beta,\\ \alpha,\beta\in R_+}} m_\alpha m_\beta(\alpha,\beta)\left(\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\cdot\frac{1+e^{-\beta}}{1-e^{-\beta}}+1\right) \equiv 0 \tag{11}$$

in the trigonometric case. Here $\alpha \not\sim \beta$ means that α is not proportional to β (let us note that in the case $BC(n,m)$, there are proportional roots).

One can verify that conditions (3) are equivalent to the following condition:

(3*) the function $(\Psi^{\mathrm{rat}}_{\mathcal{R}})^{-1}$ is a formal eigenfunction of the rational CMS operator, and the function $(\Psi^{\mathrm{tr}}_{\mathcal{R}})^{-1})$ is a formal eigenfunction of the trigonometric CMS operator.

The theorem below describes all possible admissible deformations of generalized root systems. In order to do so, one needs to use the fact that it suffices to verify the conditions (10) and (11) only for two-dimensional subsystems of generalized root systems (cf. [4, 28]). It suffices to consider only the classical root systems $A(n,m)$ and $BC(n,m)$ (from the point of view of deformations, the others are simply special cases) and the exceptional root systems $G(1,2)$, $AB(1,3)$, and $D(2,1\lambda)$. The forms B are obviously defined up to a multiple, so we will choose some normalization to avoid unnecessary constants.

One of the main results of [14] is the following theorem.

Theorem 1.7. *The following list is the complete set of admissible deformations of generalized root systems.*

In all cases, admissible forms depend on one parameter. We denote this parameter by k and choose it in such a way that the value $k=-1$ corresponds to the case of Lie superalgebras.

Classical series.

Series $A(n,m)$. The form B is equal to

$$B=\sum_{i\in I_1} e^i\otimes e^i + k\sum_{j\in I_2} e^j\otimes e^j, \tag{12}$$

where k is an arbitrary parameter. The multiplicities $m_\alpha = m(\alpha)$ of nonisotropic roots

$$m(e_i-e_j)=k,\quad i,j\in I_1,\qquad m(e_i-e_j)=k^{-1},\quad i,j\in I_2$$

(recall that isotropic roots have multiplicity 1).

The corresponding one-parameter family of deformed CMS operators has the form

$$\begin{aligned} L_{A(n-1,m-1)} &= \left(\frac{\partial^2}{\partial {x_1}^2}+\dots+\frac{\partial^2}{\partial {x_n}^2}\right)+k\left(\frac{\partial^2}{\partial {y_1}^2}+\dots+\frac{\partial^2}{\partial {y_m}^2}\right)\\ &\quad -\sum_{1\le i<j\le n}\frac{2k(k+1)}{\sinh^2(x_i-x_j)}-\sum_{1\le i<j\le m}\frac{2(k^{-1}+1)}{\sinh^2(y_i-y_j)}-\sum_{1\le i\le n}\sum_{1\le j\le m}\frac{2(k+1)}{\sinh^2(x_i-y_j)}. \end{aligned}\tag{13}$$

Note that in the case $m=n$, the vector

$$v=\sum_{i\in I_1}e_i-\sum_{i\in I_2}e_i$$

is not isotropic for the deformed form, so strictly speaking, we deform not the generalized system of the type $A(n-1,n-1)$ but its degenerate extension.

Series $BC(n,m)$. The form B is the same as before, and the multiplicities are

$$\begin{gathered} m(e_i\pm e_j)=k,\quad m(e_i)=p,\quad m(2e_i)=q,\quad i,j\in I_1,\\ m(e_i\pm e_j)=k^{-1},\quad m(e_j)=r,\quad m(2e_j)=s,\quad i,j\in I_2, \end{gathered}$$

where p, q, r, and s satisfy the relations

$$p=kr,\quad 2q+1=k(2s+1).$$

The corresponding deformed CMS operator deepens on three free parameters and can be given by the formula

$$
\begin{aligned}
L_{BC(n,m)} = &-\left(\frac{\partial^2}{\partial {x_1}^2}+\cdots+\frac{\partial^2}{\partial {x_n}^2}\right)-k\left(\frac{\partial^2}{\partial {y_1}^2}+\cdots+\frac{\partial^2}{\partial {y_m}^2}\right) \\
&+\sum_{1\le i<j\le n}\left(\frac{2k(k+1)}{\sinh^2(x_i-x_j)}+\frac{2k(k+1)}{\sinh^2(x_i+x_j)}\right)+\sum_{1\le i\le n}\sum_{1\le j\le m}\frac{2(k+1)}{\sinh^2(x_i-y_j)} \\
&+\sum_{1\le i<j\le m}\left(\frac{2(k^{-1}+1)}{\sinh^2(y_i-y_j)}+\frac{2(k^{-1}+1)}{\sinh^2(y_i+y_j)}\right)+\sum_{i=1}^{n}\frac{p(p+2q+1)}{\sinh^2 x_i} \\
&+\sum_{i=1}^{n}\frac{4q(q+1)}{\sinh^2 2x_i}+\sum_{j=1}^{m}\frac{kr(r+2s+1)}{\sinh^2 y_j}+\sum_{j=1}^{n}\frac{4ks(s+1)}{\sinh^2 2y_j}. \qquad (14)
\end{aligned}
$$

Exceptional cases.

Case $AB(1,3)$.

$$B = e^1\otimes e^1+e^2\otimes e^2+e^3\otimes e^3+3ke^4\otimes e^4,$$

the multiplicities are given by

$$m(e_i)=a=\frac{3k+1}{2},\quad m(e_4)=b=\frac{1-k}{2k},\quad m(e_i\pm e_j)=c=\frac{3k-1}{4},\quad i,j=1,2,3.$$

The deformed CMS operator has the form

$$
\begin{aligned}
L_{AB(1,3)} = &-\left(\frac{\partial^2}{\partial {x_1}^2}+\frac{\partial^2}{\partial {x_2}^2}+\frac{\partial^2}{\partial {x_3}^2}\right)-3k\frac{\partial^2}{\partial y^2}+\sum_{i=1}^{3}\frac{a(a+1)}{\sinh^2 x_i}+\frac{3kb(b+1)}{\sinh^2 y} \\
&+\sum_{1\le i<j\le 3}\left(\frac{4c(c+1)}{\sinh^2(x_i-x_j)}+\frac{4c(c+1)}{\sinh^2(x_i+x_j)}\right)+\frac{1}{4}\sum_{\pm}\frac{(3k+3)}{\sinh^2\frac{1}{2}(y\pm x_1\pm x_2\pm x_3)}, \qquad (15)
\end{aligned}
$$

where the parameters a, b, and c are given in terms of the parameter k above, and the sum is taken over all possible 8 combinations of signs. In this case, we have only one free parameter k.

Case $G(1,2)$. In the basis e_1, e_2, e_4, the form B has the form

$$B = e^1\otimes e^1+e^2\otimes e^2-\frac{1}{2}\left(e^1\otimes e^2+e^2\otimes e^1\right)+ke^4\otimes e^4.$$

The multiplicities are

$$m(e_i)=a=1+2k,\quad m(e_i-e_j)=b=\frac{2k-1}{3},\quad m(e_4)=c=\frac{1}{k}+2,$$
$$m(2e_4)=d=\frac{1}{2k}-\frac{1}{2},\quad i,j=1,2,3.$$

The corresponding deformed operator has the following form:

$$
\begin{aligned}
L_{G(1,2)} = &-\left(\frac{\partial^2}{\partial {x_1}^2}-\frac{\partial^2}{\partial x_1\partial x_2}+\frac{\partial^2}{\partial {x_2}^2}\right)-k\frac{\partial^2}{\partial y^2}+\sum_{i=1}^{3}\frac{a(a+1)}{\sinh^2 x_i}+\sum_{1\le i<j\le 3}\frac{3b(b+1)}{\sinh^2(x_i-x_j)}+ \\
&+\frac{kc(c+2d+1)}{\sinh^2 y}+\frac{4kd(d+1)}{\sinh^2 2y}+\sum_{i=1}^{3}\left(\frac{2(k+1)}{\sinh^2(x_i-y)}+\frac{2(k+1)}{\sinh^2(x_i+y)}\right), \qquad (16)
\end{aligned}
$$

where the parameters a, b, c, and d are given in terms of the parameter k above. We again have the one-parameter family.

Case $D(2,1,\lambda)$ The form B is

$$B = \lambda_1 e^1 \otimes e^1 + \lambda_2 e^2 \otimes e^2 + \lambda_3 e^3 \otimes e^3,$$

where the parameters λ_i, $i = 1,2,3$, are arbitrary but nonzero. Let us introduce the parameter

$$k = \lambda_1 + \lambda_2 + \lambda_3 - 1$$

such that for $k = -1$ we obtain a case of Lie superalgebras. The multiplicities are

$$m(2e_i) = m_i = \frac{k+1}{2\lambda_i} - 1, \quad i = 1,2,3.$$

The corresponding deformed CMS operator has the form

$$L_{D(2,1,\lambda)} = \lambda_1 \frac{\partial^2}{\partial {x_1}^2} + \lambda_2 \frac{\partial^2}{\partial x_2^2} + \lambda_3 \frac{\partial^2}{\partial {x_3}^2} + \sum_{i=1}^{3} \frac{4\lambda_i m_i(m_i+1)}{\sinh^2 2x_i} + \sum_{\pm} \frac{2(k+1)}{\sinh^2(x_1 \pm x_2 \pm x_3)}, \tag{17}$$

where the last sum is taken over all possible combinations of signs (four signs in this case). This family can be parametrized by points on the projective plane $\lambda_1 : \lambda_2 : \lambda_3$. The case of Lie superalgebras corresponds to the line $\lambda_1 + \lambda_2 + \lambda_3 = 0$, so again we have a free parameter (say, k).

Everywhere above, the hyperbolic sine is defined by the usual formula $\sinh x = \frac{1}{2}(e^x - e^{-x})$. Moreover, x_i and y_j are chosen as elements of dual space defined by the basis vectors and the form B. The next theorem shows the meaning of Harish–Chandra homomorphism.

Theorem 1.8. *Let $A(n,m)$ and $BC(n,m)$ be generalized root systems of the corresponding types, the parameter $k \neq 0$, and H_2 be the corresponding rational or trigonometric deformed CMS operator.*

Then the restriction of the Harish-Chandra homomorphism on the centralizer $C(H_2)$ is an injection and its image is contained in the algebra $\Lambda_{\mathcal{R}}^{\mathrm{rat}}$ or in the algebra $\Lambda_{\mathcal{R}}^{\mathrm{tr}}$, respectively.

The proof can be found in [14].

2. Integrability of Deformed Quantum CMS Systems

In this section, we will prove that the deformed quantum CMS systems related to classical generalized root systems, i.e., the systems of the types $A(n-1,m-1)$ and $BC(n,m)$, are integrable. The integrability of the rational deformed CMS operator for exceptional root systems has still not been proved.

Quantum analogs of the Lax pairs for usual CMS systems were proposed in 1992 by Ujino, Hikami, and Wadati in [27, 29] (see also [25]). We note that the quantum version of the Moser matrix L was used in 1975 by Calogero, Marchioro, and Ragnisco (see [2]) in order to construct integrals of the CMS system. A proof of the integrability of the usual CMS systems are based either on Dunkl operators or on Lax pairs. It appears that, in the deformed case, there are no Dunkl operators. So we use the Lax-pair method, which allows one to construct some elements from the centralizer of the given operator.

2.1. Lax pair.

The proof of integrability is based on the following Lax equation.

Theorem 2.1. *Let $\mathfrak{A}$ be an associative algebra and $a \in \mathfrak{A}$. Denote by E the identity $(n \times n)$-matrix. Let L and M be matrices of the same size with elements in $\mathfrak{A}$ such that*

$$[L, aE] = [L, M]. \tag{18}$$

Moreover, let e^* and e be $(1 \times n)$- and $(n \times 1)$-matrices with elements from $\mathfrak{A}$, respectively (i.e., a row and a column) such that

$$e^* M = Me = 0, \quad e^* a = a e^*, \quad ea = ae. \tag{19}$$

Then the elements

$$L_r = e^* L^r e$$

commute with a.

Proof. The proof follows the ideas from [27, 29]. Equality (18) can be rewritten in the form $[L, aE - M] = 0$. Therefore, $[L^r, aE - M] = 0$ and hence

$$e^*(aE - M)L^r e = e^* L^r (aE - M)e.$$

Further, $e^* a L^r e = e^* L^r aEe$ and, therefore, $ae^* L^r e = e^* L^r ea$. □

2.2. Rational case. We consider the rational deformed CMS operator of the type $A(n-1, m-1)$. It has the form

$$H_2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + k \sum_{i=1}^{m} \frac{\partial^2}{\partial y_i^2} - \sum_{i<j}^{n} \frac{2k(k+1)}{(x_i - x_j)^2} - \sum_{i<j}^{m} \frac{2(k^{-1}+1)}{(y_i - y_j)^2} - \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{2(k+1)}{(x_i - y_j)^2}.$$

For the reader's convenience, we introduce the new variables $y_j = x_{n+j}$, $j = 1, \dots, n$. Then the algebra $\mathcal{D}_{\mathcal{R}}^{\mathrm{rat}}$ can be described as the algebra generated by differentiations $\dfrac{\partial}{\partial x_i}$, $i = 1, \dots, n+m$, and multiplication operators by the functions $\dfrac{1}{x_i - x_j}$, $i \neq j$. The algebra $\Lambda_{\mathcal{R}}^{\mathrm{rat}}$ can be described as follows:

$$\Lambda_{\mathcal{R}}^{\mathrm{rat}} = \left\{ f \in \mathbb{C}[\xi_1, \dots, \xi_{n+m}]^{S_n \times S_m} \;\middle|\; \frac{\partial f}{\partial \xi_i} - \frac{\partial f}{\partial \xi_{n+j}} \in (\xi_i - k\xi_{n+j}), 1 \le i \le n,\ 1 \le j \le m \right\},$$

and the group $S_n \times S_m$ permutes the first n and the last m symbols. In this case, the Harish-Chandra homomorphism can be described by the formulas

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = \xi_i, \quad i = 1, \qquad \varphi\left(\frac{1}{x_i - x_j}\right) = 0, \quad i, j = 1, \dots, n+m.$$

Theorem 2.2. *The operator H_2 is integrable. More precisely, let us define the $(m+n) \times (m+n)$-matrix L by the equalities*

$$L = (L_{ij}), \quad L_{ii} = k^{p(i)} \frac{\partial}{\partial x_i}, \quad L_{ij} = \frac{k^{1-p(j)}}{x_i - x_j}, \ i \neq j.$$

Then the operators

$$L_r = \sum_{ij} (L^r)_{ij} k^{-p(i)} \tag{20}$$

are differential operators of order r and they commute with each other, and we also have $H_2 = L_2$.

Proof. Let us write the Lax equation in this case. We set $\mathfrak{A} = \mathcal{D}_{\mathcal{R}}^{\mathrm{rat}}$, $a = H_2$, and take the matrix

$$M_{ij} = \frac{2k^{1-p(j)}}{(x_i - x_j)^2}, \quad i \neq j, \qquad M_{ii} = -\sum_{j \neq i} \frac{2k^{1-p(j)}}{(x_i - x_j)^2}.$$

as the matrix M. We also set

$$e^* = \left(1, \ \dots, \ \frac{1}{k}, \ \dots, \ \frac{1}{k}\right) \quad (\text{or } e_i^* = k^{-p(i)}, \ i = 1, \dots, m+n),$$
$$e = (1, \ \dots, \ 1) \qquad\qquad (\text{or } e_i = 1, \ i = 1, \dots, n+m).$$

We verify that the element a and the matrices L, M, e^*, and e satisfy the conditions of Theorem 2.1. It is clear that $e^* M = Me = 0$. Let us represent the matrix L in the form $L = \partial + A$, where ∂ is the diagonal matrix with elements

$$\partial_i = k^{p(i)} \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n+m,$$

and rewrite the element H_2 in the form $H_2 = \Delta - f$, where f is a potential. Further, we have

$$[L, aE - M] = -[\partial, fE] + [A, \Delta E] - [\partial, M] - [A, M].$$

We must show that the right-hand side is zero. Let us calculate the nondiagonal elements. Let $i, j \in \{1, \dots, n+m\}$, $i \neq j$. Then it is easy to verify that

$$[\partial, fE]_{ij} = 0, \quad \big([A, \Delta E] - [\partial, M]\big)_{ij} = \frac{2k^{1-p(j)}}{(x_i - x_j)^3}\big(k^{p(i)} - k^{p(j)}\big),$$

$$[A, M]_{ij} = \frac{2k^{1-p(j)}}{(x_i - x_j)^3}\big(k^{1-p(j)} - k^{1-p(i)}\big),$$

and, therefore, $[L, aE - M]_{ij} = 0$. Similarly, we can check that $[L, aE - M]_{ii} = 0$, $i = 1, \dots, n+m$. Therefore, the elements L_r, $r = 1, 2, \dots$, commute with H_2. The Harish-Chandra theorem implies that L_r, $r = 1, 2, \dots$, commute with each other. □

Corollary 2.3. *For a general value of the parameter k, the centralizer $C(H_2)$ is generated by L_r, $r = 1, 2, \dots$, and the restriction of the Harish-Chandra homomorphism to the centralizer is an isomorphism.*

Proof. The image of the integral L_r under the Harish-Chandra homomorphism is

$$\sum_{i=1} \xi_i^r + k^{r-1} \sum_{j=1}^{m} \eta_j^r.$$

For a general value of the parameter k, such elements generate the whole algebra $\Lambda_{\mathcal{R}}^{\mathrm{rat}}$ (see [14]). □

Now we consider the rational CMS operator of the type $BC_{m,n}$. In this case, it has the form

$$H_2 = \left(\frac{\partial^2}{\partial {x_1}^2} + \dots + \frac{\partial^2}{\partial {x_n}^2}\right) + k\left(\frac{\partial^2}{\partial {y_1}^2} + \dots + \frac{\partial^2}{\partial {y_m}^2}\right) -$$
$$- \sum_{i<j}^{n}\left(\frac{2k(k+1)}{(x_i - x_j)^2} - \frac{2k(k+1)}{(x_i + x_j)^2}\right) - \sum_{i<j}^{m}\left(\frac{2(k^{-1}+1)}{(y_i - y_j)^2} - \frac{2(k^{-1}+1)}{(y_i + y_j)^2}\right)$$
$$- \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{2(k+1)}{(x_i - y_j)^2} + \frac{2(k+1)}{(x_i + y_j)^2}\right) - \sum_{i=1}^{n}\frac{q(q+1)}{x_i^2} - \sum_{j=1}^{m}\frac{ks(s+1)}{y_j^2},$$

where the parameters k, q, and s satisfy the relations

$$2q + 1 = k(2s + 1). \tag{21}$$

Again, we set $y_j = x_{n+j}$, $j = 1, \dots, n$. Then the algebra $\mathcal{D}_{\mathcal{R}}^{\mathrm{rat}}$ can be described as the subalgebra generated by differentiations $\dfrac{\partial}{\partial x_i}$, $i = 1, \dots, n+m$, and multiplication operators by the functions $\dfrac{1}{x_i - x_j}$, $\dfrac{1}{x_i + x_j}$, $i \neq j$, and $\dfrac{1}{x_i}$. The algebra $\Lambda_{\mathcal{R}}^{\mathrm{rat}}$ can be described as follows:

$$\Lambda_{\mathcal{R}}^{\mathrm{rat}} = \left\{ f \in \mathbb{C}[\xi_1^2, \dots, \xi_{n+m}^2]^{S_n \times S_m} \;\middle|\; \frac{\partial f}{\partial \xi_i} - \frac{\partial f}{\partial \xi_{n+j}} \in \big(\xi_i - k\xi_{n+j}\big),\ 1 \le i \le n,\ 1 \le j \le m \right\},$$

and the Harish-Chandra homomorphism acts by the formulas

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = \xi_i, \quad \varphi\left(\frac{1}{x_i - x_j}\right) = \varphi\left(\frac{1}{x_i + x_j}\right) = \varphi\left(\frac{1}{x_i}\right) = 0, \quad i, j = 1, \dots, n+m.$$

Theorem 2.4. *The operator H_2 is integrable. More precisely, let us define the $2(m+n) \times 2(m+n)$-matrix L as follows*:

$$L = \begin{pmatrix} L_0 & L_1 \\ -L_1 & -L_0 \end{pmatrix},$$

where

$$(L_0)_{ii} = k^{p(i)}\frac{\partial}{\partial x_i}, \quad (L_1)_{ii} = \frac{k^{p(i)}m(i)}{x_i}; \qquad (L_0)_{ij} = \frac{k^{1-p(j)}}{x_i - x_j}, \quad (L_1)_{ij} = \frac{k^{1-p(j)}}{x_i + x_j}, \quad i \neq j.$$

Then

$$L_{2r} = e^* L^{2r} e \tag{22}$$

are differential operators of the order $2r$ and they commute with each other; moreover, $H_2 = L_2$.

Proof. In this case, the proof is similar to the previous case. Let us indicate the corresponding data for the Lax pair. The matrix L is given above; $e = (1, 1, \dots, 1)^T$; $e_i^* = e_{n+m+i}^* = k^{-p(i)}$, $i = 1, \dots, n+m$; $m(i) = q$ for $i = 1, \dots, n$ and $m(i) = s$ for $i = n+1, \dots, n+m$. Let us define the matrix M by the following formulas:

$$M = \begin{pmatrix} M_0 & M_1 \\ M_1 & M_0 \end{pmatrix},$$

where

$$(M_0)_{ii} = -\sum \frac{2k^{1-p(j)}}{(x_i - x_j)^2} - \sum \frac{2k^{1-p(j)}}{(x_i - x_j)^2} - \sum \frac{m(i)k^{p(i)}}{x_i^2}, \quad (M_1)_{ii} = \frac{k^{p(i)}m(i)}{x_i^2},$$

$$(M_0)_{ij} = \frac{2k^{1-p(j)}}{x_i - x_j}, \quad (M_1)_{ij} = \frac{2k^{1-p(j)}}{(x_i + x_j)^2}, \quad i \neq j.$$

Let us verify that the element a and the matrices L, M, e^*, and e satisfy the conditions of Theorem 2.1. It is clear that $e^*M = Me = 0$. As before, we represent the matrix L in the form

$$L = \begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix} - \begin{pmatrix} A_0 & A_1 \\ -A_1 & -A_0 \end{pmatrix},$$

where ∂ is the diagonal matrix with elements

$$\partial_i = k^{p(i)}\frac{\partial}{\partial x_i}, \quad i = 1, \dots, n+m,$$

and write the element H_2 in the form $H_2 = \Delta - f$, where f is a potential. Again, as before, we have

$$[L, aE - M] = -[\partial, fE] + [A, \Delta E] - [\partial, M] - [A, M].$$

We must show that the right-hand side is zero. This is equivalent to the following two equations:

$$-[\partial, fE] + [A_0, \Delta E] - [\partial, M_0] - [A_0, M_0] - \{A_1, M_1\} = 0,$$
$$[A_1, \Delta E] - \{\partial, M_1\} - [A_1, M_0] - \{A_0, M_1\} = 0,$$

where the brackets $\{\ ,\ \}$ mean the anticommutator. Let us check the first equation. Calculate the nondiagonal elements. Let $i, j \in \{1, \dots, n+m\}$, $i \neq j$. Then

$$[\partial, fE]_{ij} = 0, \quad \big([A_0, \Delta E] - [\partial, M_0]\big)_{ij} = \frac{2k^{1-p(j)}}{(x_i - x_j)^3}\big(k^{p(i)} - k^{p(j)}\big),$$

$$\begin{aligned}
\big([A_0, M_0] + \{A_1, M_1\}\big)_{ij} &= \\
&= \frac{2k^{1-p(j)}}{(x_i - x_j)^3}\big(k^{1-p(j)} - k^{1-p(i)}\big) + \frac{2k^{1-p(j)}}{(x_i - x_j)(x_i + x_j)^2}\big(k^{1-p(j)} - k^{1-p(i)}\big) + \\
&+ \frac{k^{1-p(j)}}{x_i - x_j}\left(\frac{m(i)k^{p(i)}}{x_i^2} - \frac{m(j)k^{p(j)}}{x_j^2}\right) + \frac{2k^{1-p(j)}}{(x_i + x_j)^2}\left(\frac{m(i)k^{p(i)}}{x_i} + \frac{m(j)k^{p(j)}}{x_j}\right) + \\
&+ \frac{k^{1-p(j)}}{(x_i + x_j)}\left(\frac{m(i)k^{p(i)}}{x_i^2} + \frac{m(j)k^{p(j)}}{x_j^2}\right).
\end{aligned}$$

Therefore, in the first equation all nondiagonal elements are zeros. Similarly, we can prove that all diagonal elements in the first equation are zeros. The second equation can be verified in the same way. Therefore, the elements L_r, $r = 1, 2, \ldots$, commute with H_2. Moreover, it is easy to check that $L^{2r+1} = 0$, and Theorem 1.8 implies that the operators L_r, $r = 1, 2, \ldots$, commute with each other. $\square$

Corollary 2.5. *For a general value of the parameter k, the centralizer $C(H_2)$ is generated by operators L_r, $r = 1, 2, \ldots$, and the restriction of the Harish-Chandra homomorphism to the centralizer is an isomorphism.*

Proof. The image of the integral L_r under the Harish-Chandra homomorphism is

$$\sum_{i=1} \xi_i^{2r} + k^{2r-1} \sum_{j=1}^{m} \eta_j^{2r}.$$

For a general value of the parameter k, these elements generate the algebra $\Lambda_{\mathcal{R}}^{\mathrm{rat}}$. $\square$

2.3. Trigonometric case. In this section, we prove the integrability of trigonometric deformed quantum CMS systems corresponding to the classical series of the generalized root systems $A(n,m)$ and $BC(n,m)$. The integrability of the deformed CMS operator is still not proved in the case of the exceptional generalized root systems.

First, consider the trigonometric quantum deformed CMS operator of the type $A(n,m)$. Let us introduce the new variables by the formulas $x_i = e^{\varepsilon_i}$, $i = 1, \ldots, n$, and $y_j = e^{\delta_j}$, $j = 1, \ldots, m$. Then the operator takes the form

$$H_2 = \sum_{i=1}^{n} \left(x_i \frac{\partial}{\partial x_i} \right)^2 + k \sum_{j=1}^{m} \left(y_j \frac{\partial}{\partial y_j} \right)^2 - \\ - \sum_{i<j}^{n} \frac{2k(k+1) x_i x_j}{(x_i - x_j)^2} - \sum_{i<j}^{m} \frac{2(k^{-1}+1) y_i y_j}{(y_i - y_j)^2} - \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{2(k+1) x_i y_j}{(x_i - y_j)^2}. \tag{23}$$

Define $y_j = x_{n+j}$, $j = 1, \ldots, n$; then the algebra $\mathcal{D}_{\mathcal{R}}^{\mathrm{tr}}$ can be described as the algebra generated by differentiations $x_i \dfrac{\partial}{\partial x_i}$, $i = 1, \ldots, n+m$, and multiplication operators by the functions $\dfrac{x_i + x_j}{x_i - x_j}$, $i \neq j$. The algebra $\Lambda_{\mathcal{R}}^{\mathrm{tr}}$ can be described as follows:

$$\Lambda_{\mathcal{R}}^{\mathrm{tr}} = \Bigg\{ f \in \mathbb{C}[\xi_1, \ldots, \xi_{n+m}]^{S_n \times S_m} \ \Bigg| \\ f\left(\xi_i - \frac{1}{2},\ \xi_{n+j} + \frac{1}{2} \right) - f\left(\xi_i + \frac{1}{2},\ \xi_{n+j} - \frac{1}{2} \right) \in (\xi_i - k\xi_{n+j}), \\ 1 \le i \le n,\ 1 \le j \le m \Bigg\},$$

and the Harish-Chandra homomorphism acts by the formulas

$$\varphi\left(x_i \frac{\partial}{\partial x_i} \right) = \xi_i, \quad \varphi\left(\frac{x_i + x_j}{x_i - x_j} \right) = 1, \quad i, j = 1, \ldots, n+m,\ i < j.$$

Theorem 2.6. *The operator H_2 is integrable. Define the $(m+n) \times (m+n)$-matrix L by the formulas*

$$L = (L_{ij}), \quad L_{ii} = k^{p(i)} x_i \frac{\partial}{\partial x_i}, \quad L_{ij} = \frac{1}{2} k^{1-p(j)} \frac{x_i + x_j}{x_i - x_j}, \quad i \neq j.$$

Then

$$L_r = \sum_{i,j} (L^r)_{ij} k^{-p(i)} \tag{24}$$

are differential operators of the order r; they commute with each other and

$$H_2 = L_2 + B(\rho, \rho), \quad \rho = \frac{1}{2} \sum_{\alpha \in R^+} m(\alpha)\alpha.$$

Proof. The proof is the same as above. Let us indicate the corresponding data for the Lax pair: $a = H_2$;

$$M_{ij} = \frac{2k^{1-p(j)} x_i x_j}{(x_i - x_j)^2}, \quad i \neq j, \qquad M_{ii} = -\sum_{j \neq i} \frac{2k^{1-p(j)} x_i x_j}{(x_i - x_j)^2};$$

$e = (1, \dots, 1)^T$; $e_i^* = k^{-p(i)}$, $i = 1, \dots, n+m$. Similarly to the rational case, we can check that $[L, aE - M] = 0$, and by Theorem 2.1 the element L_r commutes with H_2. Theorem 1.8 shows that that the elements L_r, $r = 1, 2, \dots$, commute with each other. $\square$

Corollary 2.7. *For a general value of the parameter k, the centralizer is generated by the elements L_r, $r = 1, 2, \dots$, and the restriction of the Harish-Chandra homomorphism is an isomorphism.*

Proof. The highest term of the image of the integral L_r under the Harish-Chandra homomorphism has the form

$$\sum_{i=1}^{n} \xi_i^r + k^{r-1} \sum_{j=1}^{n} \eta_j^r.$$

Therefore, for a general value of the parameter k, the images of the integrals generate the algebra $\Lambda_{\mathcal{R}}^{\mathrm{tr}}$. $\square$

Now we consider the trigonometric CMS operator of the type $BC_{m,n}$. As in the previous case, we define $x_i = e^{\varepsilon_i}$, $i = 1, \dots, n$, and $y_j = e^{\delta_j}$, $j = 1, \dots, m$. Then the operator can be rewritten in the form

$$\begin{aligned} H_2 = \sum_{i=1}^{n} \left(x_i \frac{\partial}{\partial x_i} \right)^2 + k \sum_{j=1}^{m} \left(y_j \frac{\partial}{\partial y_j} \right)^2 - \sum_{i<j}^{n} \left(\frac{8k(k+1)x_i x_j}{(x_i - x_j)^2} + \frac{8k(k+1)x_i x_j}{(x_i x_j - 1)^2} \right) - \\ - \sum_{i<j}^{m} \left(\frac{8(k^{-1}+1)y_i y_j}{(y_i - y_j)^2} + \frac{8(k^{-1}+1)y_i y_j}{(y_i y_j - 1)^2} \right) - \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{8(k+1)x_i y_j}{(x_i - y_j)^2} - \\ - \sum_{i=1}^{n} \left(\frac{4p(p+2q+1)x_i}{(x_i - 1)^2} + \frac{16q(q+1)x_i^2}{(x_i^2 - 1)^2} \right) - \sum_{j=1}^{m} \left(\frac{4kr(r+2s+1)y_j}{(y_j - 1)^2} + \frac{16ks(s+1)y_j^2}{(y_j^2 - 1)^2} \right), \end{aligned} \tag{25}$$

where the parameters satisfy the relations

$$p = kr, \quad 2q + 1 = k(2s+1).$$

If we define $y_j = x_{n+j}$, $j = 1, \dots, n$, then the algebra $\mathcal{D}_{\mathcal{R}}^{\mathrm{tr}}$ can be described as the algebra generated by differentiations $x_i \dfrac{\partial}{\partial x_i}$, $i = 1, \dots, n+m$, and multiplication operators by the functions $\dfrac{x_i + x_j}{x_i - x_j}$,

$\dfrac{x_ix_j+1}{x_ix_j-1}$, $i\neq j$, $\dfrac{x_i+1}{x_i-1}$, and $\dfrac{x_i^2+1}{x_i^2-1}$. In this case, the algebra $\Lambda_{\mathcal{R}}^{\mathrm{tr}}$ can be described as follows:

$$\Lambda_{\mathcal{R}}^{\mathrm{tr}} = \Bigg\{ f\in\mathbb{C}[\xi_1^2,\dots,\xi_{n+m}^2]^{S_n\times S_m} \ \Bigg| \ f\left(\xi_i-\frac{1}{2},\ \xi_{n+j}+\frac{1}{2}\right)-f\left(\xi_i+\frac{1}{2},\ \xi_{n+j}-\frac{1}{2}\right)\in(\xi_i-k\xi_{n+j}),$$
$$1\leq i\leq n,\ 1\leq j\leq m\Bigg\},$$

and we have the following action of the Harish-Chandra homomorphism:

$$\varphi\left(\frac{\partial}{\partial x_i}\right)=\xi_i,\quad \varphi\left(\frac{x_i+x_j}{x_i-x_j}\right)=1,\quad i<j,\quad \varphi\left(\frac{x_ix_j+1}{x_ix_j-1}\right)=1,$$
$$\varphi\left(\frac{x_i+1}{x_i-1}\right)=1,\qquad \left(\frac{x_i^2+1}{x_i^2-1}\right)=1,\quad i,j=1,\dots,n+m.$$

Theorem 2.8. *The operator H_2 is integrable. In other words, let us define the $2(m+n)\times 2(m+n)$-matrix L by the rule*

$$L=\begin{pmatrix} L_0 & L_1\\ -L_1 & -L_0\end{pmatrix},$$

where

$$(L_0)_{ii}=k^{p(i)}x_i\frac{\partial}{\partial x_i},\quad (L_1)_{ii}=\frac{k^{p(i)}\mu(i)(x_i+1)}{2(x_i-1)}+\frac{k^{p(i)}\nu(i)(x_i^2+1)}{x_i^2-1},$$
$$(L_0)_{ij}=\frac{k^{1-p(j)}(x_i+x_j)}{2(x_i-x_j)},\quad (L_1)_{ij}=\frac{k^{1-p(j)}(x_ix_j+1)}{2(x_ix_j-1)},\quad i\neq j.$$

and $\mu(i)=p$, $\nu(i)=q$, $i=1,\dots,n$, $\mu(i)=r$, $\nu(i)=s$, $i=1,\dots,n+1,n+m$, and e^* and e are the same as in the rational BC case. Then

$$L_{2r}=e^*L^{2r}e$$

are differential operators of the order $2r$ and they commute with each other. We also have the equality

$$H_2=L_2+B(\rho,\rho),\quad \rho=\frac{1}{2}\sum_{\alpha\in R^+}m(\alpha)\alpha.$$

As far as the author knows, the matrix M has never been calculated before, although there is no doubt of its existence. But we will prove this theorem in the next section.

Corollary 2.9. *For a general value of the parameter k, the centralizer is generated by the elements L_{2r}, $r=1,2,\dots,$ and the restriction of the Harish-Chandra homomorphism onto centralizer is an isomorphism.*

Proof. The highest term in the image of the integral L_{2r} under the Harish-Chandra homomorphism is

$$\sum_{i=1}^n\xi_i^{2r}+k^{r-1}\sum_{j=1}^n\eta_j^{2r}.$$

Therefore, for a general value of the parameter k, the images of such integral generate the algebra $\Lambda_{\mathcal{R}}^{\mathrm{tr}}$. $\square$

3. Infinite-Dimensional Quantum CMS Systems

In this section, we define infinite-dimensional analogs of the CMS systems and show their connections to the deformed CMS systems. Under infinite-dimensional operators we mean differential operators on the algebra of symmetric functions.

In this situation, our main tool is the Dunkl operator at infinity. This approach actually does not allow one to get Dunkl operators for deformed CMS systems, but it allows us to explain naturally the appearance of the quantum Moser matrix in the deformed case.

In this section, we also obtain new formulas for quantum integrals at infinity for rational and trigonometric cases of the type A and BC. We note that the same type of approach was used recently in the trigonometric A-case by M. Nazarov and E. Sklynin in [8].

In order to construct infinite-dimensional analogs, we must rewrite the deformed CMS operators in the radial form. This means that instead of the operator L_2 we will use the operator $\mathcal{L}_2 = \Psi_{\mathcal{R}} L_2 \Psi_{\mathcal{R}}^{-1}$. The next proposition can be verified by direct calculations.

Prposition 3.1. *In the rational case, the following equalities hold*:

$$\mathcal{L}_2 = \Psi_{\mathcal{R}}^{\mathrm{rat}} L_2 \left(\Psi_{\mathcal{R}}^{\mathrm{rat}}\right)^{-1} = \Delta - \sum_{\alpha \in R^+} \frac{2m(\alpha)}{\alpha} \partial_\alpha.$$

In the trigonometric case, the following equalities hold:

$$\mathcal{L}_2 = \Psi_{\mathcal{R}}^{\mathrm{tr}} L_2 \left(\Psi_{\mathcal{R}}^{\mathrm{tr}}\right)^{-1} = \Delta - \sum_{\alpha \in R^+} m(\alpha) \frac{1+e^{-\alpha}}{1-e^{-\alpha}} \partial_\alpha$$

We also change the matrix L and the integrals by the formulas

$$\mathcal{L} = \Psi_{\mathcal{R}} L (\Psi_{\mathcal{R}})^{-1}, \quad \mathcal{L}_r = \Psi_{\mathcal{R}} L_r (\Psi_{\mathcal{R}})^{-1}.$$

Recall that the algebra of the symmetric function Λ by definition is the inverse limit of the algebras of the symmetric polynomials (see [7]). This algebra is isomorphic to the algebra of polynomials in infinitely many variables $\Lambda = \mathbb{C}[p_1, p_2, \dots]$, where $p_1, p_2, \dots$ are infinite-dimensional power sums. Let us add to this algebra one additional variable p_0 (infinite sum of zero powers) and denote this algebra by $\bar{\Lambda}$. We will also consider the algebras $\bar{\Lambda}[x]$ and $\bar{\Lambda}[x, x^{-1}]$, which are the algebras of polynomials and Laurent polynomials in one variable. We need the algebras generated by the deformed power sums as well.

Definition 3.2. Denote by $\Lambda_{n,m}$ the subalgebra in the algebra $\mathbb{C}[x_1, \dots, x_{n+m}]$ generated by the deformed power sums

$$p_r(x) = \sum_{i=1}^{n+m} k^{-p(i)} x_i^r = \sum_{i=1}^{n} x_i^r + \frac{1}{k} \sum_{i=n+1}^{n+m} x_i^r, \quad r \in \mathbb{Z}_{\geq 0}.$$

In the same way, denote by $\Lambda_{n,m}^{\pm}$ the subalgebra in the algebra $\mathbb{C}[x_1^{\pm 1}, \dots, x_{n+m}^{\pm 1}]$ generated by the deformed power sums

$$p_r(x) = \sum_{i=1}^{n+m} k^{-p(i)} x_i^r = \sum_{i=1}^{n} x_i^r + \frac{1}{k} \sum_{i=n+1}^{n+m} x_i^r, \quad r \in \mathbb{Z}.$$

3.1. Rational operator of the type A.

Definition 3.3. An *infinite-dimensional Dunkl operator* of the type A is the linear operator on the algebra $\bar{\Lambda}[x]$ defined by the formula

$$D = \partial - k\Delta,$$

where ∂ is a differentiation such that

$$\partial(x) = 1, \quad \partial(p_l) = lx^{l-1},$$
$$\Delta(x^l) = x^{l-1}p_0 + x^{l-2}p_1 + \dots + xp_{l-2} + p_{l-1}l - lx^{l-1}, \quad \Delta(x^l f) = \Delta(x^l)f,$$

where $f \in \Lambda$.

Introduce a linear operator by the formula

$$E : \bar{\Lambda}[x] \to \bar{\Lambda}, \quad E(x^l f) = p_l f, \quad f \in \bar{\Lambda}, \quad l \in \mathbb{Z}_{\geq 0}, \tag{26}$$

and also the operators

$$\mathcal{L}_r^\infty : \bar{\Lambda} \to \bar{\Lambda}, \quad \mathcal{L}_r^\infty = \operatorname{Res} E \circ D^r, \quad r \in \mathbb{Z}_+, \tag{27}$$

where Res means the restriction to $\bar{\Lambda}$. We also define the homomorphisms

$$\varphi_{n,m}^{(i)} : \bar{\Lambda}[x] \to \mathbb{C}[x_1, \dots, x_{n+m}], \quad \varphi_{n,m}^{(i)}(x) = x_i, \quad \varphi_{n,m}^{(i)}(p_l) = p_l(x) \quad \forall l \in \mathbb{Z}_{\geq 0}.$$

By F we denote the column

$$F = \left(\varphi_{n,m}^{(1)}, \ \varphi_{n,m}^{(2)}, \ \dots, \ \varphi_{n,m}^{(n+m)}\right)^T.$$

Prposition 3.4. *On $\bar{\Lambda}[x]$, the following relation holds*:

$$F \circ D = \mathcal{L}F. \tag{28}$$

Proof. It suffices to prove that

$$\varphi_{n,m}^{(i)} \circ D = k^{p(i)} \frac{\partial}{\partial x_i} \circ \varphi_{n,m}^{(i)} - \sum_{j \neq i} \frac{k^{1-p(j)}}{x_i - x_j} (\varphi_{n,m}^{(i)} - \varphi_{n,m}^{(j)}). \tag{29}$$

For any $f \in \bar{\Lambda}$ we have

$$\begin{aligned}
&\varphi_{n,m}^{(i)} \circ (\partial - k\Delta)(x^l f) = \\
&\qquad = \varphi_{n,m}^{(i)}(lx^{l-1}f + x^l \partial f - k(lx^{l-1}p_0 + x^{l-2}p_1 + \dots + xp_{l-2} + p_{l-1} - lx^{l-1})f) = \\
&\qquad = lx_i^{l-1}(1+k)\varphi_{n,m}(f) + x_i^l \varphi_{n,m}(\partial f) - k(x_i^{l-1}p_0 + x_i^{l-2}p_1 + \dots + x_i p_{l-2} + p_{l-1})\varphi_{n,m}^{(i)}(f).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&k^{p(i)} \frac{\partial}{\partial x_i} \circ \varphi_{n,m}^{(i)}(x^l f) - \sum_{j \neq i} \frac{k^{1-p(j)}}{x_i - x_j} \left(\varphi_{n,m}^{(i)} - \varphi_{n,m}^{(j)}\right)(x^l f) = \\
&\quad = k^{p(i)} l x_i^{l-1} \varphi_{n,m}^{(i)}(f) + x_i^l k^{p(i)} \partial_i \big(\varphi_{n,m}^{(i)}(f)\big) - (x_i^{l-1}(kn+m) - \\
&\qquad\qquad - k\Big(x_i^{l-2}p_1 + \dots + x_i p_{l-2} + p_{l-1} - k^{-p(i)} l x_i^{l-1}\Big)\varphi_{n,m}^{(i)}(f) = \\
&\quad = \Bigg(k^{(p(i)} + k^{1-p(i)}) l x_i^{l-1} \varphi_{n,m}^{(i)}(f) + k^{p(i)} x_i^l \partial_i \big(\varphi_{n,m}^{(i)}(f)\big) - \\
&\qquad\qquad - k\Big(x_i^l(n + k^{-1}m) + x_i^{l-2}p_1 + \dots + x_i p_{l-2} + p_{l-1}\Big)\Bigg)\varphi_{n,m}^{(i)}(f).
\end{aligned}$$

Since $k^{p(i)} + k^{1-p(i)} = 1 + k$ for all $i = 1, \dots, n+m$, it remains to prove that

$$\varphi_{n,m}^{(i)}(\partial f) = k^{p(i)} \partial_i \varphi_{n,m}^{(i)}(f). \tag{30}$$

Since both ∂ and ∂_i are differentiations, it suffices to verify the last equality for $f = p_l$; this verification is easy. $\square$

The assertion below explains the name and the formulas for infinite-dimensional Dunkl operators.

Corollary 3.5. *Let $m = 0$. Then the following diagram is commutative:*

$$\begin{array}{ccc} \bar{\Lambda}[x] & \xrightarrow{D} & \bar{\Lambda}[x] \\ \varphi_{n,0}^{(i)} \downarrow & & \downarrow \varphi_{n,0}^{(i)} \\ \Lambda_{n,0}[x_i] & \xrightarrow{D_i} & \Lambda_{n,0}[x_i] \end{array}$$

where D_i are the Dunkl operators

$$D_i = \frac{\partial}{\partial x_i} - k \sum_{j \neq i} \frac{1}{x_i - x_j}(1 - s_{ij}),$$

and s_{ij} is a transposition.

Proof. Let $m = 0$; then the formula 29 can be represented in the form

$$\varphi_{n,0}^{(i)} \circ D = \frac{\partial}{\partial x_i} \circ \varphi_{n,0}^{(i)} - k \sum_{j \neq i} \frac{1}{x_i - x_j}(1 - s_{ij}) \circ \varphi_{n,0}^{(i)}. \tag{31}$$

□

Now we are ready to formulate the main result.

Theorem 3.6. *Differential operators $\mathcal{L}_r^\infty$ commute with each other:*

$$[\mathcal{L}_r^\infty, \mathcal{L}_s^\infty] = 0,$$

and the operator $\mathcal{L}_2^\infty$ has the following form:

$$\mathcal{L}_2^\infty = \sum_{a,b \geq 1} p_{a+b-2} \partial_a \partial_b - k \sum_{a,b \geq 0} p_a p_b \partial_{a+b+2} + (1 + k) \sum_{a \geq 2} (a - 1) p_{a-2} \partial_a, \tag{32}$$

where $\partial_a = a\partial/\partial p_a$,.

Proof. In order to show that $\mathcal{L}_r^\infty$ are differential operators of order $\leq r$, recall that $\mathcal{M}$ is a differential operator on $\bar{\Lambda}$ of order $\leq r$ if $\mathrm{ad}(f)^{r+1}\mathcal{M} = 0$ for any $f \in \bar{\Lambda}$. Since E and Δ commute with multiplication by f, we have

$$\mathrm{ad}(f)^{r+1}\big(\mathcal{L}_r^\infty\big) = \mathrm{Res}\, E \circ \mathrm{ad}(f)^{r+1}(D^r).$$

Since

$$\mathrm{ad}(f)(D) = ad(f)(\partial) = -\partial f,$$

we have $\mathrm{ad}(f)^2(D) = 0$ and, therefore, $\mathrm{ad}(f)^{r+1}(D^r) = 0$. In order to prove the commutativity in the infinite-dimensional situation, let us reduce the proof to the finite-dimensional case. We have the following commutative diagram:

$$\begin{array}{ccc} \bar{\Lambda} & \xrightarrow{\mathcal{L}_r^\infty} & \bar{\Lambda} \\ \varphi_{n,m} \downarrow & & \downarrow \varphi_{n,m} \\ \Lambda_{n,m} & \xrightarrow{\mathcal{L}_r} & \Lambda_{n,m} \end{array} \tag{33}$$

where $\varphi_{n,m}(p_r) = p_r(x)$. Indeed, it is easy to verify that $e^*F = \varphi_{n,m} \circ E$, and after the restriction to $\bar{\Lambda}$ we have the equality $F = e\varphi_{n,m}$. Therefore, $F \circ D^r = \mathcal{L}^r F$ and after the restriction to $\bar{\Lambda}$ we have

$$e^*F \circ D^r = \varphi_{n,m} \circ E \circ D^r = e^*\mathcal{L}^r e\varphi_{n,m}.$$

This proves the commutativity. Since the integrals commute, we have

$$\varphi_{n,m}\Big(\big[\mathcal{L}_r^\infty, \mathcal{L}_s^\infty\big](f)\Big) = \big[\mathcal{L}_r, \mathcal{L}_s\big](\varphi_{n,m}(f)) = 0,$$

In order to to complete the proof, we need the following Lemma.

Lemma 3.7. *Let $f \in \bar{\Lambda}$. If $\varphi_{N,0}(f) = 0$ for all N, then $f = 0$.*

Proof. By definition, f is a polynomial in a finite number of variables p_r, $1 \le r \le M$, for some natural number M, and the coefficients of f polynomially depend on p_0.

Take N greater than M. Since $\varphi_{N,0}(p_r)$ are algebraically independant for $1 \le r \le M$ in $\Lambda_{N,0}$ and $\varphi_{N,0}(f) = 0$, all coefficients of f are zeros for $p_0 = N$. Since this holds for all $N > M$, the coefficients are identically zeros. Therefore, $f = 0$. This proves the lemma. □

Now, applying Lemma 3.7, we obtain the commutativity. □

3.2. Rational operator of the type B. The proofs in this case are the same as in the case A and so we omit them.

Let us define an *infinite-dimensional Dunkl operator of the type BC* as the operator

$$D : \bar{\Lambda}[x] \to \bar{\Lambda}[x], \quad D = \partial - 2k\Delta - \frac{q}{x}(1 - \tau). \tag{34}$$

Here the differentiation ∂ in $\bar{\Lambda}[x]$ are defined by the formulas

$$\partial(x) = 1, \quad \partial(p_l) = 2lx^{2l-1}, \quad l \in \mathbb{Z}_{\ge 0},$$

and the operator $\Delta : \bar{\Lambda}[x] \to \bar{\Lambda}[x]$ is defined by the formula

$$\Delta(x^l f) = \Delta(x^l)f, \quad \Delta(1) = 0, \quad f \in \bar{\Lambda}, \quad l \in \mathbb{Z}_{\ge 0},$$

where

$$\begin{aligned}\Delta(x^{2l}) &= x^{2l-1}p_0 + x^{2l-3}p_1 + \dots + x^3 p_{l-2} + xp_{l-1} - lx^{2l-1},\\ \Delta(x^{2l-1}) &= x^{2l-2}p_0 + x^{2l-4}p_1 + \dots + x^2 p_{l-2} + p_{l-1} - lx^{2l-2}, \quad l > 0;\end{aligned}$$

the involution τ can be defined by the formula

$$\tau(x^l f) = (-x)^l f, \quad f \in \bar{\Lambda}.$$

Introduce also the linear operator

$$E : \bar{\Lambda}[x] \to \bar{\Lambda} E(x^{2l} f) = p_l f, \quad E(x^{2l+1} f) = 0, \quad f \in \bar{\Lambda}, \quad l \in \mathbb{Z}_{\ge 0},$$

and the operators

$$\mathcal{L}_r^{\infty} : \bar{\Lambda} \to \bar{\Lambda}, \quad r \in \mathbb{Z}_+, \quad \mathcal{L}_r^{\infty} = \operatorname{Res} E \circ D^{2r}, \tag{35}$$

where, as above, Res means the restriction to $\bar{\Lambda}$. In this case, the homomorphisms $\varphi_{n,m}^{(i)} : \bar{\Lambda}[x] \to \mathbb{C}[x_1, \dots, x_{n+m}]$ act by the rule

$$\varphi_{n,m}^{(i)}(x) = x_i, \quad \varphi_{n,m}^{(i)}(p_r) = p_{2r}(x) \quad \forall r \in \mathbb{Z}_{\ge 0}.$$

In this case, the column F has the form

$$F = \left(\varphi_{n,m}^{(1)}, \ \dots, \ \varphi_{n,m}^{(n+m)}, \ \tau_1\varphi_{n,m}^{(1)}, \ \dots, \ \tau_{n+m}\varphi_{n,m}^{(n+m)}\right)^T,$$

where τ_i changes the sign of the variables x_i.

Prposition 3.8. *On $\bar{\Lambda}[x]$, the following relation holds*:

$$F \circ D = \mathcal{L}F. \tag{36}$$

Corollary 3.9. *Let $m = 0$. Then the following diagram is commutative*:

$$\begin{array}{ccc} \bar{\Lambda}[x] & \xrightarrow{D} & \bar{\Lambda}[x] \\ {\scriptstyle \varphi_{n,0}^{(i)}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_{n,0}^{(i)}} V, \\ \Lambda_{n,0}[x_i] & \xrightarrow{D_i} & \Lambda_{n,0}[x_i] \end{array}$$

where D_i are the Dunkl operators,

$$\frac{\partial}{\partial x_i}-\sum_{j\neq i}\frac{1}{x_i-x_j}(1-s_{ij})\sum_{j\neq i}\frac{1}{x_i-x_j}\big(1-\tau_i\tau_j s_{ij}\big)-\frac{p}{x_i}(1-\tau_i),$$

s_{ij} is a transposition, and τ_i changes the sign of x_i.

Proof. The relation (36) can be rewritten in the form

$$\varphi^{(i)}_{n,m}\circ D_\infty=k^{p(i)}\frac{\partial}{\partial x_i}\circ\varphi^{(i)}_{n,m}-\frac{k^{p(i)}m(i)}{x_i}(1-\tau_i)\varphi^{(i)}_{n,m}-\\-\sum_{j\neq i}\frac{k^{1-p(j)}}{x_i-x_j}\Big(\varphi^{(i)}_{n,m}-\varphi^{(j)}_{n,m}\Big)-\sum_{j\neq i}\frac{k^{1-p(j)}}{x_i+x_j}\Big(\varphi^{(i)}_{n,m}-\tau_j\varphi^{(j)}_{n,m}\Big).$$

If $m=0$, then the previous formula can be represented in the form

$$\varphi^{(i)}_{n}\circ D=\left(\frac{\partial}{\partial x_i}-\sum_{j\neq i}\frac{1}{x_i-x_j}(1-s_{ij})\sum_{j\neq i}\frac{1}{x_i-x_j}\big(1-\tau_i\tau_j s_{ij}\big)-\frac{p}{x_i}(1-\tau_i)\right)\circ\varphi^{(i)}_{n,m}.$$

□

As above, we have the following main result.

Theorem 3.10. *Differential operators $\mathcal{L}^\infty_r$ commute with each other:*

$$\Big[\mathcal{L}^\infty_r,\ \mathcal{L}^\infty_s\Big]=0.$$

and the operator $\mathcal{L}^\infty_2$ has the following explicit form:

$$\mathcal{L}^\infty_2=8\sum_{a,b\geq 1}p_{a+b-1}\partial_a\partial_b-4k\sum_{a,b\geq 0}p_ap_b\partial_{a+b+1}+4k\sum_{a\geq 0}(a+1)p_a\partial_{a+1}+\\+2\sum_{a\geq 0}(2a+1)p_a\partial_{a+1}-4q\sum_{a\geq 0}p_a\partial_{a+1},\quad(37)$$

where $\partial_a=a\partial/\partial p_a$.

3.3. Trigonometric operator of the type A.

It is easy to verify that

$$\mathcal{L}_2=\sum_{i=1}^n\left(x_i\frac{\partial}{\partial x_i}\right)^2+k\sum_{j=1}^m\left(y_j\frac{\partial}{\partial y_j}\right)^2-k\sum_{1\leq i<j\leq n}\frac{x_i+x_j}{x_i-x_j}\left(x_i\frac{\partial}{\partial x_i}-x_j\frac{\partial}{\partial x_j}\right)-\\-\sum_{1\leq i<j\leq m}\frac{y_i+y_j}{y_i-y_j}\left(y_i\frac{\partial}{\partial y_i}-y_j\frac{\partial}{\partial y_j}\right)-\sum_{i=1}^n\sum_{j=1}^m\frac{x_i+y_j}{x_i-y_j}\left(x_i\frac{\partial}{\partial x_i}-ky_j\frac{\partial}{\partial y_j}\right).$$

Definition 3.11. Let us define an infinite-dimensional trigonometric Dunkl operator of the type A by the formula

$$D=\partial-\frac{1}{2}k\Delta,$$

where ∂ is a differential operator such that

$$\partial(x)=x,\quad\partial(p_r)=rx^r,$$

and

$$\Delta(x^l)=x^lp_0+2x^{l-1}p_1+\cdots+2xp_{l-1}+p_ll-2lx^l,\quad\Delta(x^lf)=\Delta(x^l)f,\quad\Delta(1)=0,$$

where $f\in\Lambda$.

The operator $E : \bar{\Lambda}[x] \to \bar{\Lambda}$ can be defined by the formula

$$E(x^l f) = p_l f, \quad f \in \bar{\Lambda}, \quad l \in \mathbb{Z}_{\geq 0}, \tag{38}$$

and the operators $\mathcal{L}_r^\infty : \bar{\Lambda} \to \bar{\Lambda}$, $r \in \mathbb{Z}_+$, by the formulas

$$\mathcal{L}_r^\infty = \operatorname{Res} E \circ D^r, \tag{39}$$

where Res means the restriction to $\bar{\Lambda}$. The homomorphisms $\varphi_{n,m}^{(i)}$ and the column F are defined exactly as in the rational case of the type A.

Prposition 3.12. *We have the following relation on* $\bar{\Lambda}[x]$:

$$F \circ D = \mathcal{L} F. \tag{40}$$

Corollary 3.13. *Let* $m = 0$. *Then the following diagram is commutative*:

$$\begin{array}{ccc} \bar{\Lambda}[x] & \xrightarrow{D} & \bar{\Lambda}[x] \\ \varphi_{n,0}^{(i)} \big\downarrow & & \big\downarrow \varphi_{n,0}^{(i)} \\ \Lambda_{n,0}[x_i] & \xrightarrow{D_i} & \Lambda_{n,0}[x_i], \end{array}$$

where D_i *are the Dunkl–Heckman operators*

$$D_i = x_i \frac{\partial}{\partial x_i} - \frac{1}{2} k \sum_{j \neq i} \frac{x_i + x_j}{x_i - x_j}(1 - s_{ij})$$

and s_{ij} *is a transposition.*

Theorem 3.14 (see [1, 23, 26])**.** *The differential operators* $\mathcal{L}^{(r)}$ *commute with each other*:

$$\left[\mathcal{L}_r^\infty,\ \mathcal{L}_s^\infty\right] = 0.$$

The operator $\mathcal{L}_2^\infty$ *has the following explicit form*:

$$\mathcal{L}_2^\infty = \sum_{a,b>0} p_{a+b} \partial_a \partial_b - k \sum_{a,b>0} p_a p_b \partial_{a+b} + (1+k) \sum_{a>0} a p_a \partial_a - k p_0 \sum_{a>0} p_a \partial_a, \tag{41}$$

where $\partial_a = a\partial/\partial p_a$.

3.4. Trigonometric operator of the type B. In this case,

$$\begin{aligned} \mathcal{L}_2^{(n,m)} = {} & \sum_{i=1}^n \partial_i^2 + k \sum_{\alpha=1}^m \partial_\alpha^2 - k \sum_{i<j}^n \left(\frac{x_i + x_j}{x_i - x_j}(\partial_i - \partial_j) + \frac{x_i x_j + 1}{x_i x_j - 1}(\partial_i + \partial_j) \right) - \\ & - \sum_{\alpha<\beta}^m \left(\frac{y_\alpha + y_\beta}{y_\alpha - y_\beta}(\partial_\alpha - \partial_\beta) + \frac{y_\alpha y_\beta + 1}{y_\alpha y_\beta - 1}(\partial_\alpha + \partial_\beta) \right) - \sum_{i=1}^n \left(p \frac{x_i + 1}{x_i - 1} + 2q \frac{x_i^2 + 1}{x_i^2 - 1} \right) \partial_i - \\ & - k \sum_{\alpha=1}^m \left(r \frac{y_\alpha + 1}{y_\alpha - 1} + 2s \sum_{\alpha=1}^n \frac{y_\alpha^2 + 1}{y_\alpha^2 - 1} \right) \partial_\alpha - \sum_{i=1}^n \sum_{\alpha=1}^m \left(\frac{x_i + y_\alpha}{x_i - y_\alpha}(\partial_i - k\partial_\alpha) + \frac{x_i y_\alpha + 1}{x_i y_\alpha - 1}(\partial_i + k\partial_\alpha) \right). \end{aligned}$$

Definition 3.15. Let us define the infinite-dimensional trigonometric Dunkl operator of the type B by the formula

$$D = \partial - \frac{1}{2} k \Delta - \frac{1}{2} p \frac{x+1}{x-1}(1-\theta) - \frac{1}{2} p \frac{x^2+1}{x^2-1}(1-\theta),$$

where ∂ is a differential operator such that

$$\partial(x) = x, \quad \partial(p_r) = r(x^r - x^{-r}), \quad r \in \mathbb{Z}_{\geq 0},$$

and the homomorphisms Δ and θ of the $\bar{\Lambda}$-modules are defined by the formulas $\Delta(1) = 0$,

$$\Delta(x^l) = x^r(p_0 - 2r - 1) - 2\sum_{j=1}^{r-1} x^{r-2j} - x^{-r} + 2\sum_{j=1}^{r-1} p_j x^{r-j} + p_r,$$

$$\Delta(x^{-r}) = x^{-r}(p_0 - 2r - 1) + 2\sum_{j=1}^{r-1} x^{r-2j} + x^r - 2\sum_{j=1}^{r-1} p_j x^{-r+j} - p_r,$$

where $r > 0$ and

$$\theta(x) = x^{-1}, \quad \theta(p_r) = p_r.$$

We also introduce the linear operator $E : \bar{\Lambda}[x] \to \bar{\Lambda}$ by the formula

$$E(x^r f) = p_{|r|} f, \quad f \in \bar{\Lambda}, \quad r \in \mathbb{Z}, \tag{42}$$

and define the operators

$$\mathcal{L}_r^\infty : \bar{\Lambda} \to \bar{\Lambda}, \quad \mathcal{L}_r^\infty = \operatorname{Res} E \circ D^{2r}, \quad r \in \mathbb{Z}_+, \tag{43}$$

where Res means the restriction to $\bar{\Lambda}$. We also define the homomorphisms

$$\varphi_{n,m}^{(i)} : \bar{\Lambda}[x] \to \mathbb{C}[x_1^{\pm 1}, \dots, x_{n+m}^{\pm 1}]$$

by the rule

$$\varphi_{n,m}^{(i)}(x) = x_i, \quad \varphi_{n,m}^{(i)}(p_r) = p_r(x) + p_{-r}(x) \quad \forall r \in \mathbb{Z}_{\geq 0}.$$

The column F has the same form as in the rational BC case:

$$F = \left(\varphi_{n,m}^{(1)},\ \varphi_{n,m}^{(2)},\ \dots,\ \varphi_{n,m}^{(n+m)},\ \tau_1\varphi_{n,m}^{(1)},\ \tau_2\varphi_{n,m}^{(2)},\ \dots,\ \tau_{n+m}\varphi_{n,m}^{(n+m)}\right)^T.$$

Prposition 3.16. *On $\bar{\Lambda}[x]$, the following relation holds*:

$$F \circ D = \mathcal{L} F. \tag{44}$$

The following assertion explains the name and formulas for the infinite-dimensional Dunkl operators.

Corollary 3.17. *Let $m = 0$. Then the following diagram is commutative:*

$$\begin{array}{ccc} \bar{\Lambda}[x, x^{-1}] & \xrightarrow{D} & \bar{\Lambda}[x, x^{-1}] \\ \varphi_{n,0}^{(i)} \Big\downarrow & & \Big\downarrow \varphi_{n,0}^{(i)} \\ \Lambda_{n,0}[x_i, x_i^{-1}] & \xrightarrow{D_i} & \Lambda_{n,0}[x_i, x_i^{-1}], \end{array}$$

where D_i are trigonometric Dunkl-Heckman operators of the type BC:

$$D_i = \partial_i - \frac{1}{2}k\sum_{j\neq i}^{N}\left(\frac{x_i + x_j}{x_i - x_j}(1 - s_{ij}^+) + \frac{x_i x_j + 1}{x_i x_j - 1}(1 - s_{ij}^-)\right) - \frac{1}{2}p\frac{x_i + 1}{x_i - 1}(1 - t_i) - q\frac{x_i^2 + 1}{x_i^2 - 1}(1 - t_i);$$

here s_{ij} is a transposition, $t_i(x_i) = x_i^{-1}$, and $t_i(x_j) = x_j$, $j \neq i$.

Proof. The proof easily follows from the explicit formula for the Lax matrix. □

Theorem 3.18. *The differential operators $\mathcal{L}_r^\infty$ commute with each other*:

$$\left[\mathcal{L}_r^\infty,\ \mathcal{L}_s^\infty\right] = 0,$$

The operator $\mathcal{L}^{\infty_2}$ has the following explicit form:

$$\mathcal{L}_2^\infty = 4\sum_{a,b\geq 1}(p_{a+b}-p_{a-b})\partial_a\partial_b + 2\sum_{a\geq 1}(ak+a+k+h)p_a\partial_a +$$
$$+\,2(k-q)\sum_{a\geq 2}\left(\sum_{j=1}^{a-1}p_{a-2j}\right)\partial_a - p\sum_{a\geq 2}\left(\sum_{j=1}^{2a-1}p_{a-j}\right)\partial_a - 2k\sum_{a\geq 2}\left(\sum_{j=1}^{a-1}p_jp_{a-j}\right)\partial_a.$$

4. Algebras of Integrals and Spectral Decomposition

4.1. Trigonometric operator of the type A. In this section, we examine the action of the trigonometric algebra of integrals of the deformed quantum CMS problem. We mainly follow the paper [24].

Assume that k is a nonzero complex number. Denote by $\mathfrak{D}_{n,m}$ the subalgebra in the algebra $\mathcal{D}^{\mathrm{tr}}_{\mathcal{R}}$ generated by the integrals $\mathcal{L}_r$, $r = 1, 2, \dots$.

We examine the action of the algebra $\mathfrak{D}_{n,m}$ on the algebra $\mathfrak{A}_{n,m}$ of $(S_n \times S_m)$-invariant Laurent polynomials

$$f \in \mathbb{C}\big[x_1^{\pm 1},\ \dots,\ x_n^{\pm 1},\ y_1^{\pm 1},\ \dots,\ y_m^{\pm 1}\big]^{S_n\times S_m}$$

and satisfying the quasi-invariance condition

$$x_i\frac{\partial f}{\partial x_i} - ky_j\frac{\partial f}{\partial y_j} \in (x_i - y_j), \quad i = 1,\dots,n, \quad j = 1,\dots,m. \tag{45}$$

For any Laurent polynomial

$$f = \sum_{\mu\in X_{n,m}} c_\mu x^\mu, \quad X_{n,m} = \mathbb{Z}^n \oplus \mathbb{Z}^m,$$

consider the set $M(f)$ consisting of μ such that $c_\mu \neq 0$ and define the *support* $S(f)$ as the intersection of convex hull of $M(f)$ with $X_{n,m}$. It turns out that the algebra $\mathfrak{D}_{n,m}$ maps the algebra $\mathfrak{A}_{n,m}$ into itself.

Theorem 4.1. *The operators $\mathcal{L}_p$, $p = 1, 2, \dots$, preserve the algebra $\mathfrak{A}_{n,m}$ and the support: for any $D \in \mathfrak{D}_{n,m}$ and $f \in \mathfrak{A}_{n,m}$,*

$$S(Df) \subseteq S(f).$$

Corollary 4.2. *Denote by $\mathfrak{A}^+_{n,m}$ the subalgebra in the algebra $\mathfrak{A}_{n,m}$ consisting of ordinary polynomials. Then the operators $\mathcal{L}_p$, $p = 1, 2, \dots$, map the algebra $\mathfrak{A}^+_{n,m}$ into itself.*

In order to study the structure of $\mathfrak{A}_{n,m}$ and $\mathfrak{A}^+_{n,m}$ as modules over $\mathfrak{D}_{n,m}$, we need the following partial order on the set of weights $\lambda \in X_{n,m} = \mathbb{Z}^{n+m}$. We say that $\mu \preceq \lambda$ if and only if

$$\mu_1 \leq \lambda_1, \quad \mu_1+\mu_2 \leq \lambda_1+\lambda_2, \quad \dots, \quad \mu_1+\dots+\mu_{n+m} \leq \lambda_1+\dots+\lambda_{n+m}. \tag{46}$$

Prposition 4.3. *Let $f \in \mathfrak{A}_{n,m}$ and λ be a maximal element of $M(f)$ with respect to partial order. Then for any $D \in \mathfrak{D}_{n,m}$, there is no μ from $M(D(f))$, $\mu \neq \lambda$, such that $\lambda \preceq \mu$.*

The coefficient of x^λ in $D(f)$ is $\varphi(D)(\lambda)c_\lambda$, where c_λ is the coefficient of x^λ in f.

If λ is the unique maximal element of $M(f)$, then $\mu \preceq \lambda$ for any μ from $M(D(f))$.

Let $\chi : \mathfrak{D}_{n,m} \to \mathbb{C}$ be a homomorphism. Define the corresponding *generalized eigenspace* $\mathfrak{A}_{n,m}(\chi)$ as the set of all $f \in \mathfrak{A}_{n,m}$ such that for every $D \in \mathfrak{D}_{n,m}$, there exists $N \in \mathbb{N}$ such that $(D-\chi(D))^N(f) = 0$. If the dimension of $\mathfrak{A}_{n,m}(\chi)$ is finite, then such N can be chosen independent of f.

Prposition 4.4. *The algebra $\mathfrak{A}_{n,m}$ as a module over the algebra $\mathfrak{D}_{n,m}$ can be decomposed in the direct sum of generalized eigenspaces*

$$\mathfrak{A}_{n,m} = \oplus_\chi \mathfrak{A}_{n,m}(\chi), \tag{47}$$

where the sum is taken over the set of some homomorphisms χ (explicitly described below).

Proof. Let $f \in \mathfrak{A}_{n,m}$. We define the vector space

$$V(f) = \Big\{ g \in \mathfrak{A}_{n,m} \ \Big|\ S(g) \subseteq S(f) \Big\}.$$

By Theorem 4.1, $V(f)$ is a finite-dimensional module over $\mathfrak{D}_{n,m}$. Since the proposition is valid, the claim is also valid for all finite-dimensional modules. $\square$

Now we describe all homomorphisms χ such that $\mathfrak{A}_{n,m}(\chi) \neq 0$. We say that the integral weight $\lambda \in X_{n,m} \in \mathbb{Z}^{n+m}$ is *dominant* if

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \quad \lambda_{n+1} \geq \lambda_{n+2} \geq \cdots \geq \lambda_{n+m}.$$

The set of dominant weights is denoted $X^+_{n,m}$. For every $\lambda \in X^+_{n,m}$, we define the homomorphism

$$\chi_\lambda : \mathfrak{D}_{\mathcal{R}} \to \mathbb{C}, \quad \chi_\lambda(D) = \varphi(D)(\lambda), \quad D \in \mathfrak{D}_{\mathcal{R}},$$

where φ is the Harish-Chandra homomorphism.

Prposition 4.5.

(1) *For any $\lambda \in X^+_{n,m}$, there exists χ and $f \in \mathfrak{A}_{n,m}(\chi)$, which has a unique maximal term x^λ.*
(2) *$\mathfrak{A}_{n,m}(\chi) \neq 0$ if and only if there exists $\lambda \in X^+_{n,m}$ such that $\chi = \chi_\lambda$.*
(3) *If $\mathfrak{A}_{n,m}(\chi)$ is finite-dimensional, then its dimension is equal to the number of $\lambda \in X^+_{n,m}$ such that $\chi_\lambda = \chi$.*

Proof. Let $\mu_1 = \lambda_1, \ldots, \mu_n = \lambda_n$, $\nu_1 = \lambda_{n+1}, \ldots, \nu_m = \lambda_{n+m}$. Consider the Laurent polynomial

$$g(x, y) = s_\mu(x) s_\nu(y) \prod_{i,j} \left(1 - \frac{y_j}{x_i}\right)^2,$$

where $s_\mu(x)$ and $s_\nu(y)$ are the Schur polynomials (see [7]). It is easy to verify that g belongs to the algebra $\mathfrak{A}_{n,m}$ and has a unique maximal weight λ. By Proposition 4.4, we can write $g = g_1 + \cdots + g_N$, where g_i belong to different generalized eigenspaces. Therefore, there exists i such that $\lambda \in M(g_i)$. Since g_i can be obtained from g by some element from the algebra $\mathfrak{A}_{n,m}$ (which is the projector to the corresponding generalized eigenspace in some finite-dimensional subspace containing g), then λ is the only maximal element of $M(g_i)$ by Proposition 4.3. This proves the first part.

Let $\mathfrak{A}_{n,m}(\chi) \neq 0$. Take a nonzero element f from this subspace and choose some maximal element $\lambda^{(1)}$ from $M(f)$ and an operator $D \in \mathfrak{D}_{n,m}$. Then, according to Proposition 4.3, the element $x^{\lambda^{(1)}}$ does not belong in $f_1 = (D - \chi_{\lambda^{(1)}}(D))(f)$ and $S(f_1) \subset S(f)$. Repeating this procedure, we obtain a sequence of nonzero elements $f_0 = f$, f_1, …, f_N and numbers $a_1 = \chi_{\lambda^{(1)}}(D)$, …, $a_N = \chi_{\lambda^{(N)}}(D)$ such that

$$f_i = (D - a_i) f_{i-1}, \quad i = 1, \ldots, N, \quad (D - a_N) f_{N-1} = 0.$$

Therefore,

$$P(t) = \prod_{i=1}^{N} (t - a_i)$$

is a minimal polynomial for D in the subspace $\langle f_0, \ldots, f_{N-1} \rangle$. But this subspace lies in $\mathfrak{A}_{n,m}(\chi)$. Therefore, this polynomial is a power of $t - \chi(D)$ and hence $a_1 = a_2 = \cdots = a_N = \chi(D)$. In particular, this implies that $\chi(D) = a_1 = \chi_{\lambda^{(1)}}(D)$ for some $\lambda^{(1)} \in X^+_{n,r}$, as required.

Conversely, let $\lambda \in X^+_{n,m}$. According to the first part, there exists χ and $f \in \mathfrak{A}_{n,m}(\chi)$ such that λ is its maximal weight. Therefore, the previous considerations show that $\chi = \chi_\lambda$ and thus $\mathfrak{A}_{n,m}(\chi_\lambda) \neq 0$.

To prove the third part, assume that $\mathfrak{A}_{n,m}(\chi)$ is finite-dimensional and $\lambda^{(1)}, \ldots, \lambda^{(N)}$ are all different elements from $X^+_{n,m}$ such that $\chi_{\lambda^{(i)}} = \chi$, $i = 1, \ldots, N$. According to the first two parts, there exists $f_i \in \mathfrak{A}_{n,m}(\chi)$ with a unique maximal weight $\lambda^{(i)}$. It is easy to see that $f_1, \ldots, f_N$ are linearly independent. To show that they form a basis, we consider any $f \in \mathfrak{A}_{n,m}(\chi)$ and take a maximal weight μ from $M(f)$. According to Proposition 4.3, $\chi_\mu = \chi$ and thus μ must coincide with one of $\lambda^{(i)}$. Subtracting from f a suitable multiple of f_i and using induction, we arrive at the required result. □

Corollary 4.6.

(1) The set of homomorphisms in Proposition 4.4 *consists of* $\chi = \chi_\lambda$, $\lambda \in X^+_{n,m}$.

(2) For a general value of k, *we have*

$$\mathfrak{A}^+_{n,m} = \oplus_\chi \mathfrak{A}^+_{n,m}(\chi), \tag{48}$$

where the sum is taken over the subset χ_λ *such that*

$$\lambda_1 \geq \ldots \lambda_n \geq 0, \quad \lambda_{n+1} \geq \ldots \lambda_{n+m} \geq 0, \quad \lambda_n \leq \left|\{i \mid \lambda_{n+i} > 0\}\right|$$

and all the corresponding eigenspaces are one-dimensional.

The eigenfunctions from Corollary 4.6 are called *Jack superpolynomials* (see [15]). In the general case of the algebra $\mathfrak{A}_{n,m}$, the corresponding eigenfunctions are called *Jack–Laurent superpolynomials*.

Consider several examples.

Example 4.7. Let $n = m = 1$,

$$\mathfrak{A}_{1,1} = \left\{ f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \;\middle|\; \partial_x f - k\partial_y f \in (x-y) \right\}.$$

For integers (λ, μ), we set

$$P_{\lambda,\mu} = x^\lambda y^\mu - \frac{\lambda - k\mu}{\lambda - 1 - k(\mu+1)} x^{\lambda-1} y^{\mu+1}, \quad (\lambda,\mu) \neq (0,0),\ (1,-1),$$

$$P_{0,0} = 1, \quad P_{1,-1} = \frac{x}{y} + \frac{y}{x}.$$

So we have the decomposition into generalized eigenspaces

$$\mathfrak{A}_{1,1} = \bigoplus_{(\lambda,\mu)\neq(0,0),(1,1)} \langle P_{\lambda,\mu} \rangle \oplus \left\langle P_{(0,0)},\ P_{(1-1)} \right\rangle.$$

It is difficult to verify that the image of the algebra of integrals in generalized eigenspace $\left\langle P_{(0,0)}, \P_{(1-1)} \right\rangle$ is the algebra of dual numbers $\mathbb{C}[\varepsilon]$, $\varepsilon^2 = 0$.

If $k \in \mathbb{Q}$, the decomposition may have another form.

Example 4.8. Let $k = -1$; then

$$\mathfrak{A}_{1,1} = \left\{ f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \;\middle|\; \partial_x f + \partial_y f \in (x-y) \right\}.$$

This is a particular case of the algebra of supersymmetric polynomials. In this case,

$$\mathfrak{A}_{1,1} = \bigoplus_{\lambda+\mu\neq 0} \langle P_{\lambda,\mu} \rangle \oplus \left\langle \frac{x^a}{y^a},\ a \in \mathbb{Z} \right\rangle,$$

and all generalized eigenspaces are simply eigenspaces. One of them is infinite-dimensional and all the other are one-dimensional.

Example 4.9. Let $k = -1/2$. Then

$$\mathfrak{A}_{1,1} = \left\{ f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \;\middle|\; \partial_x f + \frac{1}{2}\partial_y f \in (x-y) \right\}.$$

In this case, we have

$$\mathfrak{A}_{1,1} = \bigoplus_{2\lambda+\mu\neq 0,1} \langle P_{\lambda,\mu} \rangle \oplus \bigoplus_{a\in\mathbb{Z}} \left\langle \frac{x^a}{y^{2a}},\ \frac{x^a}{y^{2a}}\left(\frac{x}{y} + \frac{y}{x}\right) \right\rangle,$$

Thereby, this decomposition has the same form as in the general case.

4.2. Trigonometric operator of the type B. In this section, we examine the action of the trigonometric algebra of the integrals of the type B (we follow [15]). Assume hat k is general. Let us denote by $\mathfrak{D}_{n,m}$ the subalgebra in the algerba $\mathcal{D}^{\mathrm{tr}}_{\mathcal{R}}$ generated by the integrals $\mathcal{L}_r$, $r = 1, 2, \ldots$.

Now we consider the algebra $\mathfrak{A}_{n,m}$ consisting of $W_0 = (S_n \ltimes \mathbb{Z}_2^n) \times (S_m \ltimes \mathbb{Z}_2^m)$-invariant Laurent polynomials $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_m^{\pm 1}]^{W_0}$ that satisfy the quasi-invariance conditions

$$x_i \frac{\partial f}{\partial x_i} - k y_j \frac{\partial f}{\partial y_j} \in (x_i - y_j), \quad i = 1, \ldots, n, \quad j = 1, \ldots, m. \tag{49}$$

It turns out that the algebra $\mathfrak{D}_{n,m}$ maps the algebra $\mathfrak{A}_{n,m}$ into itself.

Prposition 4.10. *The operators $\mathcal{L}_p$ map the algebra $\mathfrak{A}_{n,m}$ into itself for all $p = 1, 2, \ldots$ and they preserve the support: if $f \in \mathfrak{A}_{n,m}$ then*

$$S(Df) \subseteq S(f).$$

We say that an integral weight $\lambda \in X_{n,m} \in \mathbb{Z}^{n+m}$ is dominant if

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0, \quad \lambda_{n+1} \geq \lambda_{n+2} \geq \cdots \geq \lambda_{n+m} \geq 0, \quad \lambda_n \leq \bigl|\{i \mid \lambda_{n+i} > 0\}\bigr|.$$

Denote by $X^+_{n,m}$ the set of dominant weights. For any $\lambda \in X^+_{n,m}$, we define the homomorphism

$$\chi_\lambda : \mathfrak{D}_{\mathcal{R}} \to \mathbb{C}, \quad \chi_\lambda(D) = \varphi(D)(\lambda), \quad D \in \mathfrak{D}_{\mathcal{R}},$$

where φ is the Harish-Chandra homomorphism.

Theorem 4.11. *The algebra $\mathfrak{A}_{n,m}$ as a odule over the algebra $\mathfrak{D}_{n,m}$ for a general value of k can be decomposed into the direct sum of one dimensional eigenspaces*

$$\mathfrak{A}_{n,m} = \bigoplus_{\chi} \mathfrak{A}_{n,m}(\chi), \tag{50}$$

where the sum is taken over the set of homomorphisms $\chi = \chi_\lambda$, $\lambda \in X^+$.

Definition 4.12. An eigenfunction SJ_λ corresponding to the homomorphism χ_λ is called a *Jacobi superpolynomial*. It is easy to verify that the highest term in the lexicographic order is equal to $c_\lambda x_1^{\lambda_1} \ldots x_n^{\lambda_n} y_1^{\lambda_{n+1}} \ldots y_m^{\lambda_{n+m}}$.

5. Representation Theory and Quantum CMS Systems

In the case of finite-dimensional semi-simple Lie algebras (i.e., $m = 0$), eigenfunctions of the quantum trigonometric CMS operators for an appropriate specialization of parameters yield either characters of irreducible finite-dimensional representations or spherical functions corresponding to symmetric Lie algebras (see [7]). Connections between quantum CMS systems and representation theory of Lie superalgebras are much more complicated. A general picture is not still clear. Partial results in this directions are given below.

5.1. Representations of the Lie superalgebra $\mathfrak{gl}(n, m)$ and CMS systems. First, we consider the case of polynomial representations. Recall that a representation of the Lie superalgebra $\mathfrak{gl}(n, m)$ is said to be *polynomial* if it is a subrepresentation of the tensor algebra of the identical representation. It is known (see [11]) that the tensor algebra of the identical representation is completely reducible as a module over the Lie superalgebra $\mathfrak{gl}(n, m)$, and the corresponding irreducible representations can be labelled by partitions λ such that $\lambda_{n+1} \leq m$. We denote the category of polynomial representations by $\mathcal{F}^+$. In the general case, we denote by $\mathcal{F}$ the category of all finite-dimensional representations over the Lie superalgebra $\mathfrak{gl}(n, m)$.

Definition 5.1. The Grothendieck algebra over the field $\mathbb{C}$ of the category $\mathcal{F}$ (respectively, $\mathcal{F}^+$) is the algebra generated by classes of simple modules subject to the following relations:

$$[V] = [V_1] + [V_2] \qquad 0 \to V_1 \to V \to V_2 \to 0,$$
$$\big[V_1 \otimes V_2\big] = [V_1][V_2].$$

We denote these algebras $K(\mathcal{F})$ and $K(\mathcal{F}^+)$, respectively. It is easy to verify that the algebras $K(\mathcal{F})$ and $K(\mathcal{F}^+)$ considered as vector spaces have the basis consisting of the classes of irreducible modules.

Definition 5.2. Let V be a module from the category $\mathcal{F}$ or from the category $\mathcal{F}^+$. Assume that it can be represented as the direct sum of one-dimensional irreducible $\mathfrak{h}$-modules

$$V = \bigoplus_{\lambda \in (\mathfrak{f})^*} V(\lambda).$$

Then the function

$$\operatorname{sch}(V) = \sum_{\lambda} \operatorname{sdim}\big(V(\lambda)\big) x^{\lambda}$$

is called the *supercharacter* of V, where

$$\operatorname{sdim} V(\lambda.\mu) = \dim V(\lambda.\mu)_0 - \dim V(\lambda.\mu)_1.$$

Let $U(\mathfrak{gl}(n,m))$ be its universal enveloping algebra and $Z(\mathfrak{gl}(n,m))$ be the center of $U(\mathfrak{gl}(n,m))$. Any element from $Z(\mathfrak{gl}(n,m))$ acts as a scalar operator in each irreducible module. Therefore, we have a natural action $Z(\mathfrak{gl}(n,m))$ on $K(\mathcal{F})$. Now we can formulate the following theorem.

Theorem 5.3. *The supercharater map*

$$\operatorname{sch} : K(\mathcal{F}) \to \mathfrak{A}_{n,m}$$

is an algebra isomorphism and translates the action of $Z(\mathfrak{gl}(n,m))$ to the action of $\mathfrak{D}_{n,m}$ for $k=-1$.

Its restriction to $K(\mathcal{F}^+)$ is the isomorphism

$$\operatorname{sch} : K(\mathcal{F}^+) \to \mathfrak{A}^+_{n,m}$$

and translates the action of $Z(\mathfrak{gl}(n,m))$ to the action of $\mathfrak{D}_{n,m}$ for $k=-1$, and under this isomorphism the irreducible modules go to the Schur superfunctions (see [11]).

Now let us consider the case of the symmetric Lie superalgebra $(\mathfrak{gl}(2n+1,2m),\theta)$, where θ is an involutive authomorphism such that the orthosymplectic Lie superalgebra $\mathfrak{osp}(2n+1,2m)$ is its fixed subalgebra. There exists a natural surjective homomorphism

$$\psi : Z(\mathfrak{gl}(2n+1,2m)) \to \mathfrak{D}_{n,m},$$

which is called the homomorphism of the radial part.

Therefore, we can consider the algebra $\mathfrak{A}_{n,m}$ as a module over the algebra $Z(\mathfrak{gl}(2n+1,2m))$. In this case, the main result can formulated as follows (see [24]).

Theorem 5.4. *Let $\mathfrak{D}_{n,m}$ be the algebra of the deformed quantum operators CMS with the parameter $k=-1/2$, which acts naturally on the algebra $\mathfrak{A}_{n,m}$ of deformed Laurent polynomials with the parameter $k=-1/2$, and let*

$$\mathfrak{A}_{n,m} = \bigoplus_{\chi} \mathfrak{A}_{n,m}(\chi)$$

be its decomposition into the direct sum of generalized eigenspaces. Then for any finite-dimensional generalized eigenspace $\mathfrak{A}_{n,m}(\chi)$, there exists a unique projective indecomposable module P over $\mathfrak{gl}(2n+1,2m)$ and the natural map

$$\psi : (P^*)^{\mathfrak{b}} \to \mathfrak{A}_{n,m}(\chi),$$

which is an isomorphism of vector spaces and $Z(\mathfrak{gl}(2n+1,2m)$-modules.

5.2. Orthosymplectic Lie superalgebras and CMS systems. Consider the case of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m+1, 2n)$ (the case of the superalgebra $\mathfrak{osp}(2m, 2n)$ can be considered similarly). We briefly present two main results that show some connections of the representation theory of the Lie superalgebra $\mathfrak{osp}(2m+1, 2n)$ and the corresponding CMS system. The first result describes the Grothendieck algebra as an integrable system. Let $K(\mathcal{F})$ be the Grothendieck algebra of the category of finite-dimensional representations of the superalgebra $\mathfrak{osp}(2m+1, 2n)$ with the action of the center defined above, and $\mathfrak{A}_{n,m}$ and $\mathfrak{D}_{n,m}$ be the same as in Theorem 4.11 with the parameters $k = p = -1$ and $q = 0$.

Theorem 5.5. *The supercharacter map*

$$\mathrm{sch} : K(\mathcal{F}) \to \mathfrak{A}_{n,m}$$

is an isomorphism of algebras; it translates the action of the center $Z(\mathfrak{gl}(n, m))$ to the action of the algebra $\mathfrak{D}_{n,m}$.

The proof follows from the results of [20]. The second result shows possible specializations of Jacobi superpolynomials. For any parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ and any finite-dimensional module M over $\mathfrak{p}$, one can defined the Euler character $E^{\mathfrak{p}}(M)$ according to the superversion of the Borel–Weyl–Bott construction. By the general result of Serganova (see [5])

$$E^{\mathfrak{p}}(M) = \sum_{w \in W_0} w \left(\frac{D\, e^{\rho}\, \mathrm{sch}\, M}{\prod\limits_{\alpha \in R_{\mathfrak{p}} \cap R_1^+} \left(1 - e^{-\alpha}\right)} \right), \tag{51}$$

where

$$D = \frac{\prod\limits_{\alpha \in R_1^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right)}{\prod\limits_{\alpha \in R_0^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right)}.$$

Here ρ is the half-sum of even positive roots minus the half-sum of odd positive roots, and $R_{\mathfrak{p}}$ is the set of roots α such that $\mathfrak{g}_{\pm\alpha} \subset \mathfrak{p}$.

The main result of [19] can be formulated as follows.

Theorem 5.6. *If we take the limit of Jacobi superpolynomials SJ_{λ} as $k \to -1$ and then the limit as $(p, q) \to (-1, 0)$, then the result is well defined and coincides with the Euler character $E^{\mathfrak{p}(\lambda)}(\chi)$ for the Lie superalgebra $\mathfrak{osp}(2m+1, 2n)$, where $\mathfrak{p}(\lambda)$ is some parabolic subalgebra (depending on λ), and χ is a one-dimensional representation of it.*

6. Infinite-Dimensional Versions of Eigenfunctions of CMS Operators

Infinite-dimensional versions of quantum integrals and corresponding eigenfunctions play an important role in the theory of the deformed CMS systems. These systems have an additional parameter (an analog of the dimension) and they also have some additional symmetries (see [16, 21, 23]). As was shown in Sec. 3, integrals of the deformed CMS problem can be obtained by reduction from infinite-dimensional integrals. This was first noted in [15] in the case of the trigonometric deformed CMS system of the type A. Moreover, the question what integrable systems can also be obtained from infinite-dimensional CMS systems by the same procedure was examined in [15]. Below, we mainly follow the papers [21, 23].

In order to define Jack–Laurent symmetric functions, we introduce the Laurent version of the algebra of symmetric functions. By definition, this is the algebra $\Lambda^{\pm}$ freely generated by "infinite power sums" p_i, $i \in \mathbb{Z} \setminus \{0\}$. We also consider infinite "zero power sum" p_0 as an additional parameter.

Let $\Lambda^{\pm}[x, x^{-1}]$ be the algebra of Laurent polynomials in the variable x over the algebra $\Lambda^{\pm}$. Define the differentiation ∂ in $\Lambda^{\pm}[x, x^{-1}]$ by the formula

$$\partial(x) = x, \quad \partial(p_l) = lx^l,$$

and the operators

$$\Delta : \Lambda^{\pm}[x, x^{-1}] \to \Lambda^{\pm}[x, x^{-1}], \quad \Delta(x^l f) = \Delta(x^l) f, \quad \Delta(1) = 0, \quad f \in \Lambda^{\pm}, \quad l \in \mathbb{Z},$$
$$\Delta(x^l) = x^l p_0 + 2x^{l-1} p_1 + \cdots + 2xp_{l-1} + p_l - 2lx^l, \quad l > 0,$$
$$\Delta(x^l) = -(\Delta(x^{-l}))^*, \quad l < 0, \quad x^* = x^{-1}.$$

Introduce the infinite-dimensional analogs Dunk–Heckman operators

$$D : \Lambda^{\pm}[x, x^{-1}] \to \Lambda^{\pm}[x, x^{-1}], \quad D = \partial - \frac{1}{2} k\Delta \tag{52}$$

and the linear operator

$$E : \Lambda^{\pm}[x, x^{-1}] \to \Lambda^{\pm}, \quad E(x^l f) = p_l f, \quad f \in \Lambda, \quad l \in \mathbb{Z}, \tag{53}$$

and also the operators

$$\mathcal{L}_r^{\infty} : \Lambda^{\pm} \to \Lambda^{\pm}, \quad \mathcal{L}_r^{\infty} = E \circ D^r, \quad r \in \mathbb{Z}_+ \tag{54}$$

where the action on the right-hand side is restricted to the algebra $\Lambda^{\pm}$. We claim that these operators give the Laurent version of the quantum integrals at infinity.

Theorem 6.1. *The operator $\mathcal{L}_r^{\infty}$ is a differential operator of order r polynomially depending on p_0. Moreover, the operator $\mathcal{L}_2^{\infty}$ has the following explicit form:*

$$\mathcal{L}_2^{\infty} = \sum_{a,b \in \mathbb{Z}} p_{a+b} \partial_a \partial_b - k \left[\sum_{a,b>0} p_a p_b \partial_{a+b} - \sum_{a,b<0} p_a p_b \partial_{a+b} \right] -$$
$$- kp_0 \left[\sum_{a>0} p_a \partial_a - \sum_{a<0} p_a \partial_a \right] + (1+k) \sum_{a \in \mathbb{Z}} a p_a \partial_a, \tag{55}$$

where $\partial_a = a\partial/\partial p_a$, and this operator is the Laurent version of the CMS operator at infinity.

The operators $\mathcal{L}_r^{\infty}$ commute with each other:

$$\left[\mathcal{L}_r^{\infty},\ \mathcal{L}_s^{\infty}\right] = 0.$$

Let $\mathcal{P}$ be the set of all partitions (or Young diagrams). By *bipartition* we mean a pair of partitions $\alpha = (\lambda, \mu) \in \mathcal{P} \times \mathcal{P}$. Define the *length of a bipartition* $\alpha = (\lambda, \mu)$ by $l(\alpha) := l(\lambda) + l(\mu)$. Let $l(\alpha) \leq N$; then we define

$$\chi_N(\alpha) = \Big(\underbrace{\lambda_1,\ \ldots,\ \lambda_r,\ 0,\ \ldots,\ 0,\ \mu_s,\ \ldots,\ -\mu_1}_{N}\Big). \tag{56}$$

and the symmetric polynomial

$$P_{\chi_N(\alpha)}(x_1, \ldots, x_N) = (x_1 \ldots x_N)^{-a} P_\nu(x_1, \ldots, x_N),$$

where a is a nonnegative integer such that $\nu = \chi + a$ is a partition and $P_\nu(x_1, \ldots, x_N)$ is an ordinary Jack polynomial. Consider the algebra $\Lambda^{\pm}(p_0)$ of rational functions in p_0 with coefficients in $\Lambda^{\pm}$ and the homomorphism $\varphi_N : \Lambda^{\pm}(p_0) \to \Lambda_N^{\pm}$ defined by $\varphi_N(p_i) = x_1^i + \cdots + x_N^i$, $i \in \mathbb{Z}$, with the spacialization $p_0 = N$.

Now we can define *Jack–Lautrent symmetric functions* $P_\alpha \in \Lambda^{\pm}(p_0)$ by the following theorem.

Theorem 6.2. *If $k \notin \mathbb{Q}$ and $p_0 \neq n + k^{-1}m$ for any $m, n \in \mathbb{Z}_{>0}$, then for any bipartition α there exists a unique element $P_\alpha \in \Lambda^{\pm}(p_0)$ (called the Jack–Laurent symmetric function) such that, for any $N \in \mathbb{N}$,*

$$\varphi_N(P_\alpha) = \begin{cases} P_{\chi_N(\alpha)}(x_1, \ldots, x_N), & \text{if } l(\alpha) \leq N, \\ 0, & \text{if } l(\alpha) > N. \end{cases} \tag{57}$$

It can be easily verified that for a general value of the parameter p_0, these symmetric functions are eigenfunctions for infinite-dimensional operators from Theorem 6.1. We note that for special values of p_0 this assertion does not hold. It turns out that the function P_α is not well defined for such p_0 and, in general, instead of eigenfunctions we need to consider generalized eigenspaces.

Consider the case where $p_0 = n + k^{-1}m$, $n, m \in \mathbb{Z}_{>0}$. Denote by $\pi(n, m)$ the rectangular Young diagram of size $n \times m$ and consider the corresponding bipartition $\pi = (\pi(n, m), \pi(n, m))$. Define the central symmetry θ acting on $(ij) \in \pi(n, m)$ by the formula $\theta(ij) = (n - i + 1, m - j + 1)$. Introduce the equivalence relation on the pairs of bipartitions. Below, the set-theoretic operations on partitions are understood as the operations on Young diagrams.

Definition 6.3. We say that a bipartition $\alpha = (\lambda, \mu)$ is equivalent to a bipartition $\tilde{\alpha} = (\tilde{\lambda}, \tilde{\mu})$ if

$$\alpha \setminus \pi = \tilde{\alpha} \setminus \pi \quad \text{and} \quad \theta(\lambda \setminus \tilde{\lambda}) = \mu \setminus \tilde{\mu}, \quad \theta(\tilde{\lambda} \setminus \lambda) = \tilde{\mu} \setminus \mu. \tag{58}$$

It turns out that every equivalence class E consists of 2^r elements, where the number r depends on the class E.

Let $\mathcal{D}^\infty$ be the algebra generated by the integrals $\mathcal{L}_r^\infty$, $r = 1, 2, \ldots$. The following theorem describes the structure of $\Lambda^{\pm}$ as a module over $\mathcal{D}^\infty$, and also describes the action of $\mathcal{D}^\infty$ in generalized eigenspaces.

Theorem 6.4.

(1) *The algebra $\Lambda^{\pm}$ as a module over the algebra $\mathcal{D}^\infty$ can be decomposed into the direct sum of generalized eigenspaces*

$$\Lambda^{\pm} = \bigoplus_E \Lambda^{\pm}(E),$$

where the sum is taken over all equivalence classes.

(2) *If a class E contains 2^r elements and k is not an algebraic number, then the image of the algebra $\mathcal{D}^\infty$ in the algebra $\mathrm{End}(\Lambda^{\pm}(E))$ is isomorphic to the tensor product of r copies of the algebra of dual numbers*

$$\mathbb{C}[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r]/(\varepsilon_1^2, \varepsilon_2^2, \ldots, \varepsilon_r^2).$$

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