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THE NEUMANN PROBLEM FOR THE GENERALIZED HÉNON EQUATION

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We study the behavior of radial solutions to the boundary value problem

$$-\Delta_p u + u^{p-1} = |x|^\alpha u^{q-1} \quad \text{in} \quad B, \qquad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial B, \qquad q > p,$$

in the unit ball B and prove the existence of nonradial positive solutions for some values of parameters. We obtain multiplicity results which are new even in the case $p = 2$. Bibliography: 13 titles. Illustrations: 3 figures.

1 Statement of the Problem

Let B be the unit ball in $\mathbb{R}^n$, $n \geqslant 2$. Denote $S = \partial B$, $p, q > 1$, $\alpha > 0$ and consider the problem

$$\begin{aligned} & -\Delta_p u + u^{p-1} = |x|^\alpha u^{q-1} \quad \text{in } B, \\ & u > 0 \quad \text{in } B, \\ & \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } S, \end{aligned} \tag{1.1}$$

where $x \in B$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, and $\mathbf{n}$ is the outward unit normal to S. Solutions to the problem (1.1) can be found by examining critical points of the functional

$$\mathcal{Q}_{p,q,\alpha}(u) = \frac{\int\limits_B (|\nabla u|^p + |u|^p)dx}{\left(\int\limits_B |x|^\alpha |u|^q dx\right)^{p/q}}. \tag{1.2}$$

For the functional $\mathcal{Q}_{p,q,\alpha}$ we write the Euler equation

$$-\Delta_p u + |u|^{p-2}u = \mu |x|^\alpha |u|^{q-2} u \quad \text{in } B, \qquad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } S,$$

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where μ is a Lagrange multiplier. If u_α is a minimizer of the functional $\mathscr{Q}_{p,q,\alpha}$, then u_α is a solution to the Euler equation with

$$\mu = \inf_{u \in W_p^1(B),\ u \neq 0} \mathscr{Q}_{p,q,\alpha}(u).$$

In the case $q \neq p$, the change of variable $u_\alpha \to \mu^{\frac{1}{q-p}} u_\alpha$ transforms the Euler equation to the boundary value problem (1.1), i.e., a minimizer of the functional $\mathscr{Q}_{p,q,\alpha}$ is a weak solution to the problem (1.1).

We are interested in the question whether a minimizer of the functional $\mathscr{Q}_{p,q,\alpha}$ is radial or, in other words, under what conditions on the parameters q and α a radial function is the least energy solution to the problem (1.1).

This question is successfully studied for the Dirichlet problem (cf., for example, [1, 2] for $p = 2$ and [3] for arbitrary $p > 1$). Under certain conditions on q and α, some results on multiplicity of positive solutions to the Dirichlet problem in the ball are known (cf. [1] for $p = 2$ and [4] for arbitrary $p > 1$). Hereinafter, *multiplicity of solutions* means that it is possible to construct any prescribed number of those solutions that are obtained from each other by orthogonal transformations.

For the Neumann problem with $p = 2$ it is known [5] that positive solutions are unique (thereby radial) for q close to 2 and sufficiently large α. Furthermore, there exist nonradial solutions if q is close to the critical Sobolev embedding exponent. In this case, the behavior of a radial minimizer is described in terms of Bessel functions. However, in the case $p \neq 2$, this is not the case and the analysis of the problem becomes more complicated.

Repeating the proof of Proposition 1.1 in [3], we obtain the following assertion.

Proposition 1.1. *For $q \leqslant p$, on the set of positive functions in $W_p^1(B)$, there exists a unique (up to a multiplicative constant) critical point of the functional $\mathscr{Q}_{p,q,\alpha}$. The problem (1.1) has a unique solution, and this solution is radial.*

In this paper, we study the behavior of radial solutions to the problem (1.1) in the case $q > p$ and show that for some values of the parameters there exist nonradial positive solutions. The multiplicity results obtained in this paper are new even in the case $p = 2$.

We denote by p_m^* and p_m^{**} the critical exponents in the Sobolev embedding and trace embedding in $\mathbb{R}^{n-m+1}$ respectively:

$$\frac{1}{p_m^*} = \Big(\frac{1}{p} - \frac{1}{n-m+1}\Big)_+, \quad \frac{1}{p_m^{**}} = \Big(\frac{1}{p} - \frac{1}{n-m} + \frac{1}{p(n-m)}\Big)_+.$$

We write p^* and p^{**} instead of p_1^* and p_1^{**} for the sake of brevity. We note that $p_m^* > p^*$ and $p_m^{**} > p^{**}$ if $p < n$ and $m \geqslant 2$. Constants independent of α are denoted by c with subscripts.

2 Radial and (m, k)-Radial Solutions

Following [6], we consider the decomposition $\mathbb{R}^n = (\mathbb{R}^m)^l \oplus \mathbb{R}^k$, where $n = ml + k$, $m \geqslant 2$, $k \geqslant m$ or $k = 0$. We denote by y_j ($j = 1, \ldots, l$) points in $\mathbb{R}^m$ and by z points in $\mathbb{R}^k$. For example, $x = (y_1, \ldots, y_l, z)$. The spherical coordinates of y_j are denoted by (r_j, θ_j), $\theta_j \in S_m$ (hereinafter, S_m is the unit sphere in $\mathbb{R}^m$), the spherical coordinates of z are denoted by (r_0, θ_0). The spherical coordinates of x are denoted by (r, θ), where $r = \sqrt{r_1^2 + \ldots + r_l^2 + r_0^2}$, $\theta \in S^n$.

A function u is said to be (m,k)-*symmetric* if u is invariant under all permutations of $y_1, \dots, y_l$ and (for $k \neq 0$) depends only on r_0. A function u is said to be (m,k)-*radial* if u is (m,k)-symmetric and depends only on r_j and r_0. In particular, a radial function is an $(n,0)$-radial function. We note that nontrivial admissible decompositions exist only for $n \geqslant 4$.

Let $\mathscr{W}_{(m,k)}$ be the subspace of (m,k)-radial functions in $W_p^1(B)$. It is easy to see that the functional $\mathscr{Q}_{p,q,\alpha}$ is well defined for all $q \in (p;p^*)$. As is shown in [7, Theorem 1.1], the subspace $\mathscr{W}_{(m,k)}$ is compactly embedded into the weighted space $L_{q,\alpha} = L_q(B, |x|^\alpha dx)$ for all $q > 1$ such that $q < p_m^*$ and $q < p_\alpha^* = p^* + p\alpha/(n-p)$. Denote $\widehat{p} = \min\{p_m^*, p_\alpha^*\}$. Then $\widehat{p} > p^*$ for $p < n$.

Proposition 2.1. *Assume that $\alpha > 0$ and $q \in (p;\widehat{p})$. Then the functional $\mathscr{Q}_{p,q,\alpha}$ attains a nonzero minimum on $\mathscr{W}_{(m,k)}$ and the minimizing function $\mathbf{v}_{m,\alpha}$ (after multiplication by a suitable constant) is a positive weak solution to the problem* (1.1).

Proof. We note that the restriction of the functional $\mathscr{Q}_{p,q,\alpha}$ on $\mathscr{W}_{(m,k)}$ is well defined for all $q < \widehat{p}$. Since the functional $\mathscr{Q}_{p,q,\alpha}$ is homogeneous, it suffices to minimize over the set of functions in $\mathscr{W}_{(m,k)}$ with unit $L_{q,\alpha}$-norm. This set is weakly closed in $\mathscr{W}_{(m,k)}$, and a coercive convex functional attains the minimum on this set.

By the principle of symmetric criticality [8], the first differential $D\mathscr{Q}_{p,q,\alpha}(\mathbf{v}_{m,\alpha}; h)$ of the functional (1.2) vanishes not only at increments $h \in \mathscr{W}_{(m,k)}$, but also at all $h \in W_p^1(B)$.

We note that the Euler equation with a natural boundary condition for the functional $\mathscr{Q}_{p,q,\alpha}$ coincides, after a suitable renormalization of $\mathbf{v}_{m,\alpha}$, with the boundary value problem (1.1). Using the Harnack inequality [9, Theorem 1.1], we conclude that $\mathbf{v}_{m,\alpha}$ are positive. □

The following assertion is similar to Lemma 2.5 in [5].

Lemma 2.1. *The following relation holds*

$$(\alpha+n)\int\limits_B |x|^\alpha |u|^q dx = \int\limits_S |u|^q d\theta + o(1), \quad \alpha \to +\infty,$$

1) *uniformly on all bounded subsets of $W_p^1(B)$ for $q \in (p;p^{**})$ and*

2) *uniformly on all bounded subsets of $\mathscr{W}_{(m,k)}$ for $q \in (p;p_m^{**})$.*

In particular, for radial functions this assertion is valid for all $1 < p < q < +\infty$.

Proof. We note that $(\alpha+n)|x|^\alpha = \operatorname{div}(|x|^\alpha x)$. Integrating by parts, we find

$$(\alpha+n)\int\limits_B |u|^q|x|^\alpha dx = \int\limits_B |u|^q \operatorname{div}(|x|^\alpha x)dx = \int\limits_S |u|^q|x|^\alpha \langle x, \mathbf{n}\rangle d\theta - q\int\limits_B |u|^{q-2}u|x|^\alpha \langle \nabla u, x\rangle dx$$

$$= \int\limits_S |u|^q d\theta - q\int\limits_B |u|^{q-2}u|x|^\alpha \langle \nabla u, x\rangle dx.$$

We show that the last integral is of order $o(1)$ as $\alpha \to +\infty$. Indeed, by the Hölder inequality,

$$\left|\int\limits_B |u|^{q-2}u|x|^\alpha \langle \nabla u, x\rangle dx\right| \leqslant \left(\int\limits_B |\nabla u|^p dx\right)^{1/p} \left(\int\limits_B |u|^{(q-1)p'}|x|^{(\alpha+1)p'} dx\right)^{1/p'}$$

$$\leqslant \|u\|_{W_p^1(B)} \Bigg(\int\limits_B |u|^{(q-1)p'} |x|^{(\alpha+1)p'} dx \Bigg)^{1/p'},$$

where $p' = p/(p-1)$ is the conjugate exponent.

If $p < n$ and $q \in (p; p^{**})$, then

$$(q-1)p' < \Big(\frac{p(n-1)}{n-p} - 1\Big)\frac{p}{p-1} = p^*.$$

By the Hölder inequality and embedding theorem,

$$\int\limits_B |u|^{(q-1)p'} |x|^{(\alpha+1)p'} dx \leqslant \Bigg(\int\limits_B |u|^{p^*} dx \Bigg)^{\frac{(q-1)p'}{p^*}} \Bigg(\int\limits_B |x|^{\frac{(\alpha+1)np}{p(n-1)-q(n-p)}} dx \Bigg)^{\frac{p(n-1)-q(n-p)}{n(p-1)}}$$

$$\leqslant c_1(n,p) \|u\|^{(q-1)p'}_{W_p^1(B)} \cdot o(1),$$

and the required assertion is proved.

If $u \in \mathscr{W}_{(m,k)}$, $p < n-m+1$, and $q \in (p; p_m^{**})$, then $(q-1)p' < p_m^*$. By the Hölder inequality,

$$\int\limits_B |u|^{(q-1)p'} |x|^{(\alpha+1)p'} dx \leqslant \Bigg(\int\limits_B |u|^{p_m^*} |x|^{\delta} dx \Bigg)^{\frac{(q-1)p'}{p_m^*}} \Bigg(\int\limits_B |x|^{d_1} dx \Bigg)^{d_2}, \tag{2.1}$$

where $d_1 = d_1(\alpha, n, p, q, \delta) \asymp \alpha$ as $\alpha \to +\infty$ and $d_2 = d_2(n,p,q,\delta)$ is independent of α. We fix

$$\delta > \frac{1}{q}\Big(\frac{n}{p} - \frac{n}{q} - 1\Big)_+.$$

By the weighted embedding theorem [7, Theorem 1.1],

$$\Bigg(\int\limits_B |u|^{p_m^*} |x|^{\delta} dx \Bigg)^{\frac{(q-1)p'}{p_m^*}} \Bigg(\int\limits_B |x|^{d_1} dx \Bigg)^{d_2} \leqslant c_2(n,p) \|u\|^{(q-1)p'}_{W_p^1(B)} \cdot o(1),$$

and the required assertion is proved.

If $p \geqslant n$ or $u \in \mathscr{W}_{(m,k)}$, $p \geqslant n-m+1$, then for any $q \in (p; +\infty)$ we have $(q-1)p' < p_\alpha^*$ for sufficiently large α. Then (2.1) remains valid with p_m^* replaced by p_α^* and some fixed

$$\delta > \frac{p'}{q' p_\alpha^*}\Big(\frac{n}{p} - \frac{n}{q} - 1\Big)_+.$$

Then we argue as above. □

By Proposition 2.1, $\mu_{q,m,\alpha} = \min\limits_{v \in \mathscr{W}_{(m,k)},\ v \neq 0} \mathscr{Q}_{p,q,\alpha}(v) > 0$ is well defined for any $\alpha > 0$ and $q \in (p; \widehat{p})$. Hence any (m,k)-radial minimizer $\mathbf{v}_{m,\alpha}$ of the functional $\mathscr{Q}_{p,q,\alpha}$ such that $\|\mathbf{v}_{m,\alpha}\|_{W_p^1(B)} = 1$ is a solution to the problem

$$\begin{aligned} &-\Delta_p u + u^{p-1} = \mu_{q,m,\alpha}^{q/p} |x|^{\alpha} u^{q-1} \quad \text{in } B, \\ &\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } S. \end{aligned} \tag{2.2}$$

We consider $(n, 0)$-radial solutions (i.e., radial) in more detail. We introduce the auxiliary problem (the generalized Steklov problem)

$$
\begin{aligned}
-\Delta_p u + |u|^{p-2}u &= 0 \quad \text{in } B,\\
|\nabla u|^{p-2}\langle \nabla u; \mathbf{n}\rangle &= \lambda |u|^{p-2}u \quad \text{on } S.
\end{aligned}
$$

It is known [10] that the first nonzero eigenvalue λ_p of this problem is simple and is expressed as

$$
\lambda_p = \inf_{u \in W_p^1(B),\ u \neq 0} \frac{\|u\|^p_{W_p^1(B)}}{\|u\|^p_{L_p(S)}}. \tag{2.3}
$$

The corresponding eigenfunction φ is positive and radial in B. We assume that $\|\varphi\|_{W_p^1(B)} = 1$. The function $\varphi(r)$ is a solution to the problem

$$
\begin{aligned}
&-\frac{1}{r^{n-1}}(r^{n-1}(\varphi')^{p-1})' + \varphi^{p-1} = 0, \quad r \in (0;1),\\
&\varphi'(1) = \lambda_p^{\frac{1}{p-1}} \varphi(1).
\end{aligned} \tag{2.4}
$$

From (2.4) it follows that the function φ in a neighborhood of zero has the structure

$$
\varphi(r) = c_3(n,p) + o(1), \quad \varphi'(r) = c_3 n^{-\frac{1}{p-1}} r^{\frac{1}{p-1}} + o(r^{\frac{1}{p-1}}), \quad r \to 0. \tag{2.5}
$$

Without loss of generality we assume that $c_3 = 1$.

Theorem 2.1. *Let $q \in (p, p_m^{**})$. Then*

$$
\mu_{q,m,\alpha} \geqslant c_4(n,m,p,q)(\alpha+n)^{p/q} \tag{2.6}
$$

as $\alpha \to \infty$. Furthermore, for $m = n$ and $q \in (p, +\infty)$

$$
\mu_{q,n,\alpha} \sim (\text{mes } S)^{1-p/q} \lambda_p (\alpha+n)^{p/q}, \tag{2.7}
$$

where λ_p is defined by (2.3).

Proof. By Lemma 2.1, for an arbitrary nonnegative (m,k)-radial function $v \in \mathscr{W}_{(m,k)}$ such that $\|v\|_{W_p^1(B)} = 1$ we have

$$
\begin{aligned}
\mathscr{Q}_{p,q,\alpha}(v) \cdot (\alpha+n)^{-p/q} &= \left((\alpha+n) \int_B v^q |x|^\alpha dx \right)^{-p/q}\\
&= \left(\int_S v^q d\theta + o(1) \right)^{-p/q} = \left(\int_S v^q d\theta \right)^{-p/q} + o(1)
\end{aligned}
$$

as $\alpha \to \infty$ uniformly with respect to v. By the embedding theorem on the boundary for (m,k)-radial functions [11, Proposition 2.1], we have

$$
\left(\int_S v^q d\theta \right)^{-p/q} = \|v\|^{-p}_{L_q(S)} \geqslant c_4(n,m,p,q) \|v\|^{-p}_{W_p^1(B)} = c_4(n,m,p,q),
$$

i.e., $\mathscr{Q}_{p,q,\alpha}(v) \geqslant c_4(\alpha+n)^{p/q}$, and (2.6) is proved.

Let v be a radial function. Then

$$\Bigg(\int\limits_S v^q d\theta\Bigg)^{p/q} = (\text{mes } S)^{p/q}(v(1))^p = (\text{mes } S)^{p/q-1}\int\limits_S v^p d\theta,$$

i.e.,

$$\mathscr{Q}_{p,q,\alpha}(v)\cdot(\alpha+n)^{-p/q} = (\text{mes } S)^{1-p/q}\Bigg(\int\limits_S v^p d\theta\Bigg)^{-1} + o(1).$$

Setting $v = \mathbf{v}_{n,\alpha}$, we find

$$\mu_{p,q,\alpha}(\alpha+n)^{-p/q} = \mathscr{Q}_{p,q,\alpha}(\mathbf{v}_{n,\alpha})\cdot(\alpha+n)^{-p/q} = \big(\text{mes } S\big)^{1-p/q}\Bigg(\int\limits_S \mathbf{v}_{n,\alpha}^p d\theta\Bigg)^{-1} + o(1)$$

$$\geqslant (\text{mes } S)^{1-p/q}\cdot \inf_{v\in W_p^1(B),\ \|v\|=1}\frac{\|v\|^p_{W_p^1(B)}}{\|v\|^p_{L_q(S)}} + o(1) = (\text{mes } S)^{1-p/q}\lambda_p + o(1).$$

On the other hand,

$$\mu_{p,q,\alpha}(\alpha+n)^{-p/q} = \mathscr{Q}_{p,q,\alpha}(\mathbf{v}_{n,\alpha})\cdot(\alpha+n)^{-p/q} \leqslant \mathscr{Q}_{p,q,\alpha}(\varphi)\cdot(\alpha+n)^{-p/q}$$

$$= (\text{mes } S)^{1-p/q}\Bigg(\int\limits_S \varphi^p d\theta\Bigg)^{-p} + o(1) = (\text{mes } S)^{1-p/q}\lambda_p + o(1), \qquad (2.8)$$

where the inequality is valid since φ is a radial function. Thereby the relation (2.7) is proved. □

Theorem 2.2. *Assume that $q\in(p;+\infty)$ and $\mathbf{v}_\alpha$ such that $\|\mathbf{v}_\alpha\|_{W_p^1(B)} = 1$ is a minimizer of the functional $\mathscr{Q}_{p,q,\alpha}$ on the set of radial functions $\mathscr{W}_{(n,0)}$. Then the following assertions hold as $\alpha\to\infty$:*

1) $\mathbf{v}_\alpha \to \varphi$ *in* $W_p^1(B)$,

2) $\mathbf{v}_\alpha \to \varphi$ *in* $C(B)$,

3) $\mathbf{v}_\alpha$ *are uniformly bounded in* $C^1(B)$,

4) *for any* $\delta\in(0;1)$ $\mathbf{v}_\alpha\to\varphi$ *in* $C^1(B_\delta)$, *where* B_δ *is the ball with radius* δ *in* $\mathbb{R}^n$.

Proof. To prove the first assertion, we extract a subsequence $\mathbf{v}_\alpha$ that weakly converges in $W_p^1(B)$ to some function $\mathbf{v}\in W_p^1(B)$. This can be done because the functions $\mathbf{v}_\alpha$ are normalized. Then $\mathbf{v}_\alpha(r)\rightharpoonup\mathbf{v}(r)$ in $W_p^1((1/2,1))$ since the weighted function is separated from zero. By the compactness of one-dimensional embedding, we have $\mathbf{v}_\alpha(1)\to\mathbf{v}(1)$, i.e., $\mathbf{v}_\alpha\to\mathbf{v}$ in $L_p(S)$. Applying (2.8), we find

$$(\text{mes } S)^{1-p/q}\Bigg(\int\limits_S \mathbf{v}_\alpha^p d\theta\Bigg)^{-p} + o(1) = (\alpha+n)^{-p/q}\mathscr{Q}_{p,q,\alpha}(\mathbf{v}_\alpha) \leqslant (\text{mes } S)^{1-p/q}\lambda_p + o(1),$$

i.e.,

$$\|\mathbf{v}\|_{L_p(S)} = \|\mathbf{v}_\alpha\|_{L_p(S)} \geqslant \lambda_p^{-1/p} + o(1) > 0, \quad \mathbf{v} \not\equiv 0.$$

On the other hand,

$$\lambda_p \leqslant \frac{\|\mathbf{v}\|^p_{W_p^1(B)}}{\|\mathbf{v}\|^p_{L_p(S)}} \leqslant \frac{1}{\|\mathbf{v}\|^p_{L_p(S)}} = \lim_{\alpha\to\infty} \frac{1}{\|\mathbf{v}_\alpha\|^p_{L_p(S)}}$$

$$= (\text{mes } S)^{p/q-1} \lim_{\alpha\to\infty} ((\alpha+n)^{-p/q} \mathscr{Q}_{p,q,\alpha}(\mathbf{v}_\alpha) + o(1)) = \lim_{\alpha\to\infty} \frac{(\text{mes } S)^{p/q-1}\mu_{p,q,\alpha}}{(\alpha+n)^{p/q}} = \lambda_p,$$

where the last equality is valid in view of (2.7). Hence $\|\mathbf{v}\|_{W_p^1(B)} = 1$ and, consequently, $\mathbf{v} \equiv \varphi$ since λ_p is a simple eigenvalue, i.e., $\mathbf{v}_\alpha \rightharpoonup \varphi$ in $W_p^1(B)$ and $\|\mathbf{v}_\alpha\|_{W_p^1(B)} \to \|\varphi\|_{W_p^1(B)}$, which implies the strong convergence in $W_p^1(B)$ [12, Theorem 2.11]. Assertion 1) is proved.

To prove assertions 2)–4), we first show that $\mathbf{v}_\alpha$ are uniformly bounded in $L_\infty(B)$. We fix $\delta \in (0;1)$. Since $\|\mathbf{v}_\alpha\|_{W_p^1(B)} = 1$ and $\mathbf{v}_\alpha$ are radial functions, the embedding theorem for one-dimensional functions yields

$$\|\mathbf{v}_\alpha\|_{L_\infty(B\setminus B_\delta)} \leqslant c_5(\delta)\|\mathbf{v}_\alpha\|_{W_p^1(B\setminus B_\delta)} \leqslant c_5. \tag{2.9}$$

On the other hand, for any $x \in B\setminus\{0\}$ (cf. [2])

$$|\mathbf{v}_\alpha(x)| \leqslant c_6 \frac{\|\mathbf{v}_\alpha\|_{W_p^1(B)}}{|x|^{(n-p)/p}} = \frac{c_6}{|x|^{(n-p)/p}}.$$

We set $f_{p,q,\alpha}(x) = \mu_{p,q,\alpha}^{q/p}|x|^\alpha \mathbf{v}_\alpha^{q-1}$. From the obtained estimate and (2.7) it follows that

$$\|f_{p,q,\alpha}\|_{L_\infty(B_\delta)} \leqslant c_7\alpha|x|^\alpha \cdot \frac{1}{|x|^{\frac{(n-p)(q-1)}{p}}} \leqslant c_7\alpha \cdot \delta^{\alpha - \frac{(n-p)(q-1)}{p}} = o(1) \tag{2.10}$$

as $\alpha \to +\infty$ uniformly with respect to α. By (2.2), (2.9), and (2.10), the function $\mathbf{v}_\alpha$ is a solution to the problem

$$\begin{aligned} &-\Delta_p \mathbf{v}_\alpha + \mathbf{v}_\alpha^{p-1} = f_{p,q,\alpha} \quad \text{in } B_\delta, \\ &\mathbf{v}_\alpha \leqslant c_5 \quad \text{on } \partial B_\delta, \end{aligned} \tag{2.11}$$

where $\|f_{p,q,\alpha}\|_{L_\infty(\frac{1}{2}B)} = o(1)$ as $\alpha \to +\infty$. As is shown in [13], this means the uniform boundedness of $\mathbf{v}_\alpha$ in $C^{1,\beta}(B_\delta)$ with some $\beta(\delta) \in (0;1)$. Since $C^{1,\beta}$ is compactly embedded into C^1, we can assume that $\mathbf{v}_\alpha \rightrightarrows \varphi$ in $\overline{B_\delta}$, i.e.,

$$\mathbf{v}_\alpha(r) = \varphi(r) + o(1) \tag{2.12}$$

as $r \to 0$ uniformly with respect to α.

Let us show that

$$(\mathbf{v}'_\alpha)^{p-1} = (\varphi')^{p-1}(1+o(1)), \quad r \to 0. \tag{2.13}$$

Setting $(\mathbf{v}'_\alpha)^{p-1} = (\varphi')^{p-1}u_\alpha$ and substituting into (2.11), we obtain the first order linear equation for u_α

$$-\frac{1}{r^{n-1}}(r^{n-1}(\varphi')^{p-1}u_\alpha)' + \mathbf{v}_\alpha^{p-1} = f_{p,q,\alpha}.$$

We note that the solution $\widetilde{u}(r) = \dfrac{1}{r^{n-1}(\varphi')^{p-1}}$ to the homogeneous equation is independent of α. Then the general solution to the inhomogeneous equation can be found by the Lagrange formula

$$u_\alpha(r) = \int^r t^{n-1}(\mathbf{v}_\alpha^{p-1} - f_{p,q,\alpha})dt \cdot \widetilde{u}(r).$$

Taking into account (2.12), (2.5), and (2.10), we find

$$u_\alpha(r) = \int^r t^{n-1}(1+o(1))dt \cdot \widetilde{u}(r) = \Big(\frac{r^n}{n}(1+o(1)) + C\Big)(nr^{-n} + o(1))$$

uniformly in α. For $C \neq 0$ the function u_α does not satisfy the summability condition at zero, which means $u_\alpha(r) = 1 + o(1)$ and, consequently, (2.13) is proved. Hence $\mathbf{v}'_\alpha \rightrightarrows \varphi'$ in $\overline{B_\delta}$ and assertion 4) is proved.

In view of (2.9), it remains to show that $\mathbf{v}_\alpha$ are bounded in $C^1(\overline{B \setminus B_\delta})$. We note that $\mathbf{v}_\alpha$ is a solution to the equation

$$-\frac{1}{r^{n-1}}(r^{n-1}(\mathbf{v}'_\alpha)^{p-1})' + \mathbf{v}_\alpha^{p-1} = \mu_{q,n,\alpha}^{q/p} r^\alpha \mathbf{v}_\alpha^{q-1}, \quad \mathbf{v}'_\alpha(1) = 0.$$

Integrating this equation on $r \in [s;1]$, $s \geqslant 1/2$, we find

$$-\int_s^1 \frac{1}{r^{n-1}}(r^{n-1}(\mathbf{v}'_\alpha)^{p-1})'dr + \int_s^1 \mathbf{v}_\alpha^{p-1}dr = \mu_{q,n,\alpha}^{q/p}\int_s^1 r^\alpha \mathbf{v}_\alpha^{q-1}dr.$$

Integrating by parts in the first term, we get

$$(\mathbf{v}'_\alpha(s))^{p-1} = (n-1)\int_s^1 \frac{(\mathbf{v}'_\alpha(r))^{p-1}dr}{r} - \int_s^1 \mathbf{v}_\alpha^{p-1}dr + \mu_{q,n,\alpha}^{q/p}\int_s^1 r^\alpha \mathbf{v}_\alpha^{q-1}dr,$$

which implies

$$\begin{aligned}|\mathbf{v}'_\alpha(s)|^{p-1} &\leqslant (n-1)\int_s^1 \frac{(\mathbf{v}'_\alpha(r))^{p-1}dr}{r} + \int_s^1 \mathbf{v}_\alpha^{p-1}dr + \mu_{q,n,\alpha}^{q/p}\int_s^1 r^\alpha \mathbf{v}_\alpha^{q-1}dr \\ &\leqslant 2^{n-2}(n-1)\int_s^1 r^{n-1}\big(\mathbf{v}'_\alpha(r)\big)^{p-1}dr + 2^{n-1}\int_s^1 r^{n-1}\mathbf{v}_\alpha^{p-1}dr + c_6\alpha \cdot c_4^{q-1}\cdot \frac{r^{\alpha+1}}{\alpha+1}\Big|_s^1 \\ &\leqslant 2^{n-2}(n-1)\|\nabla \mathbf{v}_\alpha\|_{L_{p-1}(B)}^{p-1} + 2^{n-1}\|\mathbf{v}_\alpha\|_{L_{p-1}(B)}^{p-1} + c_8(n,p,q) \\ &\leqslant 2^n\|\mathbf{v}_\alpha\|_{W_p^1(B)}^{p-1} + c_8(n,p,q) = 2^n + c_8(n,p,q)\end{aligned}$$

in view of (2.9) and (2.7), which implies $|\mathbf{v}'_\alpha(s)| \leqslant c_8(n,p,q)$ for any $s \in [1/2;1]$, where the constant c_8 is independent of α.

Thus, $\mathbf{v}'_\alpha(r)$ are uniformly bounded. Consequently, $\mathbf{v}_\alpha(r)$ are equicontinuous. By the Arzeli–Ascoli theorem, $\mathbf{v}_\alpha \rightrightarrows \varphi$. □

3 Multiplicity of Solutions

Throughout the section, we assume that $p < n$.

Theorem 3.1. *Let $q \in (p^{**}; p^*)$. Then there exists $\widehat{\alpha}(p,q) > 0$ such that for all $\alpha > \widehat{\alpha}$ the (global) minimizer of the functional $\mathscr{Q}_{p,q,\alpha}$ is a nonradial function.*

Remark 3.1. In the case $p = 2$, this assertion is proved in [5].

Proof of Theorem 3.1. We consider a nonnegative function $v \in C_0^\infty(B)$. We set $x_\alpha = (1 - 1/\alpha; 0; \ldots; 0)$ and $v_\alpha(x) = v(\alpha(x - x_\alpha))$. Then supp $v_\alpha \subset B_{1/\alpha}(x_\alpha)$ and

$$\int\limits_B v_\alpha^q |x|^\alpha dx = \int\limits_{B_{1/\alpha}(x_\alpha)} v_\alpha^q |x|^\alpha dx \geqslant \Big(1 - \frac{2}{\alpha}\Big)^\alpha \int\limits_{B_{1/\alpha}(x_\alpha)} v_\alpha^q dx = \alpha^{-n}\Big(1 - \frac{2}{\alpha}\Big)^\alpha \int\limits_B v^q dx.$$

Therefore,

$$\mathscr{Q}_{p,q,\alpha}(v_\alpha) \leqslant \frac{\alpha^{p-n}\|\nabla v\|^p_{L_p(B)} + \alpha^{-n}\|v\|^p_{L_p(B)}}{\alpha^{-np/q}(1-2/\alpha)^{p\alpha/q}\|v\|^p_{L_q(B)}} \leqslant c_9(p,q)\alpha^{p-n+\frac{np}{q}},$$

i.e.,

$$\inf_{W_p^1(B)\setminus\{0\}} \mathscr{Q}_{p,q,\alpha}(u) \leqslant c_9 \alpha^{p-n+\frac{np}{q}}.$$

On the other hand, from Theorem 2.1 we have

$$\inf_{\mathscr{W}_{(n,0)}\setminus\{0\}} \mathscr{Q}_{p,q,\alpha}(u) \asymp \alpha^{\frac{p}{q}}.$$

Since $p - n + np/q < p/q$ for $q > p^{**}$, we find

$$\inf_{W_p^1(B)\setminus\{0\}} \mathscr{Q}_{p,q,\alpha}(u) < \inf_{\mathscr{W}_{(n,0)}\setminus\{0\}} \mathscr{Q}_{p,q,\alpha}(u)$$

if α is sufficiently large. □

Theorem 3.2. *Let $q \in (p^{**}; \min\{p^*, p_m^{**}\})$. Then there exists $\widehat{\alpha}_m(p,q) > 0$ such that for all $\alpha > \widehat{\alpha}_m$ the (global) minimizer of the functional $\mathscr{Q}_{p,q,\alpha}$ is not an (m,k)-radial function.*

Proof. As was shown in Theorem 3.1, for $q < p^*$ we have

$$\inf_{W_p^1(B)\setminus\{0\}} \mathscr{Q}_{p,q,\alpha}(u) \leqslant c_9 \alpha^{p-n+\frac{np}{q}}.$$

But (2.6) implies

$$\inf_{\mathscr{W}_{(m,k)}\setminus\{0\}} \mathscr{Q}_{p,q,\alpha}(u) \geqslant c_{10}(n,p,q)\alpha^{\frac{p}{q}}$$

for $q < p_m^{**}$. Therefore, for $q > p^{**}$ and sufficiently large α

$$\inf_{W_p^1(B)\setminus\{0\}} \mathscr{Q}_{p,q,\alpha}(u) < \inf_{\mathscr{W}_{(m,k)}\setminus\{0\}} \mathscr{Q}_{p,q,\alpha}(u).$$

The theorem is proved. □

Lemma 3.1. *Assume that $m < n$ and $q \in (p, p_m^*)$. Then for sufficiently large α*

$$\mathscr{Q}_{p,q,\alpha}(\mathbf{v}_{m,\alpha}) \leqslant c_{11}\alpha^{\frac{np}{q}-n+p+(m-1)\left(1-\frac{p}{q}\right)}.$$

Proof. We note that $\mathscr{Q}_{p,q,\alpha}(\mathbf{v}_{m,\alpha}) \leqslant \mathscr{Q}_{p,q,\alpha}(u)$ for any $u \in \mathscr{W}_{(m,k)}$. We consider a subset of nonnegative functions u such that $\operatorname{supp} u \subset \{r \geqslant 1 - 1/\alpha\}$. We introduce the auxiliary functional

$$Q_{p,q}(u) = \frac{\|u\|^p_{W^1_p(B)}}{\|u\|^p_{L_q(B)}}.$$

Then

$$\mathscr{Q}_{p,q,\alpha}(u) = \frac{\|u\|^p_{W^1_p(B)}}{\left(\int\limits_B |x|^\alpha u^q dx\right)^{p/q}} = \frac{\|u\|^p_{W^1_p(B)}}{\left(\int\limits_{B\setminus B_{1-1/\alpha}} |x|^\alpha u^q dx\right)^{p/q}}$$

$$\leqslant \frac{\|u\|^p_{W^1_p(B)}}{\|u\|^p_{L_q(B)}} \cdot \left(1 - \frac{1}{\alpha}\right)^{-\alpha p/q} \leqslant c_{12}(p,q) Q_{p,q}(u).$$

We make the similarity transformation $B \to \alpha B$ and denote $\xi = \alpha x$ and $\widetilde{u}(\xi) = u(\xi/\alpha)$. We extend $\widetilde{u}$ along the radial variable as follows:

$$\widetilde{v}(|\xi|, \theta) = \begin{cases} \widetilde{u}(|\xi|, \theta), & \alpha - 1 \leqslant |\xi| \leqslant \alpha, \\ \widetilde{u}(2\alpha - |\xi|, \theta), & \alpha \leqslant |\xi| \leqslant \alpha + 1. \end{cases}$$

Then

$$Q_{p,q}(u) = \frac{\alpha^{\frac{np}{q}-n+p} \int\limits_{\alpha B} (|\nabla \widetilde{u}|^p + \alpha^{-p}\widetilde{u}^p) d\xi}{\left(\int\limits_{\alpha B} \widetilde{u}^q d\xi\right)^{p/q}} \leqslant c_{13}(p,q)\alpha^{\frac{np}{q}-n+p} \frac{\int\limits_{B_{\alpha+1}\setminus B_{\alpha-1}} |\nabla \widetilde{v}|^p d\xi}{\left(\int\limits_{B_{\alpha+1}\setminus B_{\alpha-1}} \widetilde{v}^q d\xi\right)^{p/q}}.$$

Since $\widetilde{v}$ is an (m,k)-radial function supported in the annulus $B_{\alpha+1} \setminus B_{\alpha-1}$, Lemma 1.5 in [6] yields the estimate

$$\frac{\int\limits_{B_{\alpha+1}\setminus B_{\alpha-1}} |\nabla \widetilde{v}|^p d\xi}{\left(\int\limits_{B_{\alpha+1}\setminus B_{\alpha-1}} \widetilde{v}^q d\xi\right)^{p/q}} \leqslant c_{14}(p,q)\alpha^{(m-1)(1-p/q)}.$$

Finally,

$$\mathscr{Q}_{p,q,\alpha}(\mathbf{v}_{m,\alpha}) \leqslant \mathscr{Q}_{p,q,\alpha}(u) \leqslant c_{11}(n,m,p,q)\alpha^{\frac{np}{q}-n+p+(m-1)\left(1-\frac{p}{q}\right)}.$$

The theorem is proved. □

Theorem 3.3. *Assume that $p < n-m+1$ and $q \in (p^{**}_m; p^*_m)$. Then there exists $\widetilde{\alpha}_m(p,q) > 0$ such that for all $\alpha > \widetilde{\alpha}_m$ the (m,k)-minimizer of the functional $\mathscr{Q}_{p,q,\alpha}$ is not radial.*

Proof. We assume that $\mathbf{v}_{(m,k)}$ is a radial function. By Theorem 2.1,

$$\mathscr{Q}_{p,q,\alpha}(\mathbf{v}_{(m,k)}) \asymp \alpha^{\frac{p}{q}}.$$

On the other hand, by Lemma 3.1,

$$\mathscr{Q}_{p,q,\alpha}(\mathbf{v}_{(m,k)}) \leqslant c_{11}\alpha^{\frac{np}{q}-n+p+(m-1)\left(1-\frac{p}{q}\right)}.$$

However, for $q \in (p^{**}_m; p^*_m)$ we have $np/q - n + p + (m-1)(1-p/q) < p/q$. $\square$

It is easy to see that $p^{**}_m \leqslant p^*$ for $pm \leqslant n$ and $p^{**}_m > p^*$ for $pm > n$. Furthermore, in the case $pm \leqslant n$, we have $n - m + 1 - p \geqslant n - m + 1 - n/m = [(n-m)(m-1)]/m > 0$, i.e., p^{**}_m and p^*_m are finite.

As was mentioned in Section 1, renormalized minimizers over different subspaces $\mathscr{W}_{(m,k)}$ are solutions to the problem (1.1). Theorems 3.1 and 3.2 provide sufficient nonuniqueness conditions for this problem. A radial solution exists for all $1 < p < q < +\infty$. Figures 1 and 2 show intervals of values of q and α where nonradial solutions exist. It is important to note that the intervals in Figures 1 and 2 depend on m.

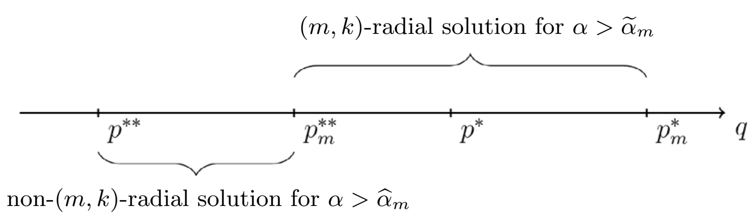

FIGURE 1. Case $pm \leqslant n$.

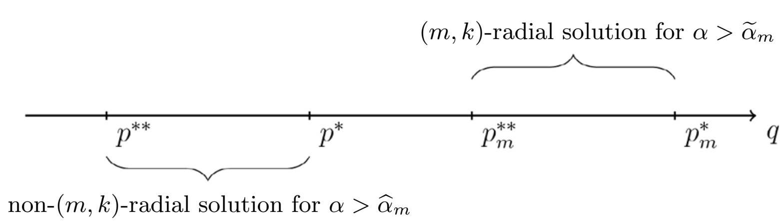

FIGURE 2. Case $pm > n$.

For $n \geqslant 4$ we consider the decomposition $\mathbb{R}^n = \mathbb{R}^2 \oplus \mathbb{R}^{n-2}$, i.e., $m = 2$, $k = n-2$. Let G be a finite subgroup of $O(2)$. We denote by $\mathbf{v}_G$ a function minimizing the functional $\mathscr{Q}_{p,q,\alpha}$ over the subspace $\mathscr{W}_G$ of all $(2, n-2)$-symmetric functions in $W^1_p(B)$ that are G-invariant with respect to y. Repeating the proof of Proposition 2.1, we find that all $\mathbf{v}_G$ are solutions to the problem (1.1).

Proposition 3.1. *For $q \in (p^{**}, \min\{p^*, p^{**}_2\})$ the function $\mathbf{v}_G$ is not $(2, n-2)$-radial.*

Proof. We consider a nonnegative $(2, n-2)$-symmetric function $v \in C_0^\infty(B)$ such that $\operatorname{supp} v \subset \{x \in B,\ |x| \geqslant 1 - 1/\alpha\}$. We define

$$u(y_1, r_0) = \sum_{g \in G} v_g(y_1, r_0),$$

where $v_g(y_1, r_0) = v((r_1, \theta_1) - (1, g\theta^0), r_0)$, θ^0 is a point of $\mathbb{S}_2$. Then $u \in \mathscr{W}_G$. Repeating the proof of Theorem 3.1, we find

$$\mathscr{Q}_{p,q,\alpha}(u) \leqslant c_{15}(p, q, G)\alpha^{p-n+np/q}.$$

Therefore,

$$\mathscr{Q}_{p,q,\alpha}(\mathbf{v}_G) \leqslant c_{15}(p, q, G)\alpha^{p-n+\frac{np}{q}}. \tag{3.1}$$

However, if $\mathbf{v}_G$ is a $(2, n-2)$-radial function, then

$$\mathscr{Q}_{p,q,\alpha}(\mathbf{v}_G) \geqslant c_{10}\alpha^{p/q}$$

in view of Theorem 2.1, which contradicts the inequality (3.1) for $q > p^{**}$. □

We extract a class G providing multiplicity of solutions to the problem (1.1). Let G_t be the group generated by rotation by an angle $2\pi/t$, $t \in \mathbb{N}$. Denote by $\mathbf{v}_t$ the minimizer of $\mathscr{Q}_{p,q,\alpha}$ on the set $\mathscr{W}_{G_t}$.

Theorem 3.4. *Let $q \in (p^{**}, \min\{p^*, p_2^{**}\})$. Then for any $t_0 \in \mathbb{N}$ there exists $\alpha_0(n, p, q, t_0)$ such that for $\alpha > \alpha_0$ and all $t = 1, 2, \ldots, t_0$ the problem (1.1) has a $(2, n-2)$-symmetric G_t-invariant solution $\mathbf{v}_t$. For different t the solutions $\mathbf{v}_t$ are different.*

Proof. We consider the functions $\mathbf{v}_t$ and $\mathbf{v}_{t'}$, $t' > t$. Let $t' = st$, $s \in \mathbb{N}$. We introduce the function

$$v_t((r_1, \theta_1), r_0) = \mathbf{v}_{st}\Big(\Big(r_1, \frac{1}{s}\theta_1\Big), r_0\Big).$$

Then $v_t \in W_p^1(B)$ is a $(2, n-2)$-symmetric function that is G_t-invariant with respect to y_1. It is easy to see that

$$\int_B |v_t|^p dx = \int_B |\mathbf{v}_{st}|^p dx, \quad \int_B |x|^\alpha |v_t|^q dx = \int_B |x|^\alpha |\mathbf{v}_{st}|^q dx.$$

Since $(v_t)_{\theta_i} = \dfrac{1}{s}(\mathbf{v}_{st})_{\theta_i}$, we have

$$\int_B |\nabla v_t|^p dx = \int_B \Big((v_t)_{r_1}^2 + \frac{1}{r_1^2}(v_t)_{\theta_1}^2 + (v_t)_{r_0}^2\Big)^{p/2} dx$$

$$= \int_B \bigg((\mathbf{v}_{st})_{r_1}^2 + \frac{1}{s^2}\frac{1}{r_1^2}(\mathbf{v}_{st})_{\theta_1}^2 + (\mathbf{v}_{st})_{r_0}^2\bigg)^{p/2} dx < \int_B |\nabla \mathbf{v}_{st}|^p dx \tag{3.2}$$

only if $(\mathbf{v}_{st})_{\theta_1} \not\equiv 0$. However, the identity $(\mathbf{v}_{st})_{\theta_1} \equiv 0$ should mean that $\mathbf{v}_{st}$ is a $(2, n-2)$-radial function, which is impossible for large α in view of Proposition 3.1. Therefore, (3.2) implies

$$\mathscr{Q}_{p,q,\alpha}(\mathbf{v}_t) \leqslant \mathscr{Q}_{p,q,\alpha}(v_t) < \mathscr{Q}_{p,q,\alpha}(\mathbf{v}_{st}) \tag{3.3}$$

and the assertion is proved in this case.

In the general case, we denote by $\widehat{t}$ the least common multiple of (t, t'). If $\mathbf{v}_t$ is equivalent to $\mathbf{v}_{t'}$, then $\mathbf{v}_t$ is $G_{\widehat{t}}$-invariant with respect to y_i and, consequently, $\mathscr{Q}_{p,q,\alpha}(\mathbf{v}_{\widehat{t}}) \leqslant \mathscr{Q}_{p,q,\alpha}(\mathbf{v}_t)$, which contradicts (3.3). $\square$

In the case $n = 2$, only the trivial decomposition of $\mathbb{R}^2$ is possible. In this case, an analogue of Figure 2 looks like Figure 3.

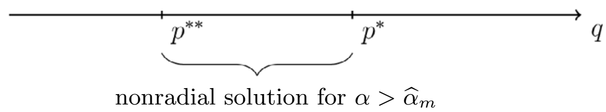

FIGURE 3. Case $n = 2$, $1 < p < 2$.

Repeating the proof of Theorem 3.4, we obtain the following assertion.

Theorem 3.5. *Assume that $n = 2$, $p \in (1, 2)$, and $q \in (p^{**}, p^*)$. Then for any $t_0 \in \mathbb{N}$ there exists $\alpha_0(p, q, t_0)$ such that for $\alpha > \alpha_0$ and all $t = 1, 2, \ldots, t_0$ the problem (1.1) has a G_t-invariant solution $\mathbf{v}_t$. For different t the solutions $\mathbf{v}_t$ are different.*

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