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On the uniform convergence of Fourier series to (ψ, β)-derivatives

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Abstract. In terms of the best approximations of a function in the space L_p, the conditions of existence of its (ψ, β)-derivatives and the uniform convergence of Fourier series to them are determined.

Keywords. (ψ, β)-derivatives, uniform convergence of Fourier series, best approximations of a function, spaces L_p.

1. Introduction

Let L_p be a space of measurable 2π-periodic functions $f(x)$ for which $\int\limits_0^{2\pi} |f(x)|^p \, dx < \infty$, $1 < p < \infty$, and let

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}$$

be its Fourier series.

Let $\psi(t) > 0$ for $t \geq 1$, and let β be any fixed real number. If the series

$$\sum_{k=-\infty}^{\infty} \frac{\hat{f}_k}{\psi(|k|)} e^{i(kx+\beta sign k)}$$

is the Fourier series of some summable function, it is called the (ψ, β)-derivative of a function f and is denoted f_β^ψ. The set of functions that satisfy these conditions is denoted by L_β^ψ.

If $f \in L_\beta^\psi$ and if $f_\beta^\psi \in \mathfrak{N}$ ($\mathfrak{N} \subset L(0, 2\pi)$), we say that the function belongs to the class $L_\beta^\psi \mathfrak{N}$ [1, pp. 142–143].

The classes introduced in such way at the fixed parameters defining them coincide with the known classes of functions $W^r; W_{\beta,p}^r; W_\beta^r H_\omega$ and similar ones.

The present work is devoted to the determination of a sufficient conditions of existence of the continuous (ψ, β)-derivative of a function f from L_p and the uniform convergence of the Fourier series of the (ψ, β)-derivative.

In Section 2, we give the definitions and theorems which are necessary for the formulation and proof of the main result of this work, and the theorem itself and its proof are presented in Section 3.

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2. Auxiliary assertions

Let $\omega_k(f,\delta)_p$ be the modulus of smoothness of the k-th order (k is a natural number) in the space $L_p(0,2\pi)$

$$\omega_k(f,\delta)_p = \sup_{|h|\le\delta}\left(\int_0^{2\pi} |\Delta_h^k f(x)|^p\,dx\right)^{\frac{1}{p}},$$

where $\Delta_h^k f(x)$ is a finite difference of the kth order of the function $f(x)$ with a step h

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k-i}\binom{k}{i} f(x+ih).$$

If $k=1$, we write $\omega(f,\delta)_p$ instead $\omega_1(f,\delta)_p$.

By $E_n(f)_p$, we denote the best approximation of the function $f(x)$ in the metric of the space L_p by means of trigonometric polynomials of the order of at most $n-1$.

The following proposition is valid (see [2]).

Theorem 2.1. *Let $\psi(t)$ be such positive nonincreasing function which is defined for all $t\ge 1$ and such that $\psi(2t)\ge c\psi(t)$ for $t\ge 1$ (c is some positive constant), and let the best approximations of the function $f\in L_p$, $1<p<\infty$, satisfy the condition*

$$\sum_{k=1}^{\infty}\frac{E_k(f)_p}{\psi(k)k}<\infty. \tag{2.1}$$

Then, for any real β, the function $f\in L_p$ has the (ψ,β)-derivative which belongs to L_p, and

$$E_n(f_\beta^\psi)_p \le c_1\left(\frac{E_n(f)_p}{\psi(n)}+\sum_{k=n+1}^{\infty}\frac{E_k(f)_p}{\psi(k)k}\right). \tag{2.2}$$

Here and below, c and c_i, $i=1,2,\dots$, are some constants depending in the general case on the function $\psi(\cdot)$. In the case where they depend not only on ψ, their defining parameters will be given in brackets.

This theorem yields the following assertion.

Corollary 2.1. *Let conditions of Theorem 2.1 be satisfied. Then, for any real β there exists the (ψ,β)-derivative of the function $f\in L_p$. It belongs to L_p, and*

$$\omega_k(f_\beta^\psi,\frac{1}{n})_p \le c(k)\left(\frac{1}{n^k}\sum_{\nu=1}^{n}\frac{E_\nu(f)_p}{\psi(\nu)}+\sum_{\nu=n+1}^{\infty}\frac{E_\nu(f)_p}{\psi(\nu)\nu}\right), \tag{2.3}$$

where k is any natural number.

Proof. It is known [3]) that if $f\in L_p$, $1\le p\le\infty$ and if k is a natural number, then

$$\omega_k(f,\frac{1}{n})_p \le c(k)\frac{1}{n^k}\sum_{\nu=1}^{n}\nu^{k-1}E_\nu(f)_p.$$

Theorem 2.1 implies that the (ψ,β)-derivative of a function f exists and belongs to L_p. Therefore, the inequality

$$\omega_k(f_\beta^\psi,\frac{1}{n})_p \le c(k)\frac{1}{n^k}\sum_{\nu=1}^{n}\nu^{k-1}E_\nu(f_\beta^\psi)_p$$

holds. Substituting the right-hand side of inequality (2.2) in this inequality instead of $E_\nu(f_\beta^\psi)_p$, we get

$$\omega_k(f,\frac{1}{n})_p \le c_1\frac{1}{n^k}\sum_{\nu=1}^{n}\left(\nu^{k-1}\frac{E_\nu(f)_p}{\psi(\nu)}+\sum_{m=\nu+1}^{\infty}\frac{E_m(f)_p}{\psi(m)m}\right)$$

$$\le c_2\left(\frac{1}{n^k}\sum_{\nu=1}^{n}\nu^{k-1}\frac{E_\nu(f)_p}{\psi(\nu)}+\frac{1}{n^k}\sum_{\nu=1}^{n}\nu^{k-1}\sum_{m=\nu+1}^{n}\frac{E_m(f)_p}{\psi(m)m}+\sum_{m=n+1}^{\infty}\frac{E_m(f)_p}{\psi(m)m}\right)$$

$$\le c_3\left(\frac{1}{n^k}\sum_{\nu=1}^{n}\nu^{k-1}\frac{E_\nu(f)_p}{\psi(\nu)}+\frac{1}{n^k}\sum_{\nu=1}^{n}\frac{E_\nu(f)_p}{\psi(\nu)\nu}\sum_{m=1}^{\nu}m^{k-1}+\sum_{m=n+1}^{\infty}\frac{E_m(f)_p}{\psi(m)m}\right)$$

$$\le c_3\left(\frac{1}{n^k}\sum_{\nu=1}^{n}\frac{E_\nu(f)_p}{\psi(\nu)}+\sum_{\nu=n+1}^{\infty}\frac{E_\nu(f)_p}{\psi(\nu)\nu}\right).$$

The corollary is proved. □

To prove the main result of the present work, we will use the Konyushkov–Stechkin inequality. Namely, we will need the following proposition (see [4]).

Theorem 2.2. *Let the function $f \in L_p(0,2\pi)$, $1 \le p < \infty$ be such that*

$$\sum_{k=1}^{\infty}E_k(f)_p k^{\frac{1}{p}-\frac{1}{q}-1} < \infty, \tag{2.4}$$

where q is some number, $p < q \le \infty$. Then $f \in L_q(0,2\pi)$, and

$$E_n(f)_q \le c_3\left(E_n(f)_p(n+1)^{\frac{1}{p}-\frac{1}{q}}+\sum_{k=n+1}^{\infty}E_k(f)_p k^{\frac{1}{p}-\frac{1}{q}-1}\right) \tag{2.5}$$

$n = 0, 1, \ldots.$

3. Main result

The main result of the present work is the following assertion.

Theorem 3.1. *Let $\psi(t)$ be a positive nonincreasing function which is defined for all $t \ge 1$ and is such that $\psi(2t) \ge c\psi(t)$ for $t \ge 1$ (c is some positive constant), and let the best approximations of the function $f \in L_p$, $1 < p < \infty$, satisfy the condition*

$$\sum_{k=1}^{\infty}\frac{k^{\frac{1}{p}}E_k(f)_p}{\psi(k)k} < \infty. \tag{3.1}$$

Then, for any real β, the function $f \in L_p$ possesses a continuous (ψ,β)-derivative whose Fourier series converges uniformly.

Proof. According to condition (3.1), the series

$$\sum_{k=1}^{\infty} \frac{E_k(f)_p}{\psi(k)k}$$

is convergent. By virtue of Theorem 2.1, this guarantees the existence of the (ψ, β)-derivative of the function f. The derivative belongs to the space L_p and satisfies inequalities (2.2) and (2.3). Substituting estimate (2.2) in the Konyushkov–Stechkin inequality (2.5) (written here for the function f_β^ψ and the $q = \infty$)

$$E_n(f_\beta^\psi)_\infty \le c_3 \left(E_n(f_\beta^\psi)_p (n+1)^{\frac{1}{p}} + \sum_{k=n+1}^{\infty} E_k(f_\beta^\psi)_p k^{\frac{1}{p}-1} \right)$$

we get

$$E_n(f_\beta^\psi)_\infty \le c_4 \left(\frac{E_n(f)_p (n+1)^{\frac{1}{p}}}{\psi(n)} + (n+1)^{\frac{1}{p}} \sum_{k=n+1}^{\infty} \frac{E_k(f)_p}{\psi(k)k} \right)$$
$$+c_4 \left(\sum_{k=n+1}^{\infty} \frac{k^{\frac{1}{p}} E_k(f)_p}{\psi(k)k} + \sum_{k=n+1}^{\infty} k^{\frac{1}{p}-1} \sum_{m=k+1}^{\infty} \frac{E_m(f)_p}{\psi(m)m} \right). \tag{3.2}$$

To prove the continuity of the function f_β^ψ, it is sufficient (and necessary) to prove the relations

$$\lim_{n\to\infty} E_n(f_\beta^\psi)_\infty = 0. \tag{3.3}$$

For this purpose, we evaluate the terms on the right-hand side of inequality (3.2).

First, we establish the relation

$$\lim_{n\to\infty} \frac{E_n(f)_p (n)^{\frac{1}{p}}}{\psi(n)} = 0. \tag{3.4}$$

Let

$$\alpha_n \equiv \sum_{k=[\frac{n}{2}]+1}^{\infty} \frac{k^{\frac{1}{p}} E_k(f)_p}{\psi(k)k} \ge \sum_{k=[\frac{n}{2}]+1}^{n} \frac{k^{\frac{1}{p}} E_k(f)_p}{\psi(k)k}. \tag{3.5}$$

Since the sequence $E_k(f)_p$ does not increases in k, we get

$$\alpha_n \ge \left(\frac{E_n(f)_p}{\psi(n)} \sum_{k=[\frac{n}{2}]+1}^{n} \frac{k^{\frac{1}{p}} \psi(n)}{k\psi(k)} \right). \tag{3.6}$$

But the values of k in the sum on the right-hand side of inequality (3.5) are larger than $[\frac{n}{2}]+1$. Hence, $k > \frac{n}{2}$. From whence and the properties of the function $\psi(t)$, we obtain the relation $\psi(n) \ge c\psi(\frac{n}{2}) \ge \psi(k)$, i.e., $\frac{\psi(n)}{\psi(k)} \ge c$ for $k \ge \frac{n}{2}$. Substituting this inequality in (3.5) and taking into account that the sequence $k^{\frac{1}{p}-1}$ decreases, and $n - [\frac{n}{2}] \ge \frac{n}{2}$, we have

$$\alpha_n \ge c\frac{E_n(f)_p}{\psi(n)} \left(n - \left[\frac{n}{2}\right] \right) n^{\frac{1}{p}-1} \ge \frac{c}{2} \frac{E_n(f)_p}{\psi(n)} n^{\frac{1}{p}}. \tag{3.7}$$

By Definition (3.5), the number α_n is a residual of the convergent series (3.1). Therefore, $\alpha_n \to 0$ as $n \to \infty$. This result and estimate(3.6) yield equality (3.4).

Let us set

$$\sigma_n \equiv \sum_{k=n+1}^{\infty} k^{\frac{1}{p}-1} \sum_{m=k+1}^{\infty} \frac{E_m(f)_p}{\psi(m)m}. \tag{3.8}$$

We now show that σ_n are finite, and

$$\lim_{n\to\infty} \sigma_n = 0. \tag{3.9}$$

Permutating the summations over k and n in the double sum (3.8) and using the inequality

$$\sum_{k=n+1}^{m} k^{\frac{1}{p}-1} \le \int\limits_{n}^{m} x^{\frac{1}{p}-1}\, dx \le pm^{\frac{1}{p}},$$

we obtain the relations

$$\sigma_n = \sum_{m=n+1}^{\infty} \frac{E_m(f)_p}{\psi(m)m} \sum_{k=n+1}^{m} k^{\frac{1}{p}-1} \le p \sum_{m=n+1}^{\infty} \frac{m^{\frac{1}{p}} E_m(f)_p}{\psi(m)m}.$$

From whence and the convergence of series (3.1), we get equality (3.9). (The legitimacy of a permutation of terms in the sums follows from the positiveness of terms and the well-known theorem of mathematical analysis).

We note that the convergence of the series on the right-hand side of identity (3.8) is equivalent to the convergence of the series

$$\sum_{k=n+1}^{\infty} \left((k+1)^{\frac{1}{p}-1} \sum_{m=k+1}^{\infty} \frac{E_m(f)_p}{\psi(m)m} \right),$$

whose terms in large parentheses monotonically decrease. Therefore,

$$\lim_{k\to\infty} \left((k+1)^{\frac{1}{p}-1} \sum_{m=k+1}^{\infty} \frac{E_m(f)_p}{\psi(m)m} \right) = 0. \tag{3.10}$$

Thus, according to equalities (3.4), (3.8)–(3.10), The first, second, and fourth terms on the right-hand side of inequality (3.2) tend to zero, as n tends to infinity. The tending of the third term to zero follows from the convergence of series (3.1). So, we have established relation (3.3) and prove the continuity of the (ψ,β)-derivative of the function f.

To complete the proof of the theorem, it remains to show the uniform convergence of the Fourier series of the continuous function f^{ψ}_{β}. To this end, we will establish the relation

$$\lim_{n\to\infty} n^{\frac{1}{p}} \omega \left(f^{\psi}_{\beta}, \frac{1}{n} \right)_p = 0. \tag{3.11}$$

For any $\varepsilon > 0$, we choose m such that

$$\sum_{k=m}^{\infty} k^{\frac{1}{p}} \frac{E_k(f)_p}{\psi(k)k} < \varepsilon.$$

Then, for all $n > m$,

$$\frac{n^{\frac{1}{p}}}{n}\sum_{k=1}^{n}\frac{E_k(f)_p}{\psi(k)} = \frac{1}{n^{1-\frac{1}{p}}}\sum_{k=1}^{m}\frac{E_k(f)_p}{\psi(k)} + \sum_{k=m}^{n}\left(\frac{k}{n}\right)^{1-\frac{1}{p}}\frac{k^{\frac{1}{p}}E_k(f)_p}{\psi(k)k} \le \frac{c_m}{n^{1-\frac{1}{p}}} + \varepsilon.$$

By virtue of the arbitrariness of ε, this yields the relation

$$\lim_{n\to\infty}\frac{n^{\frac{1}{p}}}{n}\sum_{k=1}^{n}\frac{E_k(f)_p}{\psi(k)} = 0.$$

From whence, inequality (2.3), and equality (3.10), we get (3.11). Based on (3.11) and the known properties of the modulus of continuity, we may conclude that

$$\omega(f^{\psi}_{\beta}, \delta)_p = o(\delta^{\frac{1}{p}}).$$

According to the Yano theorem [5], this this condition ensures the uniform convergence of the Fourier series of the function f^{ψ}_{β}. Thus, the theorem is completely proved. □

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