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MULTIVARIATE ANALOGS OF CLASSICAL UNIVARIATE DISCRETE DISTRIBUTIONS AND THEIR PROPERTIES

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Some discrete distributions such as Bernoulli, binomial, geometric, negative binomial, Poisson, Polya–Aeppli, and others play an important role in applied problems of probability theory and mathematical statistics. We propose a variant of a multivariate distribution whose components have a given univariate discrete distribution. In fact we consider some very general variant of the so-called reduction method. We find the explicit form of the mass function and generating function of such distribution and study their properties. We prove that our construction is unique in natural exponential families of distributions. Our results are the generalization and unification of many results of other authors.

1. Introduction

Some discrete distributions such as Bernoulli, binomial, geometric, negative binomial, Poisson, Polya–Aeppli, and others play an important role in applied problems of probability theory and mathematical statistics.

There were some attempts to consider their multivariate generalizations. See, for example, the papers [1, 10–12, 15]. However, there is still no general approach to this problem. Moreover, many authors considered only the two-dimensional case.

We propose a general approach to this problem. The main ideas of this approach have been used in our papers (see [2–4]).

In subsequent papers we intend to consider some applications of such multivariate discrete distributions in actuarial and financial mathematics and teletraffic theory. An example can be found in [5].

2. Univariate discrete distributions

First, we recall the definitions and properties of some classical univariate discrete distributions. All random variables (r.v.'s) below take only nonnegative integer values. In this case it is very useful to calculate probability generating functions (pgf).

Definition 1. A r.v. ε has *Bernoulli distribution* with parameter $p, 0 < p < 1$, if

$$\varepsilon = \begin{cases} 1, & \text{with probability } p, \\ 0, & \text{with probability } 1-p. \end{cases}$$

The corresponding pgf has the form

$$\psi_\varepsilon(s) := \mathsf{E}\left(s^\varepsilon\right) = 1 + p(s-1)\ . \tag{1}$$

It is easy to calculate

$$\mathsf{E}(\varepsilon) = p,\ \mathsf{D}(\varepsilon) = p(1-p).$$

This r.v. can be regarded as the indicator of a random event (success).

Definition 2. A r.v. X has *binomial distribution* with parameters (n, p), if

$$\mathsf{P}(X = m) = C_n^m p^m (1-p)^{n-m}\ ,\ m = \overline{0, n}\ . \tag{2}$$

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It is well known that $X = \varepsilon_1 + \ldots + \varepsilon_n$, where $\varepsilon_1, \ldots, \varepsilon_n$ are independent identically distributed (i.i.d.) r.v.'s with the Bernoulli distribution. Then the pgf of such distribution has the form

$$\psi_X(s) = [1 + p(s-1)]^n \ . \tag{3}$$

It it easy to calculate

$$\mathsf{E}(X) = np,\ \mathsf{D}(X) = np(1-p).$$

In fact this r.v. is the number of successes in the Bernoulli scheme with parameters (n, p).

Definition 3. A r.v. X has *Poisson distribution* with parameter $\lambda > 0$, if

$$\mathsf{P}(X = m) = \frac{\lambda^m}{m!} e^{-\lambda} \ ,\ m = 0, 1, 2, \ldots \ . \tag{4}$$

The pgf of this distribution has the form

$$\psi_X(s) = e^{\lambda(s-1)} \ . \tag{5}$$

It is easy to calculate

$$\mathsf{E}(X) = \lambda,\ \mathsf{D}(X) = \lambda.$$

The following *Poisson theorem* is very well known.

Theorem 1. *If the r.v. X has the binomial distribution with parameters (n, p), $n \to \infty$, $p \to 0$, and $np \to \lambda$, $0 < \lambda < \infty$, then for a given m*

$$\mathsf{P}(X = m) \to \frac{\lambda^m}{m!} e^{-\lambda}.$$

Definition 4. A r.v. X has the *geometric distribution* with parameter p, $0 < p < 1$, if

$$\mathsf{P}(X = m) = p(1-p)^{m-1} \ ,\ m = 1, 2, \ldots \ . \tag{6}$$

The pgf of this distribution has the form

$$\psi_X(s) = \frac{ps}{1 - (1-p)s} \ . \tag{7}$$

It is easy to calculate

$$\mathsf{E}(X) = \frac{1}{p},\ \mathsf{D}(X) = \frac{1-p}{p^2}.$$

Remark 1. 1. The r.v. X is the number of the Bernoulli trials with parameter p till the first success.

2. Sometimes, the r.v. $X - 1$ is used in the definition of the geometric distribution.

3. The geometric distribution is infinitely divisible. So for any $r > 0$ we can define the new distribution with pgf

$$\left[\frac{ps}{1 - (1-p)s}\right]^r, \tag{8}$$

which is called the *negative binomial distribution* with parameters (p, r).

For integer r we get the definition of the *Pascal distribution*.

4. Let a r.v. X have the Poisson distribution with parameter Λ, which is the r.v. with the gamma distribution with parameters (α, β). It is well known that in this case the r.v. X has the negative binomial distribution with parameters (p, r), where $p = \beta/(1+\beta)$, $r = \alpha$.

The geometric distribution has the following remarkable *lack-of-memory property.*

Theorem 2. *A r.v. X has the geometric distribution if and only if for any integer $m, n \geqslant 1$*

$$\mathsf{P}(X \geqslant m+n | X \geqslant n) = \mathsf{P}(X \geqslant m).$$

Let $Y = (Y_j, j \geqslant 1)$ be a sequence of i.i.d. r.v.'s, which have the geometric distribution with parameter $p \in (0,1)$. Let the r.v. N have the Poisson distribution with parameter $\lambda > 0$ and be independent on sequence Y.

Definition 5. The *geometric Poisson distribution* or *Polya–Aeppli distribution* with parameters (λ, p) is the distribution of the r.v.

$$X := \sum_{j=1}^{N} Y_j \ .$$

It is easy to calculate

$$\psi_X(s) = e^{\lambda(\psi_{Y_j}(s)-1)} = \exp\left(\frac{\lambda \cdot (s-1)}{1-(1-p)\cdot s}\right), \tag{9}$$

and

$$\mathsf{E}(X) = \frac{\lambda}{p},\ \mathsf{D}(X) = \frac{(2-p)\lambda}{p^2}.$$

3. Lévy processes

Definition 6. A stochastic process $X = (X(t), t \geqslant 0)$ is said to be a *Lévy process*, if:

1) $X(0) = 0$ a.s.;

2) X has independent and homogenous increments.

In this case the r.v. $X(1)$ has an infinitely divisible distribution. Moreover, every infinitely divisible distribution generates some Lévy process.

It can be proved that geometric, Poisson, and Polya–Aeppli distributions are infinitely divisible. So, they generate the corresponding Lévy processes which are useful in applications and in the proofs of the properties of these distributions.

We have already seen that the positive convolution powers of the geometric distribution are (by definition!) negative binomial distributions.

4. Multivariate discrete distributions

Below we define multivariate analogs of the univariate discrete distributions mentioned above. It is useful to find the correct notation. In what follows, all operations with vectors are considered coordinate-wise.

Let I be a set of vectors i of the form $i = (i_1, \ldots, i_d)$, where $i_k = 0 \vee 1$.

Define I_k as the set of $i \in I$ such that $i_k = 1$, $I_{kl} = \{i \in I : i_k = i_l = 1\}$, $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{0} = (0, \ldots, 0)$, $\bar{i} = \mathbf{1} - i$.

Definition 7. A random vector $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_d)$ has the *multivariate Bernoulli distribution* if it takes values in the set I and $P(\varepsilon = i) = p_i$.

Here the numbers p_i are the parameters of this distribution.

Remark 2. 1. Any component ε_k of this vector has the univariate Bernoulli distribution with parameter $p^{(k)} = \sum\limits_{i \in I_k} p_i$.

2. Any subvector of this vector has the multivariate Bernoulli distribution (with new parameters).

3. The components of this vector are independent if and only if for any $i \in I$

$$p_i = \mathsf{P}(\varepsilon_1 = i_1, \ldots, \varepsilon_d = i_d) = \mathsf{P}(\varepsilon_1 = i_1) \cdot \ldots \cdot \mathsf{P}(\varepsilon_d = i_d).$$

4) If $A_1, \ldots, A_d$ are some random events, then the random vector ε with the Bernoulli distribution is the multivariate indicator of the occurence of these random events.

The pgf of the multivariate Bernoulli distribution has the following form: for $s = (s_1, \ldots, s_d)$

$$\psi_{\vec{\varepsilon}}(s) := E(s_1^{\varepsilon_1} \ldots s_d^{\varepsilon_d}) = \sum_i s^i \cdot p_i, \tag{10}$$

and

$$\mathsf{E}(\varepsilon_k) = p^{(k)},\ \mathsf{D}(\varepsilon_k) = p^{(k)}(1 - p^{(k)}),\ \operatorname{cov}(\varepsilon_k, \varepsilon_l) = p^{(kl)} - p^{(k)}p^{(l)},$$

where $p^{(k)} = \sum_{i \in I_k} p_i$, $p^{(kl)} = \sum_{i \in I_{kl}} p_i$.

Let $(\vec{\varepsilon}_j, j \geqslant 1)$ be a sequence of random vectors with the Bernoulli distribution which has parameters $(p_i, i \in I)$ and are independent (*Bernoulli trials*).

Definition 8. A random vector X has the *multivariate binomial distribution* with parameters $(p_i, i \in \in I; n)$ if it has the form

$$X = \vec{\varepsilon}_1 + \ldots + \vec{\varepsilon}_n\ .$$

Remark 3. 1. Any component of this vector has the univariate binomial distribution.

2. Any subvector of this vector has the multivariate binomial distribution.

3. If $p_i \neq 0$ only for i of the form $(0, \ldots, 0)$, $(1, 0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 1)$, then we come back to the usual polynomial distribution.

Again it is easy to calculate

$$\psi_X(s) := \mathsf{E}(s_1^{X_1} \ldots s_d^{X_d}) = (\sum_i s^i \cdot p_i)^n, \tag{11}$$

and

$$\mathsf{E}(X_k) = np^{(k)},\ \mathsf{D}(X_k) = np^{(k)}(1 - p^{(k)}),\ \operatorname{cov}(X_k, X_l) = n \cdot (p^{(kl)} - p^{(k)}p^{(l)}).$$

Let $(\vec{\varepsilon}_j = (\varepsilon_{j1}, \ldots, \varepsilon_{jd}, j \geqslant 1)$ be Bernoulli trials with parameters $(p_i, i \in I)$.

Definition 9. A random vector $X = (X_1, \ldots, X_d)$ has the *multivariate geometric distribution* with parameters $(p_i, i \in I)$, if for any $k = \overline{1, d}$

$$X_k = \inf(j \geqslant 1 :\ \varepsilon_{jk} = 1).$$

Remark 4. 1. Any component of this vector has the univariate geometric distribution with parameter $p^{(k)}$.

2. Any subvector of this vector has the multivariate geometric distribution.

3. The components of this vectors are independent if and only if the corresponding univariate Bernoulli schemes are independent.

4. This distribution is infinitely divisible. So, using the pgf we can define the multivariate negative binomial distribution.

There exists another definition of this distribution. Let $(Y_i, i \in I)$ be a set of independent r.v.'s, which have the univariate geometric distributions with parameters p_i. Define

$$X_k := \min_{i \in I_k} Y_i\ . \tag{12}$$

From this definition we can be easily derive the following result.

Theorem 3. *A random vector $X = (X_1, \ldots, X_d)$ has the multivariate geometric distributions with parameters $(p_i, i \in I)$ if and only if it is representable as* (12).

For $d = 2, 3$ the probabilities and the pgf of the multivariate geometric distribution can be calculated in the explicit form. For example, for $d = 2$ we have:

1) if $1 \leqslant m_1 < m_2$, then

$$\mathsf{P}(X_1 = m_1, X_2 = m_2) = P_{00}^{m_1-1} P_{10} [P_{00} + P_{10}]^{m_2-m_1-1} [P_{01} + P_{11}];$$

2) if $1 \leqslant m < m_1$, then

$$\mathsf{P}(X_1 = m_1, X_2 = m) = P_{00}^{m-1} P_{01} [P_{00} + P_{01}]^{m_1-m-1} [P_{10} + P_{11}];$$

if $1 \leqslant m_1 = m_2 = m$, then

$$\mathsf{P}(X_1 = X_2 = m) = P_{00}^{m-1} P_{11} \ . \tag{13}$$

It is not difficult to calculate the pgf:

$$\psi_X(s_1, s_2) = \frac{s_1 s_2}{1 - P_{00} s_1 s_2} \cdot \left[P_{11} + P_{01} \cdot \frac{s_1(P_{10} + P_{11})}{1 - (P_{00} + P_{01}) s_1} + P_{10} \cdot \frac{s_2(P_{01} + P_{11})}{1 - (P_{00} + P_{10}) s_2} \right] \ . \tag{14}$$

For $d > 3$ we obtain very cumbersome expressions. But for the pgf we have the following recursive formula.

Theorem 4. *For the pgf of the multivariate geometric distribution we have the following recursive relation*:

$$\psi_X(s) := E(s^X) = E(s_1^{X_1} \cdot \ldots \cdot s_d^{X_d}) = \frac{s_1 \ldots s_d}{1 - s_1 \ldots s_d \cdot P_{\mathbf{0}}} \cdot \sum_{i \neq \mathbf{0}} P_i \cdot \psi_{\bar{i} \cdot X}(s) \ .$$

The main idea of the proof is very simple. We need to wait when all the components in the Bernoulli trials take value 1 for the first time. So we wait till the first occurence of the nonzero value. The possible value is $i \neq \mathbf{0}$. Now we have to wait till all other components take value 1 for the first time. Then we apply the total probability formula.

It is easy to calculate the moments of the orders up to the second:

$$\mathsf{E}(X_k) = \frac{1}{p^{(k)}}, \ \mathsf{D}(X_k) = \frac{1 - p^{(k)}}{(p^{(k)})^2},$$

$$\text{cov}(X_k, X_l) = \frac{1}{[1 - p_{00}^{(kl)}]^2} \cdot \left(1 + p_{00}^{(kl)} + \frac{p_{10}^{(kl)}}{p^{(l)}} + \frac{p_{01}^{(kl)}}{p^{(k)}} \right) - \frac{1}{p^{(k)}} \cdot \frac{1}{p^{(l)}} \ ,$$

where

$$p_{uv}^{(kl)} = \mathsf{P}(\varepsilon_{jk} = u, \varepsilon_{jl} = v) = \sum_{i : i_k = u, i_l = v} p_i, \ p^{(k)} = p_{10}^{(kl)} + p_{11}^{(kl)}, \ p^{(l)} = p_{01}^{(kl)} + p_{11}^{(kl)}.$$

Recently I. V. Zolotukhin [17] considered analogous problems. He defined the multivariate geometric distribution by relation (12) and found a simple explicit expression for the tails and some recursive formula for the pgf, which is similar to ours, but are more complicated. Moreover, he proved an analog of the lack-of-memory property for this distribution. Also he obtained the following relations for tails: for $m = (m_1, \ldots, m_d)$

$$P(X_1 > m_1, \ldots, X_d > m_d) = \prod_{i \neq \mathbf{0}} (1 - p_i)^{\max\limits_{1 \leqslant k \leqslant d} i_k \cdot m_k}.$$

There exists an alternative definition of the multivariate geometric distribution. Let $(\varepsilon_j, j \geqslant 1)$ be a multivariate Bernoulli scheme with parameters $(p_i, i \in I)$ where $0 < p_{\mathbf{0}} < 1$. Define the r.v.

$$N = \inf(j \geqslant 1 : \vec{\varepsilon}_j = \mathbf{0}),$$

and for every $k = \overline{1,d}$

$$X_k = \sum_{j=1}^{N-1} \varepsilon_{jk}.$$

By definition, the random vector $X = (X_1, \dots, X_d)$ has the *alternative multivariate geometric distribution.*

The new distribution has the following properties:

1. every component X_k has the alternative univariate geometric distribution with parameter $p^{(k)}$;
2. every subvector has the alternative multivariate geometric distribution (with new parameters);
3. our definition and the new one are different;
4. the pgf of this distribution has the form

$$\psi_X(s) = \frac{1}{1 - \sum\limits_{i \neq \mathbf{0}} (p_i/p_{\mathbf{0}}) \prod_{k=1}^{d} (s_k - 1)^{i_k}}$$

(see, for example, [8] and [9]);

5) this distribution is infinitely divisible. So, using the pgf we can define the alternative multivariate negative binomial distribution.

Definition 10. A random vector $Y = (Y_1, \dots, Y_d)$ has the *alternative multivariate negative binomial distribution* with parameters $(\alpha; p_i, i \in I)$, if its pgf has the form

$$\psi_Y(s) = \left(\frac{1}{1 - \sum\limits_{i \neq \mathbf{0}} (p_i/p_{\mathbf{0}}) \prod_{k=1}^{d} (s_k - 1)^{i_k}} \right)^{-\alpha}.$$

Later we will explain why our definition is better. For example, the alternative multivariate geometric distribution does not possess the lack-of-memory property.

Let $(Y_i, i \in I)$ be a set of independent r.v.'s, which have the univariate Poisson distribution with parameter λ_i. Let

$$X_k := \sum_{i \in I_k} Y_i \ .$$

Definition 11. A random vector $X = (X_1, \dots, X_d)$ of the above form has the *multivariate Poisson distribution* with parameters $(\lambda_i, i \in I)$.

We again notice that:

1. Any component of this vector has the univariate Poisson distribution.
2. Any subvector of this vector has the multivariate Poisson distribution.
3. In general, the components of this vector are not independent.
4. There exists an analog of the Poisson theorem on the approximation of the multivariate binomial distribution by the multivariate Poisson distribution (see [8]).

Let $(\varepsilon_j, j \geqslant 1)$ be a sequence of random vectors with the Bernoulli distribution which has parameters $(p_i, i \in I)$ and are independent. Let N be a r.v. independent of $(\varepsilon_j, j \geqslant 1)$ which has Poisson distribution with parameter $\lambda > 0$. Define the random vector

$$X = \sum_{j=1}^{N} \varepsilon_j \ .$$

Then this random vector X has the multivariate Poisson distribution with parameters $\lambda_i = \lambda \cdot p_i$.

This result is very useful in many applied problems.

It is easy to calculate the pgf

$$\psi_X(s) = \exp(\lambda(P_\varepsilon(s) - 1)) = \exp\left(\lambda \cdot \sum_i p_i \cdot (s^i - 1)\right), \tag{15}$$

and

$$\mathsf{E}(X_k) = \mathsf{D}(X_k) = \sum_{i \in I_k} \lambda_i,\ \operatorname{cov}(X_k, X_l) = \sum_{i \in I_{kl}} \lambda_i.$$

There exists an interesting relation between the multivariate Poisson distribution, the alternative multivariate negative binomial distribution and the univariate gamma distribution.

Theorem 5. *Let $X = (X_1(t), \ldots, X_d(t), t \geqslant 0)$ be a multivariate Poisson process with parameters $(\lambda_i, i \in I, i \neq \mathbf{0})$, let W have univariate gamma distribution with parameters (α, β). Assume that X and W are independent. Define the random vector $Y = X(W)$. Then the random vector Y has the alternative multivariate negative binomial distribution with parameters $(\alpha; \delta_i = \beta/(\beta + \lambda_i))$.*

The proof of this theorem can be found in [8, 9].

In many applied problems we need to have a multivariate analog of the Poisson process with dependent components. Let $(Y_i = (Y_i(t), t \geqslant 0), i \in I, i \neq \mathbf{0})$ be a set of independent univariate Poisson process with parameters λ_i.

Definition 12. A multivariate random process $X = ((X_1(t), \ldots, X_d(t)), t \geqslant 0)$ is the multivariate Poisson process with parameters $(\lambda_i, i \in I)$, if

$$X_k(t) := \sum_{i \in I_k} Y_i(t)\ ,\ t \geqslant 0.$$

Some applications of this process in insurance can be found in [5].

Let $(Y_j, j \geqslant 1)$ be a sequence of independent random vectors with the multivariate geometric distributions which has parameters $(p_i, i \in I)$. Let the r.v. N be independent of $(Y_j, j \geqslant 1)$ and have the Poisson distribution with parameter $\lambda > 0$. Define a new random vector

$$X = \sum_{j=1}^{N} Y_j\ .$$

By the definition, this random vector X has the *multivariate Polya–Aeppli distribution* with parameters $(\lambda; p_i, i \in I)$.

1. Any component of this vector has the univariate Polya–Aeppli distribution.
2. Any subvector of this vector has the multivariate Polya–Aeppli distribution.
3. In general, the components of this vector are not independent.
4. Using the technique from the papers by Minkova and Balakrishnan and Zolotukhin, it is possible to find some recursive formulas for the calculation of the probabilities and the pgf.

It is easy to calculate the moments up to the second order:

$$\mathsf{E}(X_k) = \frac{\lambda}{p^{(k)}} = \lambda \cdot \mathsf{E}(Y_{jk}),\ \mathsf{D}(X_k) = \lambda \cdot \frac{2 - p^{(k)}}{[p^{(k)}]^2} = \lambda \cdot \mathsf{E}([Y_{jk}]^2),\ \operatorname{cov}(X_k, X_l) = \lambda \cdot \mathsf{E}(Y_{jk} \cdot Y_{jl}).$$

The multivariate geometric, Poisson, and Polya–Aeppli distributions are infinitely divisible. So, they generate some multivariate Lévy processes (with dependent components!).

Remark 5. It is well known that optimal estimators exist only in the so-called exponential families (EF) of distributions. It can be proved that the multivariate Bernoulli, binomial, negative binomial, geometric, and Poisson (and, possibly, Polya–Aeppli) distributions introduced above under some re-parametrization constitute such families of distributions. And within the EF there are no other distributions with such marginals. The proof of the last assertion can be found in [2, 3].

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