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ESTIMATES OF FUNCTIONS, ORTHOGONAL TO PIECEWISE CONSTANT FUNCTIONS, IN TERMS OF THE SECOND MODULUS OF CONTINUITY

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The paper is devoted to the problem of finding the exact constant W_2^* in the inequality $\|f\| \le K \cdot \omega_2(f,\,1)$ for bounded functions f with the property

$$\int_k^{k+1} f(x)\,dx = 0, \quad k \in \mathbb{Z}.$$

Our approach allows us to reduce the known range for the desired constant as well as the set of functions involved in the extremal problem for finding the constant in question. It is shown that W_2^* also turns out to be the exact constant in a related Jackson–Stechkin type inequality. Bibliography: 3 titles.

1. Introduction

1.1. The problem. Let us denote by F^0 the space of measurable bounded functions with the property

$$\int_k^{k+1} f(x)\,dx = 0, \qquad k \in \mathbb{Z}. \tag{1.1}$$

All the functions are considered real-valued and defined at any point. Let us equip the space F^0 with the norm

$$\|f\| = \sup_{x\in\mathbb{R}} |f|. \tag{1.2}$$

The second modulus of continuity of a function f with step h is defined by

$$\omega_2(f,\,h) = \sup_{|t|\le h} \|f(x-t) - 2f(x) + f(x+t)\|\,.$$

Here we mention some properties of $\omega_2(f,\,h)$ which we use very soon:

$$\omega_2(f,\,h) \le \omega_2(g,\,h) + 4\|f-g\|; \tag{1.3}$$

$$\omega_2(\alpha f,\,h) = |\alpha|\omega_2(f,\,h); \tag{1.4}$$

$$\omega_2(f(\,\cdot\,+1),\,h) = \omega_2(f(\,\cdot\,),\,h); \tag{1.5}$$

$$\omega_2(f(-\,\cdot\,),\,h) = \omega_2(f(\,\cdot\,),\,h). \tag{1.6}$$

We denote by W_2^* the exact constant for the inequality

$$\|f\| \le K \cdot \omega_2(f,\,1) \tag{1.7}$$

in the space F^0.

In [1], Yu. Kryakin showed that

$$0.5058 \le W_2^* \le 0.6244.$$

Unfortunately, the proof of the lower estimate $(0.5810 < W_2^*)$ given in [1] contains a mistake. We replaced it by the value actually established.

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Kryakin also considered an analogous problem for higher order moduli. The best results on this topic obtained so far are given in [2].

Let us introduce the notation

$$F_b = \left\{ f \in F^0 \middle| f\left(\frac{1+b}{2}\right) = \|f\| = 1 \right\} \quad \text{and} \quad F^* = \bigcup_{b\in[0,1]} F_b.$$

The goal of the present work is to study the magnitude $\inf\limits_{f\in F_b} \omega_2(f,\, 1)$ for $b \in [0,\, 1]$. The motivation for this is explained by the following statement.

Proposition 1. *The following relation holds:*

$$W_2^* = \sup_{f\in F^*} \frac{\|f\|}{\omega_2(f,\, 1)} = \frac{1}{\inf\limits_{f\in F^*} \omega_2(f,\, 1)}.$$

Proof. Let us show that

$$\sup_{f\in F^0} \frac{\|f\|}{\omega_2(f,\, 1)} = \sup_{f\in F^*} \frac{\|f\|}{\omega_2(f,\, 1)}.$$

By definition, for any $n \in \mathbb{N}$ there exists a point $x_n \in \mathbb{R}$ such that

$$|f(x_n)| > \|f\| - \frac{1}{n}.$$

Without loss of generality, we may assume that $f(x_n) > 0$. Let us consider the functions

$$f_n(x) = \begin{cases} f(x), & x \neq x_n, \\ \|f\|, & x = x_n. \end{cases}$$

Inequality (1.3) implies that

$$\omega_2(f_n,\, 1) \le \omega_2(f,\, 1) + \frac{1}{n}.$$

Consequently, it is sufficient to consider only functions that attain their norm at some point.

On the other hand, because of (1.4), it is sufficient to consider only the case $f \in F_b$, $b \in \mathbb{R}$.

Finally, due to (1.5) and (1.6), we may assume that $b \in [0,\, 1]$. □

Let us denote by F the space of measurable bounded functions with the property

$$\int_k^{k+1} f(x)\, dx = \int_0^1 f(x)\, dx, \quad k \in \mathbb{Z},$$

and the norm defined above.

Proposition 2. *Let f be in F, let $E_0(f)$ be the best approximation of f by constant functions, and let J_2^* be the exact constant in the Jackson type inequality*

$$E_0(f) \le K \cdot \omega_2(f,\, 1).$$

Then

$$J_2^* = W_2^*.$$

Proof. Let $f \in F^*$. Take

$$f_n(x) = \begin{cases} f(x), & x \le 1, \\ \frac{n-k}{n} f(x), & x \in (k,\, k+1],\ k = 1, \dots, n, \\ 0, & x > n+1. \end{cases}$$

Consider the functions

$$\widetilde{f}_n = f_n(x) - f_n(2n+2-x) \quad \text{and} \quad \widetilde{F} = \left\{ \widetilde{f}_n \middle| f \in F^*,\, n \in \mathbb{N} \right\}.$$

It is easy to see that

$$E_0(\widetilde{f}_n) = \|\widetilde{f}_n\| = \|f\| = 1 \quad \text{and} \quad \omega_2(\widetilde{f}_n,\, 1) \le \omega_2(f) + \frac{4}{n}.$$

Consequently,

$$W_2^* = \sup_{F^*} \frac{\|f\|}{\omega_2(f,\, 1)} = \sup_{\widetilde{F}} \frac{\|f\|}{\omega_2(f,\, 1)} = \sup_{\widetilde{F}} \frac{E_0(f)}{\omega_2(f,\, 1)} \le \sup_{F} \frac{E_0(f)}{\omega_2(f,\, 1)} = J_2^*.$$

On the other hand, if $f \in F$, then $(f - \int_0^1 f(x)\,dx) \in F^0$ and

$$E_0(f) \le \|f - \int_0^1 f(x)\,dx\| \le W_2^* \cdot \omega_2(f,\, 1);$$

hence,

$$J_2^* \le W_2^*. \qquad \square$$

1.2. Main result. We use the function

$$q(b) = \begin{cases} \frac{96}{27b^3-27b^2+9b+55}, & b \in \left[0,\, \frac{1}{3}\right], \\ \frac{8(11b^2+66b-13)}{3b^4-69b^3+17b^2+385b-80}, & b \in \left[\frac{1}{3},\, 1\right]. \end{cases}$$

Theorem 1. *Let f be in F_b. Then*

$$q(b) \le \omega_2(f,\, 1).$$

The function $q(b)$ is continuous on the segment $[0,\, 1]$, has precisely two intervals of monotonicity, and attains its minimum between the points 0.43 and 0.44 with

$$q(b) > 1.6721.$$

In [3], we have presented a function from the space F^* whose second modulus of continuity is equal to $\frac{37861}{20548} - \frac{37\sqrt{8545}}{20548} < 1.6762$. Therefore,

$$1.6721 \le q(b) \le \inf_{f \in F^*} \omega_2(f,\, 1) \le 1.6762.$$

These inequalities, Theorem 1, and Proposition 1 imply the following statement.

Theorem 2. *The inequalities*

$$0.5965 \le W_2^* \le 0.5981$$

are valid. Moreover,

$$W_2^* = \sup_{b \in [b_0,\, b_1]} \frac{1}{\inf\limits_{f \in F_b} \omega_2(f,\, 1)},$$

where the points b_0 and b_1 are roots of the equation

$$q(b) = \frac{37861}{20548} - \frac{37}{20548}\sqrt{8545}.$$

2. A few lemmas and a little bit about the method

We use the following notation for the mean value of a function f on a segment $\left[x - \frac{r}{2},\, x + \frac{r}{2}\right]$, where $r > 0$:

$$f_r(x) = \frac{1}{r} \int\limits_{x-r/2}^{x+r/2} f(s)\, ds,$$

assuming that $f_0(x) = f(x)$.

To prove Theorem 1, we formulate several inequalities connecting some mean values of a function $f \in F_b$ with its second order modulus. Then we solve a linear programming problem

$$\omega_2(f,\, 1) \to \inf$$

with these mean values and $\omega_2(f,\, 1)$ taken as abstract variables.

The following lemma describes some basic properties of elements belonging to the set F_b.

Lemma 1. *Let f be in F_b and let $\tau,\, h \geq 0$. Then*

$$f_{2\tau}\left(\frac{1+b}{2} - 2h\right) - 2f_\tau\left(\frac{1+b}{2} - h\right) + 1 \leq \omega_2\left(f,\, h + \frac{\tau}{2}\right); \tag{2.1}$$

$$-f_\tau\left(\frac{1+b}{2} - h\right) - f_\tau\left(\frac{1+b}{2} + h\right) + 2 \leq \omega_2\left(f,\, h + \frac{\tau}{2}\right); \tag{2.2}$$

$$-2f_{2\tau}\left(\frac{1+b}{2}\right) + 2 \leq \omega_2\left(f,\, \frac{\tau}{2}\right); \tag{2.3}$$

$$-f_{\frac{1-b}{2}}\left(-\frac{1-b}{4}\right) + f_{\frac{1-b}{2}}\left(1 + \frac{3+b}{4}\right) + \frac{4}{1-b} \leq \frac{2}{1-b} \cdot \omega_2(f,\, 1). \tag{2.4}$$

Proof. Since $f\left(\frac{1+b}{2}\right) = 1$, inequalities (2.1) and (2.2) can be obtained by integrating

$$f\left(\frac{1+b}{2} - 2h + 2\tau s\right) - 2f\left(\frac{1+b}{2} - h + \tau s\right) + f\left(\frac{1+b}{2}\right) \leq \omega_2\left(f,\, h + \frac{\tau}{2}\right)$$

and

$$-f\left(\frac{1+b}{2} - h + \tau s\right) - 2f\left(\frac{1+b}{2} + h + \tau s\right) + f\left(\frac{1+b}{2}\right) \leq \omega_2\left(f,\, h + \frac{\tau}{2}\right),$$

respectively, with $s \in \left[-\frac{1}{2},\, \frac{1}{2}\right]$.

Further, (2.3) can be deduced from (2.2) be noticing that

$$2f_{2\tau}\left(\frac{1+b}{2}\right) = f_\tau\left(\frac{1+b}{2} - \frac{\tau}{2}\right) + f_\tau\left(\frac{1+b}{2} + \frac{\tau}{2}\right).$$

Finally, according to (2.3), we conclude that

$$-2f_2\left(\frac{1+b}{2}\right) + 2 \leq \omega_2(f,\, 1).$$

To prove (2.4), it is sufficient to take into account (1.1) and to notice that

$$\begin{aligned} 2f_2\left(\frac{1+b}{2}\right) &= \frac{1-b}{2} f_{\frac{1-b}{2}}\left(-\frac{1-b}{4}\right) + f_1\left(\frac{1}{2}\right) + \frac{1+b}{2} f_{\frac{1+b}{2}}\left(1 + \frac{1+b}{4}\right) \\ &= \frac{1-b}{2} f_{\frac{1-b}{2}}\left(-\frac{1-b}{4}\right) - \frac{1-b}{2} f_{\frac{1-b}{2}}\left(1 + \frac{3+b}{4}\right). \end{aligned}$$

□

More complicated results can be obtained in the following way. First one proves a statement of the kind

$$\left| f_\tau(x) - 2f_\sigma(y) + \sum_{i=1}^{n} \delta_i f_{\rho_k}(z_k) \right| \le \omega_2(f,\, h) \tag{2.5}$$

with

$$\sum_{i=1}^{n} \delta_i = 1, \quad \delta_i > 0, \quad x + \sum_{i=1}^{n} \delta_i z_k = 2y.$$

This result is then modified using (1.1) if needed. The simplest example is provided by the following statement.

Lemma 2. *Let f be in $L_{\mathrm{loc}}(\mathbb{R})$ and let $a \in \mathbb{R}$, h, τ, $\sigma \ge 0$.*
Then

$$\left| f_\tau(a-h) - 2f_\sigma(a) + f_{|2\sigma-\tau|}(a+h) \right| \le \omega_2\left(f,\, h + \frac{|\sigma-\tau|}{2}\right).$$

Proof. It is sufficient to show that

$$f_\tau(a-h) - 2f_\sigma(a) + f_{|2\sigma-\tau|}(a+h) \le \omega_2\left(f,\, h + \frac{|\sigma-\tau|}{2}\right)$$

for any function f in $L_{\mathrm{loc}}(\mathbb{R})$. Let us introduce

$$x(s) = a - \frac{\sigma}{2} + \sigma s \quad \text{and} \quad t(s) = h + (\sigma-\tau)s - \frac{\sigma-\tau}{2}$$

It is easy to see that

$$f(x(s)-t(s)) - 2f(x(s)) + f(x(s)+t(s)) \le \omega_2\left(f,\, h + \frac{|\sigma-\tau|}{2}\right)$$

for any $s \in [0,1]$. The proof can be completed now by integrating over $s \in [0,1]$. □

Proofs of the following lemmas can be found in [3]. They are based on the described method.

Lemma 3. *Assume that $\delta \in \left[0,\, \frac{1}{2}\right]$ and $f \in F^0$. Then*

$$\left| f_\delta\left(k \pm \frac{\delta}{2}\right) \right| \le \frac{2-\delta^2}{4}\omega_2(f,\, 1), \quad k \in \mathbb{Z}.$$

Lemma 4. *Assume that $b \in [\frac{1}{3},\, 1]$ and $f \in F_b$. Then*

$$\frac{3+b}{1+b} f_{\frac{1-b}{2}}\left(-\frac{1-b}{4}\right) - \frac{1+b}{4}\left(f_{\frac{3b-1}{4}}\left(\frac{3b-1}{8}\right) + f_{\frac{3b-1}{4}}\left(\frac{5b+1}{8}\right)\right) - \frac{1-b}{2} f_{1-b}\left(\frac{1+b}{2}\right) \le \omega_2(f,\, 1).$$

Lemma 5. *Assume that $b \in [\frac{1}{3},\, 1]$ and $f \in F_b$. Then*

$$\frac{1}{2}\left(f_{\frac{3b-1}{4}}\left(1 + \frac{3b-1}{8}\right) + f_{\frac{3b-1}{4}}\left(1 + \frac{5b+1}{8}\right)\right) - 2f_{\frac{1-b}{2}}\left(1 + \frac{3+b}{4}\right) \le \omega_2(f,\, 1).$$

It is noteworthy that the main idea of the proof of these statements does not refer to special properties of the functions under consideration. Therefore, statements similar to (2.5) may be true for any measurable function. Only a number of particular cases have been proven in the frame of this study. However, it is the author's opinion that further research in this direction could be useful in solving numerous problems.

3. Proof of theorem 1 in the case $b \in \left[0,\, \frac{1}{3}\right]$

Let us take $b \in \left[0,\, \frac{1}{3}\right]$ and $f \in F_b$. By Lemma 1,

$$f_b\left(-\frac{1-2b}{2}\right) - 2f_{\frac{b}{2}}\left(\frac{3b}{4}\right) + 1 \le \omega_2(f,\, 1), \tag{3.1}$$

$$-f_{\frac{b}{2}}\left(\frac{b}{4}\right) - f_{\frac{b}{2}}\left(1+\frac{3b}{4}\right) + 2 \le \omega_2(f,\, 1), \tag{3.2}$$

$$-f_{\frac{1-3b}{2}}\left(\frac{1+b}{4}\right) - f_{\frac{1-3b}{2}}\left(\frac{3+3b}{4}\right) + 2 \le \omega_2(f,\, 1), \tag{3.3}$$

$$-2f_{2b}\left(\frac{1+b}{2}\right) + 2 \le \omega_2(f,\, 1), \tag{3.4}$$

and

$$-f_{\frac{1-b}{2}}\left(-\frac{1-b}{4}\right) + f_{\frac{1-b}{2}}\left(1+\frac{3+b}{4}\right) + \frac{4}{1-b} \le \frac{2}{1-b}\cdot\omega_2(f,\, 1). \tag{3.5}$$

By Lemma 2,

$$-f_{2b}\left(\frac{1+b}{2}\right) + 2f_{\frac{b}{2}}\left(1+\frac{3b}{4}\right) - f_b\left(1+\frac{1+2b}{2}\right) \le \omega_2(f,\, 1). \tag{3.6}$$

By Lemma 3,

$$f_{\frac{1-3b}{2}}\left(-\frac{1-3b}{4}\right) - f_{\frac{1-3b}{2}}\left(1+\frac{3+3b}{4}\right) \le \frac{7+6b-9b^2}{8}\omega_2(f,\, 1). \tag{3.7}$$

Mean values of the function f are related by the equalities

$$f_{\frac{1-b}{2}}\left(-\frac{1-b}{4}\right) = \frac{2b}{1-b}f_b\left(-\frac{1-2b}{2}\right) + \frac{1-3b}{1-b}f_{\frac{1-3b}{2}}\left(-\frac{1-3b}{4}\right), \tag{3.8}$$

$$f_{\frac{1-b}{2}}\left(1+\frac{3+b}{4}\right) = \frac{2b}{1-b}f_b\left(1+\frac{1+2b}{2}\right) + \frac{1-3b}{1-b}f_{\frac{1-3b}{2}}\left(1+\frac{3+3b}{4}\right), \tag{3.9}$$

and

$$\frac{b}{2}f_{\frac{b}{2}}\left(\frac{b}{4}\right) + \frac{b}{2}f_{\frac{b}{2}}\left(\frac{3b}{4}\right) + \frac{1-3b}{2}f_{\frac{1-3b}{2}}\left(\frac{1+b}{4}\right) + 2bf_{2b}\left(\frac{1+b}{2}\right) + \frac{1-3b}{2}f_{\frac{1-3b}{2}}\left(\frac{3+3b}{4}\right) = 0. \tag{3.10}$$

Now, by solving the linear programming problem

$$\omega_2(f,\, 1) \to \inf$$

with respect to variables

$$\begin{array}{lll} f_{\frac{b}{2}}\left(\frac{3b}{4}\right), & f_b\left(1+\frac{1+2b}{2}\right), & f_{\frac{1-b}{2}}\left(-\frac{1-b}{4}\right), \\ f_{\frac{b}{2}}\left(\frac{b}{4}\right), & f_b\left(-\frac{1-2b}{2}\right), & f_{\frac{1-b}{2}}\left(1+\frac{3+b}{4}\right), \\ f_{2b}\left(\frac{1+b}{2}\right), & f_{\frac{1-3b}{2}}\left(1+\frac{3+3b}{4}\right), & \omega_2(f,\, 1), \\ f_{\frac{b}{2}}\left(1+\frac{3b}{4}\right), & f_{\frac{1-3b}{2}}\left(\frac{1+b}{4}\right), & \end{array}$$

and inequality constrains (3.1)–(3.10), we conclude that

$$\frac{96}{27b^3-27b^2+9b+55} \le \omega_2(f,\, 1).$$

□

4. Proof of theorem 1 in the case $b \in \left[\frac{1}{3}, 1\right]$

Let us take $b \in \left[\frac{1}{3}, 1\right]$ and $f \in F_b$. By Lemma 1,

$$f_{\frac{1-b}{2}}\left(-\frac{1-b}{4}\right) - 2f_{\frac{1-b}{4}}\left(\frac{1+3b}{4}\right) + 1 \le \omega_2(f, 1), \tag{4.1}$$

$$-f_{\frac{3b-1}{4}}\left(\frac{3b-1}{8}\right) - f_{\frac{3b-1}{4}}\left(1+\frac{5b+1}{8}\right) + 2 \le \omega_2(f, 1), \tag{4.2}$$

$$-f_{\frac{3b-1}{4}}\left(\frac{5b+1}{8}\right) - f_{\frac{3b-1}{4}}\left(1+\frac{3b-1}{8}\right) + 2 \le \omega_2(f, 1), \tag{4.3}$$

$$-f_{\frac{1-b}{4}}\left(\frac{5b-1}{8}\right) - f_{\frac{1-b}{4}}\left(1+\frac{3b+1}{8}\right) + 2 \le \omega_2(f, 1), \tag{4.4}$$

$$-2f_{1-b}\left(\frac{1+b}{2}\right) + 2 \le \omega_2(f, 1), \tag{4.5}$$

and

$$-f_{\frac{1-b}{2}}\left(-\frac{1-b}{4}\right) + f_{\frac{1-b}{2}}\left(1+\frac{3+b}{4}\right) + \frac{4}{1-b} \le \frac{2}{1-b}\cdot\omega_2(f, 1). \tag{4.6}$$

By Lemma 2,

$$-f_{1-b}\left(\frac{1+b}{2}\right) + 2f_{\frac{1-b}{4}}\left(1+\frac{1+3b}{8}\right) - f_{\frac{1-b}{2}}\left(1+\frac{3+b}{4}\right) \le \omega_2(f, 1). \tag{4.7}$$

By Lemma 4,

$$\frac{3+b}{1+b}f_{\frac{1-b}{2}}\left(\frac{1+b}{2}\right) - \frac{1+b}{4}\left(f_{\frac{3b-1}{4}}\left(\frac{3b-1}{8}\right) + f_{\frac{3b-1}{4}}\left(\frac{5b+1}{8}\right)\right) - \frac{1-b}{2}f_{1-b}\left(\frac{1+b}{2}\right) \le \omega_2(f, 1). \tag{4.8}$$

By Lemma 5,

$$\frac{1}{2}f_{\frac{3b-1}{4}}\left(\frac{7+3b}{8}\right) + \frac{1}{2}f_{\frac{3b-1}{4}}\left(\frac{9+5b}{8}\right) - 2f_{\frac{1-b}{2}}\left(1+\frac{3+b}{4}\right) \le \omega_2(f, 1). \tag{4.9}$$

Mean values of the function f are related by the equalities

$$\frac{3b-1}{4}f_{\frac{3b-1}{4}}\left(\frac{3b-1}{8}\right) + \frac{1-b}{4}f_{\frac{1-b}{4}}\left(\frac{5b-1}{8}\right) + \frac{1-b}{4}f_{\frac{1-b}{4}}\left(\frac{1+3b}{8}\right) + \frac{3b-1}{4}f_{\frac{3b-1}{4}}\left(\frac{5b+1}{8}\right) + (1-b)f_{1-b}\left(\frac{1+b}{2}\right) = 0. \tag{4.10}$$

Now, by solving the linear programming problem

$$\omega_2(f, 1) \to \inf$$

with respect to variables

$$f_{\frac{3b-1}{4}}\left(\tfrac{3b-1}{8}\right),\quad f_{1-b}\left(\tfrac{1+b}{2}\right),\quad f_{\frac{1-b}{2}}\left(1+\tfrac{3+b}{4}\right),$$
$$f_{\frac{1-b}{4}}\left(\tfrac{5b-1}{8}\right),\quad f_{\frac{3b-1}{4}}\left(1+\tfrac{3b-1}{8}\right),\quad f_{\frac{1-b}{2}}\left(-\tfrac{1-b}{4}\right),$$
$$f_{\frac{1-b}{4}}\left(\tfrac{3b+1}{8}\right),\quad f_{\frac{1-b}{4}}\left(1+\tfrac{3b+1}{8}\right),\quad \omega_2(f, 1),$$
$$f_{\frac{3b-1}{4}}\left(\tfrac{5b+1}{8}\right),$$

and inequality constrains (4.1)–(4.10), we conclude that

$$\frac{8(11b^2+66b-13)}{3b^4-69b^3+17b^2+385b-80} \le \omega_2(f,\,1).$$

□

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