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# ON THE RADIUS OF STARLIKENESS FOR HARMONIC MAPPINGS

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*In this paper, we obtain a criterion of starlikeness for the image of the disk with center at the origin and radius $r \in (0, 1)$ under a univalent harmonic mapping by a function that maps the unit disk onto a convex domain. This criterion is similar to the criterion of image convexity, and it is expressed in terms of starlikeness in one direction. As a corollary, we obtain a new estimate for the radius of starlikeness of the class of univalent harmonic mappings that take the unit disk onto a convex domain. Bibliography: 10 titles.*

## 1. Introduction and the main result

In this paper, we study the value of the radius of starlikeness for harmonic mappings of the unit disk (i.e., for univalent complex-valued functions that are harmonic in the unit disk) with the norming

$$f(0) = 0, \quad f_z(0) = 1; \tag{1}$$

here and below, the lower indices $z$ and $\bar{z}$ indicate the corresponding Cauchy–Riemann derivatives.

A simply connected domain $U \subset \mathbb{C}$ is called starlike with respect to a point $a \in U$ if for any point $z \in U$, the segment $[a, z]$ that connects $a$ and $z$ belongs to $U$. In what follows, we only work with domains that are starlike with respect to the origin and call such domains just starlike. The boundary of a Jordan starlike domain is called a starlike curve. It is easily seen that an analytic Jordan curve $\gamma$ is starlike if and only if $\arg w$ does not decrease when the point $w$ moves along $\gamma$ in the positive direction.

For a given class of univalent functions that are defined in a neighborhood of the origin, we define the radius of starlikeness as the maximal number $R > 0$ (if such a number exists) with the following property: Any disk $D_r$ of radius $0 < r \le R$ centered at the origin is mapped by any function of the class onto a starlike domain. The study of the radius of starlikeness is closely related to the study of the radius of convexity which is defined in a similar way (the image of the disk $D_r$ under the corresponding mappings is a convex domain).

Since any convex domain is starlike, the radius of convexity gives one a lower estimate (maybe, not sharp) of the radius of starlikeness for the same class of mappings.

In what follows, we consider the class $\mathcal{S}_H$ of univalent harmonic mappings $f$ of the unit disk $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$ with norming (1) that preserve the orientation of boundaries. It is known (see [1]) that any harmonic mapping $f(z)$ can be represented in the form $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are holomorphic functions called the holomorphic components of the harmonic mapping $f$. For $f \in \mathcal{S}_H$, these functions, holomorphic in the disk $\mathbb{D}$, satisfy the following norming:

$$h(0) = 0, \qquad h'(0) = 1, \qquad g(0) = 0. \tag{2}$$

Thus, functions of the considered class have representations

$$\mathcal{S}_H \ni f(z) = h(z) + \overline{g(z)}, \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{3}$$

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The mapping $f$ is univalent and preserves orientation; hence (see, for example, [1]), its Jacobian $J_f$ is positive everywhere in $\mathbb{D}$, i.e.,

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0, \quad z \in \mathbb{D}. \tag{4}$$

Condition (4) is a criterion of local univalence of a mapping $f$.

Consider one more class $\mathcal{S}_H^0 := \{f \in \mathcal{S}_H \colon f_{\bar z}(0) = 0\}$. There is a known relation between the classes $\mathcal{S}_H^0$ and $\mathcal{S}_H$ (see [2]): Any function $f \in \mathcal{S}_H$ can be represented in the form

$$f = F + \bar b_1 \bar F, \tag{5}$$

where $F \in \mathcal{S}_H^0$. Indeed, for an arbitrary function $f \in \mathcal{S}_H$, one can set

$$F = \frac{f - b_1 \bar f}{1 - |b_1|^2},$$

and the function $F$ is properly defined since it follows from (4) for $z = 0$ that $|b_1| < 1$. One can show (see [1]) that the class $\mathcal{S}_H^0$ is a compact family.

A well studied subclass of the class $\mathcal{S}_H$ is the class $\mathcal{S}$ of conformal mappings of the unit disk $\mathbb{D}$ that satisfy the norming conditions $f(0) = 0$ and $f'(0) = 1$ (see, for example, [3]). We also need the subclass $\mathcal{C}_H$ of the class $\mathcal{S}_H$ which consists of harmonic mappings of the unit disk $\mathbb{D}$ onto convex domains; in addition, set $\mathcal{C}_H^0 := \{f \in \mathcal{C}_H \colon f_{\bar z}(0) = 0\}$.

Recall that a domain $U$ is called convex in the horizontal direction if its intersection with an arbitrary horizontal line is either connected or empty. In other words, for any line that is parallel to the real axis, its intersection with the domain is either an interval (possibly, infinite) or the empty set.

In a similar way, one defines the convexity in any other direction. A domain $U$ is convex if and only if it is convex in any direction.

The following criterion of the convexity of the image of a disk under a harmonic mapping was proved in [2].

**Theorem 1** (Clunie and Sheil-Small, 1984)**.** *Let a harmonic function $f = h + \bar g$ be locally univalent in a disk $D_R$, $R > 0$. This function univalently maps the disk onto a convex domain if and only if for any $\beta \in [0, 2\pi)$, the function $\varphi_\beta(z) := h(z) + e^{i\beta} g(z)$ conformally maps $D_R$ onto a domain that is convex in the horizontal direction.*

Let $\gamma$ be a simple, closed, analytic curve such that $0 \not\in \gamma$. We say that $\gamma$ is starlike in direction $\beta$ if the ray starting at the origin at angle $\beta$ with respect to the positive direction of the real axis intersects $\gamma$ not externally at not more than one point.

We say that a curve $\gamma$ intersects a line not externally at some point of intersection if any neighborhood of this point contains points of $\gamma$ that belong to both half-planes with respect to the line. The Jordan domain $U$ bounded by such a curve is called starlike in the fixed direction $\beta$. It is natural to call the starlikeness in the directions $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ the starlikeness in the vertical direction.

A Jordan domain $U$ with analytic boundary is starlike if and only if it is starlike in any direction $\beta \in [0, 2\pi)$.

An analog of Theorem 1 for starlike domains is not valid for an arbitrary harmonic mapping $f$ of the class $\mathcal{S}_H$ (see, for example, [1, Sec. 6.7]). Nevertheless, we prove in this paper that if a function $f$ belongs to a more narrow class $\mathcal{C}_H$, then the following statement is valid.

**Theorem 2.** *A function $f = h + \bar g \in \mathcal{C}_H$ maps a disk $D_r$ of radius $r \in (0, 1)$ onto a starlike domain if and only if for any $\beta \in [0, 2\pi)$, the function $\varphi_\beta(z) = h(z) + e^{i\beta} g(z)$ maps the circle $T_r = \{|z| = r\}$ onto a curve that is starlike in the vertical direction.*

This statement implies that for the class $\mathcal{C}_H$, the following estimate of the radius of starlikeness is valid:

$$R_s(\mathcal{C}_H) \geq R_s(\mathcal{S}) = \operatorname{th}\frac{\pi}{4} \approx 0.65.$$

This estimate is the best known at the moment. One can find other estimates of radii of convexity and starlikeness for various classes of univalent conformal and harmonic mappings in the papers [4–10].

## 2. Proofs

Let $f = h + \overline{g}$ be a harmonic in $\mathbb{D}$ complex-valued function. It is shown in [1] that the image $f(T_r)$ of a circle is convex if and only if the analytic condition

$$\frac{\partial}{\partial\theta} \arg\left\{\frac{\partial f(re^{i\theta})}{\partial\theta}\right\} \geq 0 \tag{6}$$

is fulfilled for all $\theta \in [0, 2\pi)$. One can write the above condition in terms of the analytic components $h$ and $g$ as follows:

$$\operatorname{Re}\left\{\frac{z^2h''(z) + \overline{z}^2\overline{g''(z)}}{zh'(z) - \overline{z}\overline{g'(z)}} + \frac{zh'(z) + \overline{z}\overline{g'(z)}}{zh'(z) - \overline{z}\overline{g'(z)}}\right\} \geq 0, \tag{7}$$

where $z = re^{i\theta}$. In particular, if $f$ is a holomorphic function, i.e., if $g \equiv 0$, then inequality (7) becomes the well-known condition of convexity (see [3]),

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq 0. \tag{8}$$

The image $f(T_r)$ of the circle $T_r$ is starlike if and only if

$$\frac{\partial}{\partial\theta} \arg\left\{f(re^{i\theta})\right\} \geq 0 \tag{9}$$

for all $\theta \in [0, 2\pi)$ (see [1]). One can write this condition in terms of the holomorphic components $h$ and $g$ as follows:

$$\operatorname{Re}\left\{\frac{zh'(z) - \overline{z}\overline{g'(z)}}{h(z) + \overline{g(z)}}\right\} \geq 0, \quad \text{or} \quad \left|\arg\left\{\frac{zh'(z) - \overline{z}\overline{g'(z)}}{h(z) + \overline{g(z)}}\right\}\right| \leq \frac{\pi}{2}. \tag{10}$$

If $f$ is a holomorphic function ($g \equiv 0$), we get the known conditions (see [3])

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 0, \qquad \text{or} \qquad \left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| \leq \frac{\pi}{2}. \tag{11}$$

The classical Alexander theorem states that the image of a domain $U$ under a holomorphic function $f$ is convex if and only if its image under the function $zf'(z)$ is a starlike domain. This result is generalized to the case of harmonic functions $f$ (see a similar statement in [1, p. 108]).

**Lemma 1.** *Let $f = h + \overline{g}$ and $F = H + \bar{G}$ be two harmonic complex-valued functions whose holomorphic components satisfy the relations*

$$zH'(z) = h(z) \quad \text{and} \quad zG'(z) = -g(z). \tag{12}$$

*Then the image $f(T_r)$ of a circle $T_r$ is a starlike curve if and only if $F(T_r)$ is a convex curve.*

*Proof.* Indeed, differentiate relations (12) to show that

$$h'(z) = zH''(z) + H'(z) \quad \text{and} \quad -g'(z) = zG''(z) + G'(z).$$

Substitute these relations and (12) into the first formula in (10) to show that

$$\mathrm{Re}\left\{\frac{zh'(z) - \bar{z}\overline{g'(z)}}{h(z) + \overline{g(z)}}\right\} = \mathrm{Re}\left\{\frac{z^2H''(z) + \bar{z}^2\overline{G''(z)}}{zH'(z) - \bar{z}\overline{G'(z)}} + \frac{zH'(z) + \bar{z}\overline{G'(z)}}{zH'(z) - \bar{z}\overline{G'(z)}}\right\}.$$

The left-hand side of the above equality is nonnegative if and only if the curve $f(T_r)$ is starlike (see formula (10)), and the right-hand side is nonnegative if and only if the curve $F(T_r)$ is convex. This completes the proof. □

*Proof of Theorem* 1. For the function $f$, we define the corresponding function $F$ by relations (12) with additional norming conditions $H(0) = G(0) = 0$. Since $f \in \mathcal{C}_H$, the inequality $|h(z)| > |g(z)|$ holds for $0 < |z| < 1$ (see [2]). Then the Jacobian $J_F(z)$ of the function $F$ satisfies the relations

$$J_F(z) = |H'(z)|^2 - |G'(z)|^2 = \frac{|h(z)|^2 - |g(z)|^2}{|z|^2} > 0,$$

which means that the mapping $F$ is locally univalent. By Lemma 1, the curve $f(T_r)$ is starlike if and only if the curve $F(T_r)$ is convex. Since the function $F$ is locally univalent in $\mathbb{D}$, it follows from Theorem 1 that the domain $F(D_r)$ and the corresponding curve $F(T_r)$ are convex if and only if for any $\beta \in [0, 2\pi)$, the holomorphic function $\Phi_\beta(z) = H(z) - e^{i\beta}G(z)$ conformally maps the disk $D_r$ onto a domain that is convex in the horizontal direction.

In what follows, we assume that the curve $\Phi_\beta(T_r)$ does not contain horizontal rectilinear parts. Consider the function $V(\theta) := \mathrm{Im}\,\{\Phi_\beta(re^{i\theta})\}$ that is not constant on any segment and has period $2\pi$. Without loss of generality, we assume that this function increases in neighborhoods of the points $\theta = \pm\pi$. Let us show that the image $\Phi_\beta(D_r)$ is convex in the horizontal direction if and only if $V(\theta)$ has precisely one point of strict local maximum and precisely one point of strict local minimum in the segment $[-\pi, \pi]$.

Let $\Phi_\beta(D_r)$ be convex in the horizontal direction. Assume that $V(\theta)$ has two distinct points $\theta_1$ and $\theta_2$ of strict local maximum and $V(\theta_1) \le V(\theta_2)$. Then there exists a point $\theta_{\min}$ between these points such that $V(\theta_{\min}) < V(\theta_1)$. In a neighborhood of the point $\theta_1$, there exist two distinct points $\theta_1'$ and $\theta_1''$ such that $V(\theta_{\min}) < V(\theta_1') = V(\theta_1'') < V(\theta_1) \le V(\theta_2)$. Since the continuous on the segment $[\theta_1'', \theta_2]$ function $V(\theta)$ attains on this segment all intermediate values from $V(\theta_{\min})$ to $V(\theta_2)$, there exists a point $\theta_2' > \theta_1''$ such that $V(\theta_2') = V(\theta_1') = V(\theta_1'')$, which contradicts the convexity of the curve $F(T_r)$ in the horizontal direction. The case of a local minimum is treated similarly.

Now let us assume that the function $V(\theta)$ has on the segment $[-\pi, \pi]$ a unique point $\theta_{\max}$ of strict local maximum and a unique point $\theta_{\min}$ of strict local minimum. Since $V(\theta)$ increases in neighborhoods of the points $\pm\pi$, $\theta_{\max} < \theta_{\min}$. The interval $(-\pi, \pi)$ contains a point $\theta_0$ such that $V(\theta_0) = V(-\pi) = V(\pi)$ and $\theta_{\max} < \theta_0 < \theta_{\min}$.

Let us show that the function $V(\theta)$ attains on the segment $[-\pi, \theta_0]$ any its value not more than twice. Indeed, if the function attains the same value at distinct points $\theta_1 < \theta_2 < \theta_3$, then the segment $[\theta_1, \theta_3]$ contains a point of strict local minimum, which contradicts the fact that the whole segment $[-\pi, \pi]$ contains a unique point of minimum.

A similar reasoning is applicable in the case of the segment $[\theta_0, \pi]$.

Since the ranges of the function $V(\theta)$ on the intervals $(-\pi, \theta_0)$ and $(\theta_0, \pi)$ are disjoint, the function $V(\theta)$ attains any its value on the whole segment $[-\pi, \pi]$ not more than twice, which means that the curve $\Phi_\beta(D_r)$ is convex in the horizontal direction.

Hence, there exist precisely two values $\theta = \theta_{\min}$ end $\theta = \theta_{\max}$ for which

$$\frac{\partial}{\partial\theta}\text{Im}\left\{\Phi_\beta(re^{i\theta})\right\} = 0,$$

and in this case, $V'(\theta) = \frac{\partial}{\partial\theta}\text{Im}\left\{\Phi_\beta(re^{i\theta})\right\}$ changes sign passing the points $\theta_{\min}$ and $\theta_{\max}$. Indeed, if there exists one more point $\theta'$ such that $V'(\theta') = 0$, then $V'(\theta)$ does not change sign passing this point (otherwise, $\theta'$ is a point of extremum.)

Relations (12) imply that if $z = re^{i\theta}$, then

$$\begin{aligned}\frac{\partial}{\partial\theta}\text{Im}\,\{\Phi_\beta(z)\} &= \text{Im}\left\{iz\Phi'_\beta(z)\right\} = \text{Re}\,\left\{z\Phi'_\beta(z)\right\} \\ &= \text{Re}\,\left\{z(H'(z) - e^{i\beta}G'(z))\right\} = \text{Re}\,\{\varphi_\beta(z)\}\,.\end{aligned}$$

It follows from the above reasoning that

$$\text{Re}\,\{\varphi_\beta(re^{i\theta})\} = 0$$

only for $\theta = \theta_{\min}$ and $\theta = \theta_{\max}$ , and the value $\text{Re}\,\{\varphi_\beta(re^{i\theta})\}$ has different signs to the right and left of $\theta_{\min}$ and $\theta_{\max}$. For other points $\theta$ at which $\text{Re}\,\{\varphi_\beta(re^{i\theta})\} = 0$, the value $\text{Re}\,\{\varphi_\beta(re^{i\theta})\}$ does not change sign passing these points.

This means that the imaginary axis intersects the curve $\varphi_\beta(T_r)$ not externally precisely at two points $w_{\max} = \varphi_\beta(re^{i\theta_{\max}})$ and $w_{\min} = \varphi_\beta(re^{i\theta_{\min}})$; to all other points of intersection there correspond values $\theta$ such that $\text{Re}\,\{\varphi_\beta(re^{i\theta})\}$ does not change sign passing these values, i.e., the curve $\varphi_\beta(T_r)$ externally intersects the imaginary axis at the points $\varphi_\beta(re^{i\theta})$. Hence, $\varphi_\beta(T_r)$ is starlike in the vertical direction.

If the curve $\Phi_\beta(T_r)$ contains horizontal rectilinear parts, then to such parts there correspond vertical rectilinear parts of the curve $\varphi_\beta(T_r)$, which does not contradict the starlikeness of the latter curve in the vertical direction. The theorem is proved. □

**Corollary 1.** *For the class $\mathcal{C}_H$, the following estimate of the radius of starlikeness is valid:*

$$R_s(\mathcal{C}_H) \geq \text{th}\,\frac{\pi}{4} \approx 0.65.$$

*Proof.* First let $f = h + \overline{g} \in \mathcal{C}^0_H$. By Theorem 1, for any $\beta \in [0, 2\pi)$, the function $\varphi_\beta(z) = h(z) + e^{i\beta}g(z)$ is conformal in the whole unit disk $\mathbb{D}$. In addition, the norming condition (2) for the class $\mathcal{S}_H$ and the additional condition $g'(0) = 0$ defining the subclass $\mathcal{S}^0_H$ imply that $\varphi_\beta \in \mathcal{S}$. As was mentioned above, for the class $\mathcal{S}$ of conformal mappings, the radius of starlikeness is

$$R_s(\mathcal{S}) = \text{th}\,\frac{\pi}{4} \approx 0.65.$$

Hence, for any $r \leq R_s(\mathcal{S})$, the domain $\varphi_\beta(D_r)$ is starlike (in any direction, including the vertical one). By Theorem 2, the domain $f(D_r)$ is starlike as well. Relation (5) between functions of the classes $\mathcal{C}_H$ and $\mathcal{C}^0_H$ implies that the domain $f(D_r)$ is starlike for any function $f \in \mathcal{C}_H$. This proves the corollary. □

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