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METHOD OF GUIDING FUNCTIONS FOR EXISTENCE PROBLEMS FOR PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Abstract. We provide a review and systematic explanation of various generalizations of the guiding function method. The current state of the said method and its applications to various kinds of problems for nonlinear periodic systems described by differential and functional differential equations are considered.

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Introduction

Geometric and topological methods of analysis are applied to problems of nonlinear oscillations of dynamical systems since Poincaré, Brouwer, Aleksandrov, Hopf, Leray, and Schauder. Later on, these methods were developed by Krasnosel'skii, Bobylev, Borisovich, Bulgakov, Gango, Gel'man, Zabreiko, Zvyagin, Kamenskii, Myshkis, Obukhovskii, Perov, Povolotskii, Rachinskii, Sadovskii, Sapronov, Strygin, Filippov, Deimling, Górniewicz, Mawhin, Nistri, Papageorgiou, Zecca, and other researchers; the said methods demonstrate their high efficiency.

In particular, a very fruitful direction related to the notion of guiding functions is founded by Krasnosel'skii and Perov (see, e.g., [7, 8]). This method is closely related to the Poincaré operator (translation operator along system trajectories) and its fixed point (see ibid). Students of Mathematics can find basic points of the guiding functions method in [14, 17].

Note that properties of guiding functions are close to properties of Lyapunov functions, but they are used in problems not related to the stability, e.g., the existence problem for periodic solutions.

The most important achievements in the development of this direction are the Rachinskii method of many-sheeted guiding functions (see [15, 16]), the Fonda method of integral guiding functions (see [2]), and the Mawhin method of averaged and asymptotically averaged guiding functions (see [12]), proposed for differential equations. Various generalizations of the specified methods are provided in works of Borisovich, Gel'man, Myshkis, Obukhovskii, Kornev, Loi, and Zecca. In particular, the method of the above classes of guiding functions is extended to the case of differential inclusions (see, e.g., [1, 3, 5]) and generalized for the case of systems described by differential inclusions in infinite-dimensional Hilbert spaces (see, e.g., [13]).

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The present work provides a review and systematic explanation of the current state of the above generalizations and their applications to various types of nonlinear oscillation problems described by differential and functional differential equations.

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1. Krasnosel'skii–Perov Guiding Functions

Consider a differential equation of the kind

$$x'(t) = f(t, x), \tag{1.1}$$

assuming that $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map satisfying the following condition:

(f_t) the function f is T-periodic ($T > 0$) with respect to the first independent variable:

$$f(t, x) = f(t + T, x) \quad \text{for all } t \in \mathbb{R} \quad \text{and } x \in \mathbb{R}^n$$

(obviously, under this condition, one can treat f as a map defined on $[0, T] \times \mathbb{R}^n$).

Also, assume that any initial condition

$$x(s) = x_0 \tag{1.2}$$

uniquely defines a solution

$$x(t) = p(t; s, x_0) \tag{1.3}$$

of Eq. (1.1) such that it is uniquely extended to the segment $[0, T]$.

Then, following [7], one can define the operator U_T of the translation beyond the period T along the trajectories of Eq. (1.1):

$$U_T x = p(T; 0, x).$$

The existence of T-periodic solutions of Eq. (1.1) is equivalent to the existence of fixed points of the translation operator U_T (see, e.g., [7]).

To prove the existence of fixed points of the translation operator, it is suggested to find a bounded domain Ω such that the map

$$g : \overline{\Omega} \to \mathbb{R}^n, \quad g(x) = x - U_T x,$$

does not vanish on the boundary $\partial\Omega$ and to compute (or estimate) its topological degree $\deg(g, \overline{\Omega}, 0)$ then.

It is hard to compute the topological degree of the map g because the explicit form of the translation operator U_T is not known. Therefore, special tools are needed to compute the topological degree of this map.

One such tool is the method of guiding functions suggested by Krasnosel'skii and Perov (see, e.g., [7–9]). Recall its main points.

Definition 1.1. A point x_0 from $\mathbb{R}^n$ is called a *T-nonreturnability point* of trajectories if

$$x(t) = p(t; 0, x_0) \neq x_0 \quad (0 < t \leq T), \tag{1.4}$$

where $x(t) = p(t; 0, x_0)$ is the solution of Eq. (1.1), satisfying the initial condition $x(0) = x_0$.

If $p(t; 0, x_0) \neq x_0$ for all positive t, then we say that the *nonreturnability* (i.e., no period is mentioned) or nonlocal extendability of trajectories takes place.

Lemma 1.1. *Let all points of the boundary $\partial\Omega$ of a bounded domain Ω be T-nonreturnability points of trajectories and the map*

$$\psi : \overline{\Omega} \to \mathbb{R}^n, \quad \psi(x) = -f(0, x), \quad x \in \overline{\Omega}, \tag{1.5}$$

do not vanish on $\partial\Omega$. Then the degrees $\deg(\psi, \overline{\Omega}, 0)$ and $\deg(g, \overline{\Omega}, 0)$ are defined and

$$\deg(\psi, \overline{\Omega}, 0) = \deg(g, \overline{\Omega}, 0).$$

Proof. Consider the following family of maps on $\overline{\Omega}$:

$$H(t,x) = x - p(t;0,x), \quad t \in (0,T].$$

Condition (1.4) implies that $H(t,x) \neq 0,\ t \in (0,T],\ x \in \partial\Omega$. This and the continuous dependence of solutions $p(t;0,x)$ of Eq. (1.1) on the initial data x and on t imply that H is a homotopy connecting all maps $H(t,\cdot) = i - p(t,0,\cdot),\ t \in (0,T]$, where i is the identity map. Then degrees $\deg(H(t,\cdot), \overline{\Omega}, 0)$ are defined and equal to each other for all maps $H(t,\cdot),\ t \in (0,T]$. Therefore, it suffices to show that the degrees $\deg(H(t,\cdot), \overline{\Omega}, 0)$ and $\deg(\psi, \overline{\Omega}, 0)$ are equal to each other for small positive t.

To prove that, we show that, for small positive t, the directions of the maps ψ and $H(t,\cdot)$ are not opposite to each other on $\partial\Omega$. Otherwise, there exist sequences $x_n \subset \partial\Omega$ and $t_n \to 0$ such that

$$x_n - p(t_n;0,x_n) = \alpha_n f(0,x_n), \quad \alpha_n > 0,$$

whence

$$\frac{p(t_n;0,x_n) - x_n}{t_n} = -\frac{\alpha_n}{t_n} f(0,x_n)$$

and one can assume (without loss of generality) that the sequence x_n converges to a point x_0 from $\partial\Omega$. Passing to the limit, we obtain the contradictory inequality

$$f(0,x_0) = -\alpha f(0,x_0), \quad \alpha > 0.$$

□

Theorem 1.1. *Let all solutions of Eq.* (1.1) *be nonlocally extendable. Let all points of the boundary* $\partial\Omega$ *of a bounded domain* Ω *be* T*-nonreturnability points for trajectories. Let the topological degree* $\deg(\psi, \overline{\Omega}, 0)$ *be different from zero. Then there exists at least one* T*-periodic solution of Eq.* (1.1).

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function.

Definition 1.2. We say that $V(x)$ is *nondegenerate* if there exists a positive r_0 such that

$$\mathrm{grad} V(x) = \Big(\frac{\partial V(x)}{\partial x_1}, \frac{\partial V(x)}{\partial x_2}, \dots, \frac{\partial V(x)}{\partial x_n}\Big) \neq 0, \quad \|x\| \geq r_0.$$

Definition 1.3. The *index* $\mathrm{ind}\, V$ of a nondegenerate function $V(x)$ is the number

$$\mathrm{ind}\, V = \deg(\mathrm{grad}\, V, \overline{B(0,r)}, 0),$$

where $r \geq r_0$ and $B(0,r) = \{x \in \mathbb{R}^n : \|x\| < r\}$.

Note that the value of the above degree is the same for all r from $[r_0, +\infty)$.

1.1. Strict and generalized guiding functions.

Definition 1.4. A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *strict guiding function* for Eq. (1.1) if there exists a positive r_0 such that

$$(\mathrm{grad} V(x), f(t,x)) > 0, \quad 0 \leq t \leq T; \quad \|x\| \geq r_0. \tag{1.6}$$

From Condition (1.6), it follows that, defining guiding functions, we assume that they are nondegenerate.

Remark 1.1. If there exists a strict guiding function $V(x)$ of index $\mathrm{ind}\, V$ for Eq. (1.1), then

$$\deg(\psi, \overline{B(0,r)}, 0) = (-1)^n \mathrm{ind}\, V. \tag{1.7}$$

For the existence principle for a periodic solution provided below, we provide a scheme of a proof (a complete proof can be found in [7]).

Theorem 1.2. *Let there exist a strict guiding function* $V(x)$ *for Eq.* (1.1) *such that either*

$$\lim_{\|x\|\to\infty} V(x) = +\infty \tag{1.8}$$

or

$$\lim_{\|x\|\to\infty} V(x) = -\infty. \tag{1.9}$$

Then Eq. (1.1) *has at least one* T*-periodic solution.*

Proof. Let Condition (1.9) be satisfied. Then it is easy to see that the said condition implies the strict decreasing of the function $V_1(x) = -V(x)$ for all values of t such that $\|x(t)\| \geq r_0$. Hence, any solution is bounded and, therefore, nonlocally extendable.

By virtue of (1.9) and Remark 1.1, the topological degree $\deg(\psi, \overline{B(0, r_1)}, 0)$ is different from zero provided that r_1 is sufficiently large.

Assign $M = \max\limits_{\|x\|\leq r_0} V_1(x)$. Then Condition (1.6) implies the T-nonreturnability of points of the boundary of the ball $B(0, r_1)$ provided that r_1 is sufficiently large to guarantee that $V_1(x) \geq M$ for $\|x\| \geq r_1$.

All conditions of Theorem 1.1 are satisfied and Eq. (1.1) has at least one T-periodic solution. ☐

Remark 1.2. If Condition (1.8) is satisfied, then it suffices to seek periodic solutions $x(t)$ of the form $x(t) = y(-t)$ and to replace Eq. (1.1) by the equation

$$y'(t) = -f(-t, y).$$

Strengthening Condition (1.6), we obtain an existence principle for T-periodic solutions.

Definition 1.5. A nondegenerate function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *generalized guiding function* for Eq. (1.1) if there exists a positive r_0 such that

$$(\mathrm{grad} V(x), f(t, x)) \geq 0, \quad 0 \leq t \leq T; \quad \|x\| \geq r_0.$$

The following assertion is valid.

Theorem 1.3. *Let there exist a generalized guiding function* $V(x)$ *for Eq.* (1.1) *such that either Condition* (1.8) *or Condition* (1.9) *is valid. Then Eq.* (1.1) *has at least one* T*-periodic solution.*

1.2. Collections of guiding functions. As we see above, even a single guiding function allows us to select domains consisting of points possessing the T-nonreturnability properties. Therefore, an existence of several guiding functions linked by certain relations might be useful for a proof of the existence theorem for periodic solutions.

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a strict guiding function in the sense of Definition 1.4. Assign

$$m = \min_{\|x\|\leq r_0} V(x) \quad \text{and} \quad M = \max_{\|x\|\leq r_0} V(x).$$

By Ω and Ω^* we denote the sets of points x from $\mathbb{R}^n$ such that $\|x\| \geq r_0$ and the following conditions are satisfied, respectively:

$$V(x) \leq m \quad \text{and} \quad V(x) \geq M.$$

Then the following assertion is valid (see [7]).

Lemma 1.2. *If* $x_0 \in \Omega \cup \Omega^*$, *then* x_0 *is a nonreturnability point for trajectories.*

Lemma 1.2 has a clear geometric interpretation: moving along trajectories, level surfaces of the function $V(x)$ in the domains Ω and Ω^* intersect in the same direction.

Definition 1.6. If continuously differentiable functions

$$V_0(x), V_1(x), \ldots, V_k(x), \quad k \geq 1, \tag{1.10}$$

are such that

$$\lim_{\|x\|\to\infty} [|V_0(x)| + |V_1(x)| + \ldots + |V_k(x)|] = \infty, \tag{1.11}$$

then they form a *complete collection of strict guiding functions* for Eq. (1.1) provided that the following conditions are satisfied:

$$(\mathrm{grad} V_i(x), f(t,x)) > 0, \quad i = 0, 1, \ldots, k, \quad 0 \le t \le T, \quad \|x\| \ge r_0. \tag{1.12}$$

Theorem 1.4. *Let Eq.* (1.1) *have a complete collection* $\{V_0(x), V_1(x), \ldots, V_k(x)\}$ *of strict guiding functions. Let the topological index* $\mathrm{ind}\, V_0$ *of the function* $V_0(x)$ *be different from zero, i.e.,*

$$\mathrm{ind}\, V_0 \neq 0. \tag{1.13}$$

Then Eq. (1.1) *has at least one* T*-periodic solution.*

Remark 1.3. It follows from Conditions (1.12) that the maps $\mathrm{grad} V_i(x)$ $(i = 0, 1, \ldots, k)$ are homotopic to $f(0,x)$ for $\|x\| \ge r_0$. Therefore, they are homotopic to each other and their topological degrees coincide each other. Therefore, the topological indices of all guiding functions (1.10) coincide with each other.

Proof of Theorem 1.4.

1. First, suppose that solutions of Eq. (1.1) are nonlocally extendable. Let

$$m_i = \min_{\|x\| \le r_0} V_i(x), \quad M_i = \max_{\|x\| \le r_0} V_i(x), \quad i = 0, 1, \ldots, k,$$

and

$$M^* = \sum_{i=0}^{k} (|m_i| + |M_i|).$$

By virtue of (1.11), there exists r^* such that

$$|V_0(x)| + |V_1(x)| + \ldots + |V_k(x)| > M^* \quad (\|x\| \ge r^*). \tag{1.14}$$

By Ω_i and Ω_i^* $(i = 0, 1, \ldots, k)$ denote the sets of points x from $\mathbb{R}^n$ such that $\|x\| \ge r_0$ and the following conditions are satisfied, respectively:

$$V_i(x) \le m_i \quad \text{and} \quad V_i(x) \ge M_i.$$

If we assume that $\|x\| \ge r^*$, then (1.14) implies the existence of $i = i(x)$ such that $x \in \Omega_i \cup \Omega_i^*$. Then Lemma 1.2 implies that x is a nonreturnability point for trajectories.

By virtue of Remark 1.1 and Condition (1.13), the topological degree of map (1.5) is different from zero. Thus, all conditions of Theorem 1.1 are satisfied and Eq. (1.1) has a T-periodic solution.

2. We pass to the general case: assume that there exists an initial condition such that the corresponding solution "goes to infinity" within a small time segment, i.e., that solutions cannot be extended to the segment $[0, T]$.

Note that T-periodic solutions $x(\cdot)$ of all systems satisfying Conditions (1.12) satisfy the inequality

$$\|x(t)\| < r^* \quad (0 \le t \le T) \tag{1.15}$$

because a point x and $i \in \overline{1,k}$ would exist otherwise such that x belongs to the set $\Omega_i \cup \Omega_i^*$, but is not a nonreturnability point for trajectories.

The existence of estimate (1.15), which is an a priori estimate, shows the way to prove the theorem for the general case. We have to construct an auxiliary equation

$$x'(t) = f^*(t,x) \tag{1.16}$$

such that its right-hand side satisfies the following conditions:

1^0. $f^*(t,x) = f(t,x), \quad x \in \mathbb{R}^n, \ \|x\| \le r^*$;

2^0. $f^*(t,x)$ satisfies Conditions (1.12);

3^0. all solutions of Eq. (1.16) are extendable to $[0, T]$.

By virtue of Conditions 2^0 and 3^0, Eq. (1.16) has at least one T-periodic solution. It satisfies estimate (1.15). Therefore, it satisfies Eq. (1.1) as well. □

Strengthening Condition (1.12), one can obtain an existence principle for T-periodic solutions.

Definition 1.7. Nondegenerate functions (1.10) possessing property (1.11) form a *complete and acute collection of generalized guiding functions* for Eq. (1.1) if

$$(\mathrm{grad} V_i(x), f(t,x)) \geq 0, \quad i = 0, 1, \ldots, k, \quad 0 \leq t \leq T, \quad \|x\| \geq r_0,$$

and the set

$$K(x) = \left\{ y \in \mathbb{R}^n : y = \sum_{i=0}^{k} \gamma_i \mathrm{grad} V_i(x), \ \gamma_0, \ldots, \gamma_k \geq 0 \right\}$$

is a Krein cone (i.e., the belonging of y and $-y$ to $K(x)$ implies that $y = 0$) for any fixed x from $\mathbb{R}^n$ such that $\|x\| \geq r_0$.

The following assertion is valid.

Theorem 1.5. *Let Eq.* (1.1) *have a complete and acute collection* $\{V_0(x), V_1(x), \ldots, V_k(x)\}$ *of generalized guiding functions and the topological index* $\mathrm{ind}\, V_0$ *of the function* $V_0(x)$ *be different from zero. Then Eq.* (1.1) *has at least one* T*-periodic solution.*

Definition 1.8. A nondegenerate function $V(x)$ is called a *proper guiding function* for Eq. (1.1) if

$$(\mathrm{grad} V(x), f(t,x)) \geq \alpha_0 \|\mathrm{grad} V(x)\| \, \|f(t,x)\|,$$

where $\alpha_0 > 0$, $0 \leq t \leq T$, and $\|x\| \geq r_0$, and there exists a continuously differentiable function $V_1(x)$ such that

$$\|\mathrm{grad} V(x)\| \geq \|\mathrm{grad} V_1(x)\| \quad (\|x\| \geq r_0)$$

and

$$\lim_{\|x\| \to \infty} |V_1(x)| = \infty.$$

The following assertion is valid.

Theorem 1.6. *Let Eq.* (1.1) *have a proper guiding function* $V(x)$ *and the topological index of* $V(x)$ *be different from zero at infinity*:

$$\mathrm{ind}\,(V, \infty) \neq 0.$$

Then Eq. (1.1) *has at least one* T*-periodic solution.*

2. Many-Sheeted Vector Guiding Functions

Consider the following periodic problem for a differential equation in the space $\mathbb{R}^n$ $(n > 2)$:

$$z'(t) = f(t,z), \tag{2.1}$$

where $f(t,z)$ is a continuous function with respect to the (vector) independent variable (t,z), continuously differentiable with respect to z, and T-periodic with respect to t $(T > 0)$.

Suppose that a two-dimensional plane $\mathbb{R}^2$ is selected in the space $\mathbb{R}^n$ and $\mathbb{R}^{n-2}$ is the subspace complementary to that plane. Let q be the projection operator to the plane $\mathbb{R}^2$ along the subspace $\mathbb{R}^{n-2}$ and $p = I - q$. In the sequel, elements of $\mathbb{R}^2$ are denoted by ξ and elements of $\mathbb{R}^{n-2}$ are denoted by ζ. Thus, any element z of $\mathbb{R}^n$ is uniquely representable as $z = \xi + \zeta$, where $\xi = qz$ and $\zeta = pz$.

Let (φ, ρ) be the polar coordinates in $\mathbb{R}^2$. Consider the many-sheeted Riemannian surface

$$\Pi = \{(\varphi, \rho) : \varphi \in (-\infty, \infty), \rho \in (0, \infty)\}.$$

Let a scalar continuously differentiable function $W(\varphi, \rho)$ be defined on Π such that

$$\frac{\partial}{\partial \varphi} W(\varphi, \rho) > 0, \quad (\varphi, \rho) \in \Pi, \tag{2.2}$$

and

$$W(\varphi + 2\pi, \rho) = W(\varphi, \rho) + 2\pi, \quad (\varphi, \rho) \in \Pi. \tag{2.3}$$

The last relation implies the identity $\text{grad}W(\varphi + 2\pi, \rho) \equiv \text{grad}W(\varphi, \rho)$. Therefore, the vector field of gradients $\text{grad}W(\varphi, \rho)$ is defined on $\mathbb{R}^2 \setminus \{0\}$.

Let a smooth scalar function $V(\zeta)$ be defined on the subspace $\mathbb{R}^{n-2}$ such that

$$\lim_{\|\zeta\| \to \infty} V(\zeta) = \infty. \tag{2.4}$$

By virtue of (2.4), the domains $\{\zeta \in \mathbb{R}^{n-2} : V(\zeta) < \vartheta\}$ are nonempty and bounded provided that $\vartheta > \vartheta_0 = \min V(\zeta)$.

For $\rho_2 > \rho_1 \geq 0$ and $\vartheta > \vartheta_0$, select the domain

$$\Omega(\vartheta, \rho_1, \rho_2) = \{z \in \mathbb{R}^n : V(pz) < \vartheta, \ \rho_1 < \|qz\| < \rho_2\}.$$

Assume that continuous functions $\alpha_{\vartheta,\rho_1,\rho_2}(\cdot)$ and $\beta_{\vartheta,\rho_1,\rho_2}(\cdot)$ are defined on $[0, T]$ such that

$$\sup_{z \in \Omega(\vartheta,\rho_1,\rho_2)} (\text{grad}W(qz), qf(t, z)) = \alpha_{\vartheta,\rho_1,\rho_2}(t) \tag{2.5}$$

and

$$\inf_{z \in \Omega(\vartheta,\rho_1,\rho_2)} (\text{grad}W(qz), qf(t, z)) = \beta_{\vartheta,\rho_1,\rho_2}(t). \tag{2.6}$$

2.1. Strict and generalized many-sheeted vector guiding functions.

Definition 2.1. A pair $\{V(\zeta),\ W(\varphi, \rho)\}$ of functions possessing properties (2.2)–(2.4) is called a *strict many-sheeted vector guiding function* for Eq. (2.1) with respect to the domain $\Omega(\vartheta, \rho_1, \rho_2)$ if the following conditions are satisfied:

$$\sup_{t \in [0,T]} \frac{|(qf(t, z), qz)|}{\|qz\|} < \frac{\rho_2 - \rho_1}{2T}, \quad z \in \Omega(\vartheta, \rho_1, \rho_2), \tag{2.7}$$

$$(\text{grad}V(pz), pf(t, z)) < 0, \quad V(pz) \geq \vartheta, \ \|qz\| \leq \rho_2, \tag{2.8}$$

and

$$2\pi(N - 1) < \int_0^T \alpha_{\vartheta,\rho_1,\rho_2}(\tau) d\tau, \quad \int_0^T \beta_{\vartheta,\rho_1,\rho_2}(\tau) d\tau < 2\pi N, \tag{2.9}$$

where N is an integer and $\alpha_{\vartheta,\rho_1,\rho_2}(t)$ and $\beta_{\vartheta,\rho_1,\rho_2}(t)$ are defined by (2.5)-(2.6).

Define the domain $G(\vartheta, \rho) = \{z \in \mathbb{R}^n : V(pz) < \vartheta, \|qz\| < \rho\}$. It can be represented as $G(\vartheta, \rho) = G_\zeta(\vartheta) \times G_\xi(\rho)$, where $G_\zeta(\vartheta) = \left\{\zeta \in \mathbb{R}^{n-2} : V(\zeta) < \vartheta\right\}$ and $G_\xi(\rho) = \left\{\xi \in \mathbb{R}^2 : \|\xi\| < \rho\right\}$.

For the existence principle for a periodic solution provided below we provide a scheme of a proof (a complete proof can be found in [16]).

Theorem 2.1. *Let there exist a strict many-sheeted vector guiding function for Eq.* (2.1) *with respect to the domain* $\Omega(\vartheta, \rho_1, \rho_2)$. *Then Eq.* (2.1) *has at least one* T*-periodic solution* $z_*(\cdot)$ *such that*

$$z_*(t) \in G(\vartheta, \rho_0), \ \ t \in [0, T], \rho_0 = (\rho_1 + \rho_2)/2.$$

Proof. First, we show that a continuous translation operator U_T with respect to trajectories of Eq. (2.1) is defined on the set $\overline{G(\vartheta, \rho_0)}$ and nondegenerate on the boundary $\partial G(\vartheta, \rho_0)$ and, therefore, the topological degree $\deg(i - U_T, \overline{G(\vartheta, \rho_0)}, 0)$ is defined.

To do that, we have to prove that any solution of Eq. (2.1) with any initial value z_0 from $\overline{G(\vartheta, \rho_0)}$ is extendable to the segment $[0, T]$ and possesses the T-nonreturnability property.

Let us show that any solution of Eq. (2.1) with any initial value z_0 from $\overline{G(\vartheta, \rho_0)}$ is extendable to the segment $[0, T]$ and

$$z(t) \in G(\vartheta, \rho_2), \quad t \in (0, T]. \tag{2.10}$$

First, we prove that $z(t) \in G(\vartheta, \rho_2)$ for small positive t.

Consider the component $\zeta(t)$. If $\zeta(0)$ is an interior point of the domain $G_\zeta(\vartheta)$, then there exists a positive number ε_1 such that

$$\zeta(t) \in G_\zeta(\vartheta), \quad t \in (0, \varepsilon_1). \tag{2.11}$$

Now, let $\zeta(0) \in \partial G_\zeta(\vartheta)$. Then $V(\zeta(0)) = \vartheta$. Since $\|\xi(0)\| \le \rho_0 < \rho_2$, it follows from (2.8) that the estimate

$$(\operatorname{grad} V(\zeta(0)), pf(0, \zeta(0), \xi(0))) < 0$$

is valid. This and the relation $V(\zeta(0)) = \vartheta$ imply that $V(\zeta(t)) < \vartheta$ for small positive t. Therefore, there exists a positive ε_1 such that (2.11) is valid.

Obviously, there exists a positive ε_2 such that

$$\xi(t) \in G_\xi(\rho_2) \quad \text{for} \quad t \in (0, \varepsilon_2). \tag{2.12}$$

By virtue of (2.11) and (2.12), $z(t) \in G(\vartheta, \rho_2)$ provided that $0 < t < \min\{\varepsilon_1, \varepsilon_2\}$. Therefore, the number $t_* = \sup\{t > 0 : z(t) \in G(\vartheta, \rho_2)\}$ is defined and positive.

Inclusion (2.10) is equivalent to the estimate $t_* > T$.

Using relation (2.7), one can easily show that $t_* > T$. Therefore, any solution $z(\cdot)$ of Eq. (2.1) with an initial condition on $G(\vartheta, \rho_0)$ satisfies inclusion (2.10).

Let us show that these trajectories possess the T-nonreturnability property.

Since $G(\vartheta, \rho_0) = G_\zeta(\vartheta) \times G_\xi(\rho_0)$, it follows that

$$\partial G(\vartheta, \rho_0) = \left(\partial G_\zeta(\vartheta) \times \overline{G_\xi(\rho_0)}\right) \cup \left(\overline{G_\zeta(\vartheta)} \times \partial G_\xi(\rho_0)\right).$$

First, assume that $z(0) \in \partial G_\zeta(\vartheta) \times \overline{G_\xi(\rho_0)}$. Then $\zeta(0) \in \partial G_\zeta(\vartheta)$. However, $\zeta(t) \in G_\zeta(\vartheta)$ for $t \in (0, T]$ due to (2.10). Therefore, $\zeta(t) \ne \zeta(0)$ for $t \in (0, T]$ or, which is the same,

$$pz(t) \ne pz(0), \quad t \in (0, T]. \tag{2.13}$$

Now, let $z(0) \in \overline{G_\zeta(\vartheta)} \times \partial G_\xi(\rho_0)$. Then

$$z(t) \in \Omega(\vartheta, \rho_1, \rho_2), \quad t \in (0, T].$$

Using estimates (2.9), one can show that $\xi(T) \ne \xi(0)$. Therefore, taking into account (2.13), we conclude that the relation

$$z(t) \ne z(0)$$

is valid for any trajectory $z(\cdot)$ such that its initial value satisfies the inclusion $z(0) \in G(\vartheta, \rho_0)$.

Further, using (2.8), we prove that

$$\deg(i - U_T, \overline{G(\vartheta, \rho_0)}, 0) = \deg(\operatorname{grad} V(pz) + qz, \overline{G(\vartheta, \rho_0)}, 0).$$

Now, applying the theorem on products of degrees and the normalization property of topological degrees and taking into account Condition (2.4), we obtain that

$$\deg(i - U_T, \overline{G(\vartheta, \rho_0)}, 0) \ne 0.$$

By virtue of the nonzero degree principle, the field $z - U_T z$ has at least one singular point z_0 in the domain $G(\vartheta, \rho_0)$. Therefore, the solution $z(t, z_0)$ of Eq. (2.1) is T-periodic. It is easy to see that the trajectory of that solution does not intersect the boundary of the domain $G(\vartheta, \rho_0)$; therefore, the inclusion

$$z(t, z_0) \in G(\vartheta, \rho_0), \quad t \in [0, T]$$

is valid. □

Strengthening Condition (2.8), one can obtain an existence principle for T-periodic solutions.

Definition 2.2. A pair $\{V(\zeta),\ W(\varphi,\rho)\}$ of functions possessing properties (2.2)–(2.6) is called a *generalized many-sheeted vector guiding function* for Eq. (2.1) with respect to the domain $\Omega\,(\vartheta,\rho_1,\rho_2)$ if the function $V(\zeta)$ is nondegenerate and the following condition is satisfied apart from Conditions (2.7) and (2.9):

$$(\mathrm{grad}V(pz), pf(t,z)) \le 0, \quad V(pz) \ge \vartheta,\ \|qz\| \le \rho_2. \tag{2.14}$$

The following assertion is valid.

Theorem 2.2. *Let there exist a generalized many-sheeted vector guiding function for Eq.* (2.1) *with respect to the domain* $\Omega\,(\vartheta,\rho_1,\rho_2)$. *Then Eq.* (2.1) *has at least one* T*-periodic solution* $z_*(\cdot)$ *such that*

$$z_*(t) \in G\,(\vartheta,\rho_0)\,,\ \ t\in[0,T].$$

2.2. Collections of many-sheeted vector guiding functions. In [3], to develop ideas of [6, 7, 16], complete collections of strict many-sheeted vector guiding functions, complete and acute collections of generalized many-sheeted vector guiding function, and proper many-sheeted vector guiding functions are introduced for Eq. (2.1). We provide these definitions.

Let scalar continuously differentiable functions

$$V_1(\zeta), V_2(\zeta), \dots, V_k(\zeta), \quad \zeta\in\mathbb{R}^{n-2},\ k\ge 1, \tag{2.15}$$

be defined in the subspace $\mathbb{R}^{n-2}$.

Fix a positive r_0 and denote

$$m_i = \min_{\|\zeta\|\le r_0} V_i(\zeta),\ \ M_i = \max_{\|\zeta\|\le r_0} V_i(\zeta),\ \ i=1,\dots,k,$$

and

$$M^* = \sum_{i=1}^{k}(|m_i|+|M_i|).$$

In the sequel, we assume that functions (2.15) satisfy the nondegeneration condition

$$\mathrm{grad}V_i(\zeta)\ne 0 \quad \text{for all} \quad \zeta\in\mathbb{R}^{n-2} : \|\zeta\|\ge r_0,\ \ i=1,\dots,k.$$

Let functions (2.15) satisfy the condition

$$\lim_{\|\zeta\|\to\infty}[|V_1(\zeta)|+|V_2(\zeta)|+\ldots+|V_k(\zeta)|]=\infty, \quad k\ge 1. \tag{2.16}$$

By virtue of Condition (2.16), there exists r^* such that

$$|V_1(\zeta)|+|V_2(\zeta)|+\ldots+|V_k(\zeta)| > M^*, \quad \zeta\in\mathbb{R}^{n-2}:\|\zeta\|\ge r^*,\ k\ge 1. \tag{2.17}$$

For $\rho_2>\rho_1\ge 0$, select the domain

$$\Omega\,(r^*,\rho_1,\rho_2)=\{z\in\mathbb{R}^n : \|pz\|<r^*, \rho_1<\|qz\|<\rho_2\}$$

in $\mathbb{R}^n$ and assign

$$\alpha_{r^*,\rho_1,\rho_2}(t) = \sup_{z\in\Omega(r^*,\rho_1,\rho_2)}(\mathrm{grad}W(qz), qf(t,z)) \tag{2.18}$$

and

$$\beta_{r^*,\rho_1,\rho_2}(t) = \inf_{z\in\Omega(r^*,\rho_1,\rho_2)}(\mathrm{grad}W(qz), qf(t,z)). \tag{2.19}$$

Introduce a complete collection of strict many-sheeted vector guiding functions.

Definition 2.3. Functions $\{V_1(\zeta),\dots,V_k(\zeta),W(\varphi,\rho)\}$ possessing properties (2.2), (2.3), and (2.16) form a *complete collection* of strict many-sheeted vector guiding functions for Eq. (2.1) with respect to the domain $\Omega\,(r^*,\rho_1,\rho_2)$ if the following conditions are satisfied:

$$\sup_{t\in[0,T]}\frac{|(qf(t,z),qz)|}{\|qz\|} < \frac{\rho_2-\rho_1}{2T}, \quad z\in\Omega(r^*,\rho_1,\rho_2), \tag{2.20}$$

$$(\text{grad}V_i(pz), pf(t,z)) < 0, \quad \|pz\| \geq r_0,\ \|qz\| \leq \rho_2,\ i = 1, \ldots, k, \tag{2.21}$$

and

$$2\pi(N-1) < \int_0^T \alpha_{r^*,\rho_1,\rho_2}(\tau)d\tau, \int_0^T \beta_{r^*,\rho_1,\rho_2}(\tau)d\tau < 2\pi N, \tag{2.22}$$

where N is an integer and $\alpha_{r^*,\rho_1,\rho_2}(t)$ and $\beta_{r^*,\rho_1,\rho_2}(t)$ are functions defined by (2.18)-(2.19).

For $\rho_0 = (\rho_1 + \rho_2)/2$, introduce the notation

$$G(r^*, \rho_0) = \{z \in \mathbb{R}^n : \|pz\| < r^*,\ \|qz\| < \rho_0\}.$$

Applying methods of [7, 16] and the theory of topological degrees of maps, we provide a scheme of a proof of the following existence principle for periodic solutions (a complete proof can be found in [3]).

Theorem 2.3. *Let there exist a complete collection* $\{V_1(\zeta), \ldots, V_k(\zeta), W(\varphi, \rho)\}$ *of strict many-sheeted vector guiding functions for Eq.* (2.1) *with respect to the domain* $\Omega(r^*, \rho_1, \rho_2)$. *Let the topological index of the function* $V_1(\zeta)$ *be different from zero at infinity, i.e.,*

$$\text{ind}(V_1, \infty) \neq 0.$$

Then Eq. (2.1) *has at least one* T*-periodic solution* $z_*(\cdot)$ *such that* $z_*(t) \in G(r^*, \rho_0)$, $t \in [0, T]$.

Remark 2.1. Note that Conditions (2.21) imply the homotopy of the vector fields $\text{grad}V_i(pz)$, $i = 1, \ldots, k$, and $-pf(0,z)$, $z \in \mathbb{R}^n : \|pz\| \geq r_0,\ \|qz\| \leq \rho_2$. Therefore, all fields $\text{grad}V_i(\zeta)$, $i = 1, \ldots, k$, are homotopic to each other and have the same topological degree. Therefore,

$$\text{ind}(V_i, \infty) \neq 0, \quad i = 1, \ldots, k.$$

In the space $\mathbb{R}^n$, by Q_i and Q_i^*, $i = 1, \ldots, k$, denote the sets $\{z : \|pz\| \geq r_0\}$ such that the following inequalities are satisfied, respectively:

$$V_i(pz) \leq m_i \quad \text{and} \quad V_i(pz) \geq M_i, \quad i = 1, \ldots, k.$$

To prove Theorem 2.3, we need the following assertions.

Lemma 2.1. *All points of the set* $\{z \in \mathbb{R}^n : \|pz\| = r^*\}$ *are nonreturnability points for trajectories of Eq.* (2.1).

Proof. Assume that $\|pz\| = r^*$. It follows from (2.17) that there exists $i = i(pz)$ such that $z \in Q_i \cup Q_i^*$. Then, following [7, Lemma 6.7], we conclude that z is a nonreturnability point for trajectories of Eq. (2.1). ☐

Let $G_\zeta(r^*) = \left\{\zeta \in \mathbb{R}^{n-2} : \|\zeta\| < r^*\right\}$ and $G_\xi(\rho_0) = \left\{\xi \in \mathbb{R}^2 : \|\xi\| < \rho_0\right\}$.

Lemma 2.2. *For any trajectory* $z(\cdot)$ *of Eq.* (2.1) *with an initial value* $z(0)$ *from* $\overline{G_\zeta(r^*)} \times \partial G_\xi(\rho_0)$, *the relation*

$$qz(T) \neq qz(0)$$

is valid.

Proof of Theorem 2.3.

1. First, assume that solutions of Eq. (2.1) are extendable to $[0, T]$. By virtue of conditions imposed on the function $f(t, z)$, the Cauchy problem for Eq. (2.1) is uniquely solvable for any initial value z_0 from $\mathbb{R}^n$. By $z(t, z_0)$ denote the solution such that $z(0) = z_0$.

Under the imposed assumptions, a continuous translation operator $U_T z_0 = z(T; z_0)$ is defined on the set $\overline{G(r^*, \rho_0)}$. By virtue of Lemmas 2.1 and 2.2, this operator is nondegenerate on $\partial G(r^*, \rho_0)$. Hence, the topological degree $\deg(i - U_T, \overline{G(r^*, \rho_0)}, 0)$ is defined. In this case, it is known (see, e.g., [7]) that the existence proof for T-periodic solutions of Eq. (2.1) is reduced to the proof of the relation

$$\deg(i - U_T, \overline{G(r^*, \rho_0)}, 0) \neq 0.$$

Following [16], we prove that the specified topological degree satisfies the condition

$$\deg(i - U_T, \overline{G(r^*, \rho_0)}, 0) = \deg(\text{grad} V_i(pz) + qz, \overline{G(r^*, \rho_0)}, 0), \quad i = 1, \ldots, k. \tag{2.23}$$

Any field $\text{grad} V_i(pz) + qz,\ z \in G(r^*, \rho_0),\ i = 1, \ldots, k$, is the direct sum of the vector fields $\text{grad} V_i(\zeta) \in \mathbb{R}^{n-2}$, where $\zeta \in G_\zeta(r^*),\ i = 1, \ldots, k$, and $\xi \in \mathbb{R}^2,\ \xi \in G_\xi(\rho_0)$. Therefore, by virtue of properties of products of degrees, we have the relation

$$\deg(\text{grad} V_i(pz) + qz, \overline{G(r^*, \rho_0)}, 0) = \deg(\text{grad} V_i(\zeta), \overline{G_\zeta(r^*)}, 0) \deg(\xi, \overline{G_\xi(\rho_0)}, 0),$$

where $i = 1, \ldots, k$. Since the point $\xi = 0$ belongs to the domain $G_\xi(\rho_0)$, it follows from the normalization property of topological degrees that

$$\deg(\xi, \overline{G_\xi(\rho_0)}, 0) = 1.$$

Taking into account Remark 2.1, we see that

$$\deg(\text{grad} V_i(pz) + qz, \overline{G(r^*, \rho_0)}, 0) \neq 0;$$

then

$$\deg(i - U_T, \overline{G(r^*, \rho_0)}, 0) \neq 0.$$

by virtue of (2.23).

By virtue of the nonzero degree principle, the field $z - U_T z$ has at least one singular point z_0 in the domain $G(r^*, \rho_0)$. The solution $z(t, z_0)$ of Eq. (2.1) is T-periodic. It is easy to see that the trajectory of this solution does not intersect the boundary of the domain $G(r^*, \rho_0)$ and, therefore, the inclusion

$$z(t, z_0) \in G(r^*, \rho_0), \quad t \in \mathbb{R},$$

holds.

2. To consider the general case where there exist initial conditions such that the corresponding solutions "go to infinity" within small time segments (i.e., those solutions cannot be extended to the segment $[0, T]$), one can use the Krasnosel'skii scheme (see, e.g., [7]).

To provide an a priori estimate for T-periodic solutions $z(\cdot)$ of Eq. (2.1) satisfying Conditions (2.20)-(2.21), we note that such periodic solutions with respect to the p-projection satisfy the inequality

$$\|pz(t)\| < r^*,\ t \in [0, T], \tag{2.24}$$

because otherwise there would exist a point z belonging to a set $Q_i \cup Q_i^*,\ i = 1, \ldots, k$, such that it is not a nonreturnability point for trajectories.

Let us show that $\|qz(t)\| < \rho_2,\ t \in [0, T]$, if $\|qz(0)\| \leq \rho_0$. Since $\rho_0 < \rho_2$, it follows that

$$\|qz(t)\| < \rho_2, \quad t \in [0, \varepsilon).$$

Therefore, a positive number $t_* = \sup\{t > 0 : \|qz(t)\| < \rho_2\}$ is defined.

Let us show that $t_* > T$.

Since $\|qz(t_*)\| = \rho_2$ and $\|qz(0)\| = \rho_0$, it follows that

$$\|qz(t_*)\| - \|qz(0)\| \geq \rho_2 - \rho_0 = (\rho_2 - \rho_1)/2.$$

The last inequality can be represented as follows:

$$\rho(t_*) = \rho(0) \geq (\rho_2 - \rho_1)/2.$$

Therefore, we have the inequality

$$\max_{t \in [0, t_*]} |\rho'(t)| \geq (\rho_2 - \rho_1)/2t_*. \tag{2.25}$$

On the other hand, since $\rho'(t) = \frac{(qf(t, \zeta(t), \xi(t)), \xi(t))}{\|\xi(t)\|}$ and $z(t) \in G(r^*, \rho_2)$ for $t \in (0, t_*)$, it follows from (2.20) that

$$\max_{t \in [0, t_*]} |\rho'(t)| < (\rho_2 - \rho_1)/2T. \tag{2.26}$$

Comparing (2.25) and (2.26), we see that $t_* > T$. Therefore, we have the estimate

$$\|qz(t)\| < \rho_2, \quad t \in [0,T]. \tag{2.27}$$

The existence of estimates (2.24) and (2.27), which are a priori estimates, shows the way to prove Theorem 2.3 for the general case. We have to construct an auxiliary equation

$$z'(t) = f^*(t, z(t)) \tag{2.28}$$

such that its right-hand side satisfies the following three requirements:

1^0. $f^*(t,z) = f(t,z), \quad z \in \overline{\Omega(r^*, \rho_1, \rho_2)}$.
2^0. $f^*(t,z)$ satisfies Conditions (2.21);
3^0. all solutions of Eq. (2.28) are extendable to $[0,T]$.

Then, as we proved above, Eq. (2.28) has at least one T-periodic solution $z(\cdot)$. This solution satisfies estimates (2.24) and (2.27). Therefore, it is a solution of Eq. (2.1) as well. □

Definition 2.4. Functions $\{V_1(\zeta), \ldots, V_k(\zeta),\ W(\varphi,\rho)\}$ possessing properties (2.2), (2.3), and (2.16) form a *complete and acute collection of generalized many-sheeted vector guiding function* for Eq. (2.1) with respect to the domain $\Omega\,(r^*, \rho_1, \rho_2)$ if the functions $V_i(\zeta)$ are nondegenerate, Conditions (2.20), (2.22), and

$$(\mathrm{grad} V_i(pz), pf(t,z)) \le 0, \quad \|pz\| \ge r_0,\ \|qz\| \le \rho_2,\ i = 1, \ldots, k, \tag{2.29}$$

are satisfied, and the set

$$K(\zeta) = \left\{ \eta \in \mathbb{R}^{n-2} : \eta = \sum_{i=1}^{k} \gamma_i \mathrm{grad} V_i(\zeta),\ \gamma_1, \ldots, \gamma_k \ge 0 \right\}$$

is a Krein cone for any fixed ζ from $\mathbb{R}^{n-2}$ such that $\|\zeta\| \ge r_0$.

The following assertion is valid.

Theorem 2.4. *Let there exist a complete and acute collection $\{V_1(\zeta), \ldots, V_k(\zeta),\ W(\varphi,\rho)\}$ of generalized many-sheeted vector guiding functions of Eq.* (2.1) *with respect to the domain $\Omega\,(r^*, \rho_1, \rho_2)$. Let the topological index of the function $V_1(\zeta)$ be different from zero at infinity, i.e.,*

$$\mathrm{ind}\,(V_1, \infty) \ne 0.$$

Then Eq. (2.1) *has at least one T-periodic solution $z_*(\cdot)$ such that $z_*(t) \in G\,(r^*, \rho_0)$, $t \in [0,T]$.*

To prove this, we need the following assertion (see, e.g., [6]).

Lemma 2.3. *Let $V_1(\zeta), V_2(\zeta), \ldots, V_k(\zeta), \zeta \in \mathbb{R}^{n-2}$, $k \ge 1$, be an acute collection of functions. Then there exists a locally Lipschitz function $g : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ such that*

$$(\mathrm{grad} V_i(\zeta), g(\zeta)) < 0, \quad i = 1, \ldots, k,\ \|\zeta\| \ge r_0. \tag{2.30}$$

Proof of Theorem 2.4. By virtue of Lemma 2.3, there exists a function $g : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ satisfying Condition (2.30). Consider the family of auxiliary differential equations

$$z'(t) = f(t, z(t)) + \varepsilon g(pz(t)), \quad \varepsilon > 0. \tag{2.31}$$

It is easy to see that any complete and acute collection $\{V_1(\zeta), \ldots, V_k(\zeta), W(\varphi,\rho)\}$ of generalized many-sheeted vector guiding functions for Eq. (2.1) with respect to the domain $\Omega\,(r^*, \rho_1, \rho_2)$ is a complete collection of strict many-sheeted vector guiding functions for Eq. (2.31) with respect to the same domain provided that ε is sufficiently small. Then, due to Theorem 2.3, for any sufficiently small $\varepsilon = \varepsilon_m$, $m \ge 1$, there exists at least one T-periodic solution $z_m(\cdot)$ of Eq. (2.31). Passing to the limit as $\varepsilon_m \to 0$, we obtain the sought solution $z(\cdot)$ of Eq. (2.1) as the limit point of the sequence $\{z_m(\cdot)\}$. □

Let $\{V(\zeta), W(\varphi,\rho)\}$ be a strict many-sheeted vector guiding function for Eq. (2.1), i.e.,

$$(\mathrm{grad}V(pz), pf(t,z)) < 0, \quad \|pz\| \geq r_0,\ \|qz\| \leq \rho_2.$$

Following [7], we show that another function $\widetilde{V}(\zeta)$ (depending on the function $V(\zeta)$) can be constructed to satisfy the condition

$$\lim_{\|\zeta\|\to\infty} \left\{|V(\zeta)| + \left|\widetilde{V}(\zeta)\right|\right\} = \infty \tag{2.32}$$

provided that the angle between $\mathrm{grad}V(\zeta)$ and $-pf(t,z)$ is acute and bounded from above by a number that is less than $\pi/2$.

Definition 2.5. A pair $\{V(\zeta),\ W(\varphi,\rho)\}$ of functions possessing properties (2.2) and (2.3) is called a proper *many-sheeted vector guiding function* for Eq. (2.1) with respect to the domain $\Omega\,(r^*,\rho_1,\rho_2)$ if $V(\zeta)$ is a nondegenerate function, the condition

$$(\mathrm{grad}V(pz), pf(t,z)) \leq \delta_0 \left\|\mathrm{grad}V(pz)\right\| \left\|pf(t,z)\right\|, \tag{2.33}$$

where $\delta_0 < 0,\ 0 \leq t \leq T, \|pz\| \geq r_0,\ \|qz\| \leq \rho_2$, is satisfied (apart from Conditions (2.7) and (2.9)), and there exists a continuously differentiable function $V_1(\zeta)$ such that

$$\|\mathrm{grad}V(\zeta)\| \geq \|\mathrm{grad}V_1(\zeta)\| \quad (\|\zeta\| \geq r_0) \tag{2.34}$$

and

$$\lim_{\|\zeta\|\to\infty} |V_1(\zeta)| = \infty. \tag{2.35}$$

The following assertion is valid.

Theorem 2.5. *Let there exist a proper many-sheeted vector guiding function* $\{V(\zeta), W(\varphi,\rho)\}$ *for Eq.* (2.1) *with respect to the domain* $\Omega\,(r^*,\rho_1,\rho_2)$, *and let the topological index of the function* $V(\zeta)$ *be different from zero at infinity, i.e.,*

$$\mathrm{ind}\,(V,\infty) \neq 0.$$

Then Eq. (2.1) *has at least one* T*-periodic solution* $z_*(\cdot)$ *such that* $z_*(t) \in G\,(r^*,\rho_0)\,,\ t \in [0,T]$.

Proof. Suppose that $V_1(\zeta)$ satisfies Conditions (2.34)-(2.35). Assign $\widetilde{V}(\zeta) = V(\zeta) + \frac{\delta_0}{2}V_1(\zeta)$. It is easy to verify that $\widetilde{V}(\zeta)$ is a strict many-sheeted vector guiding function for Eq. (2.1).

It follows from (2.35) that Condition (2.32) is satisfied. Thus, the collection $\left\{V(\zeta), \widetilde{V}(\zeta),\ W(\varphi,\rho)\right\}$ is a complete collection of strict many-sheeted vector guiding functions for Eq. (2.1) with respect to the domain $\Omega\,(r^*,\rho_1,\rho_2)$. Therefore, due to Theorem 2.3, there exists at least one T-periodic solution $z_*(\cdot)$ of Eq. (2.1) such that

$$z_*(t) \in G\,(r^*,\rho_0)\,,\ t \in [0,T].$$

□

Remark 2.2. Following the Krasnosel'skii smoothing method for right-hand sides, one can show that Theorems 2.1–2.5 are valid for Eq. (2.1) as well provided that the right-hand side $f(t,z)$ is continuous with respect to the (vector) independent variable (t,z).

3. Integral Guiding Functions

For $\tau > 0$, denote the space $\mathcal{C}([-\tau,0];\mathbb{R}^n)$ by $\mathcal{C}$. For any function $x(\cdot) \in \mathcal{C}([-\tau,T];\mathbb{R}^n)$, $T > 0$, the symbol x_t denotes the following function from $\mathcal{C}$: $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau,0]$.

Consider the following periodic problem for a functional differential equation:

$$x'(t) = f(t,x_t), \tag{3.1}$$

$$x(0) = x(T). \tag{3.2}$$

We assume that the map $f : \mathbb{R}\times\mathcal{C} \to \mathbb{R}^n$ satisfies the following conditions:

(f_t) the function f is T-periodic with respect to the first independent variable, i.e.,

$$f(t,\varphi) = f(t+T,\varphi) \quad \text{for all } t \in \mathbb{R},\ \varphi \in \mathcal{C}$$

(obviously, this condition allows us to treat the map f defined on $[0,T]\times\mathcal{C}$);

(f_1) for any φ from $\mathcal{C}$, the function $f(\cdot,\varphi) : [0,T] \to \mathbb{R}^n$ is measurable;

(f_2) for almost any (a. a.) t from $[0,T]$, the map $f(t,\cdot) : \mathcal{C} \to \mathbb{R}^n$ is continuous;

(f_3) for any positive ρ, there exists a function $\alpha_\rho(\cdot)$ from $L^1_+([0,T],\mathbb{R})$ such that

$$\|f(t,\varphi)\| \le \alpha_\rho(t)$$

for any φ from $\mathcal{C}$ such that $\|\varphi\| \le \rho$ and for a. e. t from $[0,T]$.

To investigate problem (3.1)-(3.2), we use the theory of the coincidence topology degree for a map pair in the following situation (see, e.g., [10]).

Let X_1 and X_2 be normed spaces, $U \subset X_1$ be a bounded open set, $l :\ \operatorname{dom} l \subseteq X_1 \to X_2$ be a linear zero-index Fredholm operator,

$$N_l = \{x \in \operatorname{dom} l :\ lx = 0\},$$

and

$$R_l = \{lx :\ x \in \operatorname{dom} l\}.$$

Let $p :\ X_1 \to X_1$ and $q :\ X_2 \to X_2$ be continuous projection operators such that $R_p = N_l$ and $N_q = R_l$ and l_p denote the restriction of l to $\operatorname{dom} l \cap N_p$, where $N_p = \operatorname{Ker} p$.

Further, let $k_{p,q} = l_p^{-1}(i-q) :\ X_2 \to \operatorname{dom} l \cap N_p$ be a continuous operator.

Definition 3.1. We say that a map $G : \overline{U} \to X_2$ is *l-compact* if

(1) $G(U)$ is a bounded set;

(2) $k_{p,q} \circ G : \overline{U} \to K(E_1)$ is a compact map.

Remark 3.1. The above definition of l-compact maps does not depend on the choice of linear projection operators $p : E_1 \to E_2$ and $q : E_1 \to E_2$.

Let $G : \overline{U} \to X_2$ be a closed l-compact operator such that $lx \ne Gx$ for all x from ∂U. Then the degree of the coincidence is defined: it is an integer (denoted by $\deg(l,G,\overline{U})$) such that if it is different from zero, then there exists a coincidence point x_0 from U, $l(x_0) = G(x_0)$.

The following assertion can be proved by means of the coincidence and topological degrees of maps (see, e.g., [10]).

Theorem 3.1. *Let the operators qG and $k_{p,q}G$ be compact and the following conditions be satisfied*:

(1) $lx \ne \lambda Gx$ *for all* λ *from* $(0,1)$ *and* x *from* $\operatorname{dom} l \cap \partial U$;

(2) $0 \ne qGx$ *for all* x *from* $\operatorname{Ker} l \cap \partial U$.

Then the equation $lx = Gx$ has a solution from $\operatorname{dom} l \cap \overline{U}$ *if* $\deg(qG\,|_{\operatorname{Ker} l \cap U}, \overline{\operatorname{Ker} l \cap U}, 0) \ne 0$.

Let C_T denote the space of continuous T-periodic functions $x : \mathbb{R} \to \mathbb{R}^n$ with norm $\|x\|_C = \sup\limits_{t\in[0,T]} \|x(t)\|$.

Let $\|x\|_2$ denote the norm of the function x in the space L^2, $\|x\|_2 = \left(\int\limits_0^T \|x(s)\|^2\,ds\right)^{\frac{1}{2}}$.

3.1. Strict integral guiding functions.

In [2], the following notion is introduced.

Definition 3.2. A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called an *integral guiding function* of problem (3.1)-(3.2) if there exists a positive N such that

$$\int\limits_0^T (\operatorname{grad} V(x(s)), f(s,x_s))\,ds > 0 \tag{3.3}$$

for any continuously differentiable function x from C_T such that $\|x\|_2 \geq N$ and $\|x'(t)\| \leq \|f(t, x_t)\|$ $(t \in [0, T])$.

The following assertion is valid.

Theorem 3.2. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be an integral guiding function of problem* (3.1)-(3.2) *such that*

$$\deg(\operatorname{grad} V; \overline{B_N}, 0) \neq 0, \tag{3.4}$$

where $B_N \subset \mathbb{R}^n$ is the ball of radius N centered at the origin. Then problem (3.1)-(3.2) *has a solution.*

Remark 3.2. Note that the assumptions of the theorem are satisfied if, e.g., the function V is even or $\lim\limits_{\|x\| \to \infty} V(x) = \pm\infty$.

We provide a scheme of a proof of Theorem 3.2 (for a detailed proof, see [2]).

Proof. To use Theorem 3.1, consider the operators

$$l : \operatorname{dom} l := \{x \in C_T : x \in C^1\} \to C_T, \quad lx = x',$$

and

$$G : C_T \to C_T, \quad (Gx)(t) = f(t, x_t).$$

It is easy to verify (see, e.g., [10]) that the Nemytskii operator G is an l-compact operator, l is a linear zero-index Fredholm operator, and $\operatorname{Ker} l = \mathbb{R}^n$. The projection $q : C_T \to \mathbb{R}^n$ can be defined by the relation $qx = \frac{1}{T} \int\limits_0^T x(s)\, ds$. It is easy to verify that pG and $k_{p,q}G$ are compact operators.

Note that if $\lambda \in (0, 1)$, then the solution $x \in \operatorname{dom} l$ of the equation $l(x) = \lambda G(x)$ satisfies the problem

$$x'(t) = \lambda f(t, x_t), \tag{3.5}$$

$$x(0) = x(T). \tag{3.6}$$

Then

$$\int\limits_0^T (\operatorname{grad} V(x(s)), f(s, x_s))\, ds = \frac{1}{\lambda} \int\limits_0^T \big(\operatorname{grad} V(x(s)), x'(s)\big)\, ds$$

$$= \frac{1}{\lambda} \int\limits_0^T V'(x(s))\, ds = \frac{1}{\lambda}(V(x(T)) - V(x(0))) = 0,$$

which implies that

$$\|x\|_2 < N.$$

On the other hand, it follows from Condition (f_3) that there exists a positive M such that $\|x'\|_2 < M$. Then there exists a positive M' such that

$$\|x\|_C < M'.$$

Let U be a ball $B_r \subset C_T$ of radius $r = \max\{N, M', NT^{-1/2}\}$. Then

$$l(x) \neq \lambda G(x)$$

for all $x \in \partial U$.

Now, take an arbitrary u from $\partial U \cap \operatorname{Ker} l$. Since $\|u\| \geq NT^{-1/2}$, it follows from the definition of integral guiding functions that

$$\int\limits_0^T (\operatorname{grad} V(u), f(s, u))\, ds > 0$$

and, therefore, $qGx \neq 0$ for any x from $\operatorname{Ker} l \cap \partial U$.

This means that, by virtue of (3.4), we have the relation

$$\deg(qG|_{U_{\mathrm{Ker}\, l}}, \overline{U_{\mathrm{Ker}\, l}}, 0) = \deg(\mathrm{grad} V, \overline{U_{\mathrm{Ker}\, l}}, 0) \neq 0,$$

where $U_{\mathrm{Ker}\, l} = U \cap \mathrm{Ker}\, l$. Thus, all conditions of Theorem 3.1 are satisfied and problem (3.1)-(3.2) has a solution. □

Examples.

3.1.1. Differential equations with delays. Consider the periodic problem for a differential equations with delay

$$x'(t) = f(t, x(t-\tau)), \tag{3.7}$$

$$x(0) = x(T), \tag{3.8}$$

where the map $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies Conditions $(f_t), (f_1)$, and (f_2).

Theorem 3.3. *Suppose that there exist positive $\overline{N}$ and C such that*

$$(x, f(t, x)) \geq C$$

for any x, $\|x\| \geq \overline{N}$. If

$$\|f(t, x)\| \leq M$$

and $C - \tau M^2 > 0$, then problem (3.7)-(3.8) has a solution.

Proof. We show that $V(x) = \frac{1}{2}\|x\|^2$ is an integral guiding function for problem (3.7)-(3.8). Then, taking into account Remark 3.2, we deduce the assertion of the present theorem from Theorem 3.2. Indeed, if a positive N is sufficiently large, then conditions $\|x'(t)\| \leq M$ $(t \in [0, T])$ and $\|x\|_2 \geq N$ imply that $\|x(t)\| \geq \overline{N}$ for any t from $[0, T]$ and any continuously differentiable function $x(\cdot)$ from C_T. Then such a function $x(\cdot)$ satisfies the relation

$$\int_0^T (\mathrm{grad} V(x(s)), f(s, x(s-\tau)))\, ds = \int_0^T (x(s), f(s, x(s-\tau)))\, ds$$

$$= \int_0^T (x(s-\tau), f(s, x(s-\tau)))\, ds + \int_0^T (x(s) - x(s-\tau), f(s, x(s-\tau)))\, ds$$

$$\geq CT - \tau M^2 T = (C - \tau M^2)T > 0.$$

□

3.1.2. Semilinear functional differential equations. Consider the periodic problem

$$x'(t) = Ax(t) + f(t, x_t), \tag{3.9}$$

$$x(0) = x(T), \tag{3.10}$$

where the map $f : \mathbb{R} \times C \to \mathbb{R}^n$ satisfies Conditions (f_t) and (f_1)–(f_3), while $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator.

Theorem 3.4. *Let there exist a positive ε such that the quadratic form (Ax, x) satisfies the condition*

$$(Ax, x) \geq \varepsilon \|x\|^2$$

for all $x \in \mathbb{R}^n$. If

$$\overline{\lim_{\|x\|_2 \to +\infty}} \frac{\|\tilde{G}x\|_2}{\|x\|_2} < \varepsilon$$

for all x from C_T, where $\tilde{G}$ is the Nemytskii operator generated by f, i.e.,

$$\tilde{G} : C_T \to C_T : [\tilde{G}(x)](t) = f(t, x_t),$$

then problem (3.9)-(3.10) has a solution.

Proof. Similarly to the previous example, we show that $V(x) = \frac{1}{2}\|x\|^2$ is an integral guiding function for problem (3.9)-(3.10). Indeed, if $\|x\|_2$ is sufficiently large, then

$$\int_0^T (\mathrm{grad}V(x(s)), Ax(s) + f(s, x_s))\; ds = \int_0^T (Ax(s), x(s))\; ds + \int_0^T (x(s), f(s, x_s))\; ds$$

$$\geq \varepsilon\|x\|_2^2 - \|x\|_2\|\tilde{G}x\|_2 > 0.$$

□

3.1.3. Gradient functional differential equations. Consider a periodic problem of the kind

$$x'(t) = \mathrm{grad}\, g(x(t)) + f(t, x_t), \tag{3.11}$$

$$x(0) = x(T), \tag{3.12}$$

where f is a map satisfying Conditions (f_t) and (f_1)–(f_3), while $\mathrm{grad}\, g$ is the gradient of a C_1-function $g : \mathbb{R}^n \to \mathbb{R}$.

Theorem 3.5. *Let the following conditions be satisfied:*

(1) *there exist positive constants* ε, K, *and* β, $\beta \geq 1$, *such that*

$$\|\mathrm{grad}\, g(x)\| \geq \varepsilon\|x\|^\beta - K$$

for all x *from* $\mathbb{R}^n$;

(2) $\displaystyle\overline{\lim_{\|x\|_2\to+\infty}} \frac{\|\tilde{g}x\|_2}{\|x\|_2^\beta} < \varepsilon T^{(1-\beta)/2}$ *for all* x *from* $\in C_T$;

(3) *the gradient* $\mathrm{grad}\, g$ *has a nonzero topological index, i.e.,*

$$\deg(\mathrm{grad}\, g, \overline{B_N}, 0) \neq 0$$

provided that N *is positive and sufficiently large.*

Then problem (3.11)-(3.12) *has a solution.*

Proof. Similarly to previous examples, we show that g is an integral guiding function for problem (3.11)-(3.12). Note that the embedding $L^{2\beta} \subset L^2$ yields the following estimate for any continuously differentiable function $x(\cdot) \in C_T$:

$$\|\mathrm{grad}\, g(x(\cdot))\|_2 \geq \varepsilon\|x\|_{2\beta}^\beta - K\sqrt{T} \geq \varepsilon T^{(1-\beta)/2}\|x\|_2^\beta - K\sqrt{T}.$$

Then the inequality

$$\int_0^T (\mathrm{grad}\, g(x(s)), \mathrm{grad}\, g(x(s)) + f(s, x_s))\; ds \geq \|\mathrm{grad}\, g(x(\cdot))\|_2 \left(\|\mathrm{grad}\, g(x(\cdot))\|_2 - \|\tilde{g}x\|_2\right)$$

$$\geq \|\mathrm{grad}\, g(x(\cdot))\|_2 \left(\varepsilon T^{(1-\beta)/2} - \frac{K\sqrt{T}}{\|x\|_2^\beta} - \frac{\|\tilde{g}x\|_2}{\|x\|_2^\beta}\right)\|x\|_2^\beta > 0$$

holds provided that $\|x\|_2$ is sufficiently large. □

3.2. Generalized integral guiding functions. Strengthening Condition (3.3), one can obtain an existence principle for T-periodic solutions.

Definition 3.3. A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *generalized integral guiding function* of problem (3.1)-(3.2) if there exists a positive N such that

$$\int_0^T (\mathrm{grad} V(x(s)), f(s, x_s))\, ds \geq 0 \tag{3.13}$$

for any continuously differentiable function x from C_T such that $\|x\|_2 \geq N$ and $\|x'(t)\| \leq \|f(t, x_t)\|$ $(t \in [0, T])$.

The following assertion is valid.

Theorem 3.6. *Suppose that $V : \mathbb{R}^n \to \mathbb{R}$ is a generalized integral guiding function of problem* (3.1)-(3.2)*,* $\mathrm{grad}\, V(x) \neq 0$ *provided that $x \in \mathbb{R}^n$ and $\|x\| \geq N$, and*

$$\deg(\mathrm{grad} V, \overline{B_N}, 0) \neq 0, \tag{3.14}$$

where $B_N \subset \mathbb{R}^n$ is the ball of radius N centered at the origin. Then problem (3.1)-(3.2) *has a solution.*

4. Guiding Functions on Given Sets

Consider the following periodic problem for a differential equation:

$$x'(t) = f(t, x(t)) \quad \text{a. e.} \quad t \in [0, T], \tag{4.1}$$

$$x(0) = x(T). \tag{4.2}$$

It is assumed that the map $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions:

(f_t) the function f is T-periodic with respect to the first independent variable, i.e.,

$$f(t, x) = f(t + T, x) \quad \text{for all } t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^n$$

(obviously, this condition allows us to assume that the map f is defined on $[0, T] \times \mathbb{R}^n$);

(f_1) the function $f(\cdot, x) : [0, T] \to \mathbb{R}^n$ is measurable for any x from $\mathbb{R}^n$;

(f_2) for a. a. t from $[0, T]$, the map $f(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous;

(f_3) for any positive ρ, there exists a function $\alpha_\rho(\cdot)$ from $L^1_+([0, T], \mathbb{R})$ such that if $x \in \mathbb{R}^n$ and $\|x\| \leq \rho$, then

$$\|f(t, x)\| \leq \alpha_\rho(t)$$

for a. a. t from $[0, T]$.

We say that an absolutely continuous function $x(\cdot)$ is a solution of problem (4.1)-(4.2) if it almost everywhere (a. e.) satisfies Condition (4.2) and Eq. (4.1).

To investigate problem (4.1)-(4.2), we use the theory of the coincidence topology degree for a pair of maps in a situation similar to Sec. 3.

Let $D \subset \mathbb{R}^n$ be a nonempty set and C_T be the space of continuous T-periodic functions $x : \mathbb{R} \to \mathbb{R}^n$ with the norm $\|x\|_C = \sup_{t \in [0,T]} \|x(t)\|$.

Introduce the notation

$$\Gamma(D) := \{x \in C_T : x(t) \in D \quad \text{for all } \ t \in [0, T]\}.$$

For any function $V : \mathbb{R}^n \to \mathbb{R}$, $M \subset \mathbb{R}$, and any real r, introduce

$$V^{-1}(M) := \{x \in \mathbb{R}^n : V(x) \in M\}$$

and

$$\mathcal{V}_r := \{x \in \mathbb{R}^n : V(x) < r\}.$$

4.1. Generalized guiding functions on given sets. In [12], the following notion is introduced.

Definition 4.1. A continuously differentiable function $V : D \to \mathbb{R}$ is called a *generalized guiding function* on D for Eq. (4.1) if the inequality

$$(\mathrm{grad} V(x), f(t,x)) \leq 0 \tag{4.3}$$

is satisfied for any x from D and any t from $[0,T]$.

The case where $D = \mathbb{R}^n \setminus B(r)$, $r > 0$, is similar to the one considered in Sec. 1.1 (see [10] for details).

The following assertion is valid.

Theorem 4.1. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that the following conditions are satisfied*:

(1) $\mathcal{V}_0$ *is a nonempty, open, and bounded set*;
(2) V *is a generalized guiding function for Eq.* (4.1) *on the set* $V^{-1}(0)$;
(3) $\mathrm{grad}\, V(x) \neq 0$ *for all* $x \in V^{-1}(0)$;
(4) $\deg(\mathrm{grad}\, V, \overline{\mathcal{V}_0}, 0) \neq 0$.

Then Eq. (4.1) *has at least one T-periodic solution* $x(\cdot) \in \Gamma(\overline{\mathcal{V}_0})$.

We provide a scheme of a proof of Theorem 4.1 (a detailed proof can be found in [11, 12]).

Proof.

(a). First, we prove that the assertion of the theorem is valid if Conditions 2 and 3 are assumed to be satisfied on a set $V^{-1}([0,\varepsilon])$, $\varepsilon > 0$.

Define the homotopy $H : [0,T] \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ as follows:

$$H(t,x,\lambda) = -(1-\lambda)\mathrm{grad}\, V(x(t)) + \lambda f(t, x(t)). \tag{4.4}$$

Consider the periodic problem

$$x'(t) = H(t,x,\lambda) \quad \text{a.e. in} \quad \lambda \in [0,1), \tag{4.5}$$

$$x(0) = x(T). \tag{4.6}$$

Let $\lambda \in [0,1)$ and x be a solution of (4.5)-(4.6) such that $x \in \Gamma(\overline{\mathcal{V}_0})$. Let us show that $x \in \Gamma(\mathcal{V}_0)$, i.e., $V(x(t)) < 0$, for all t from $[0,T]$. Suppose that there exists τ from $[0,T]$ such that $V(x(\tau)) = 0$. This means that $x(\tau) \in V^{-1}(0)$ and, by virtue of Condition 3, the relation $\mathrm{grad}\, V(x(\tau)) \neq 0$ is satisfied. Hence, there exists a positive δ such that

$$\mathrm{grad}\, V(x(\tau)) \neq 0 \quad \text{for all} \;\; t \in [\tau-\delta, \tau+\delta] \cap [0,T]. \tag{4.7}$$

Without loss of generality, assume that $\tau - \delta \in (0,T)$ and $V(x(t)) \in [-\varepsilon, 0]$ for all t from $[\tau-\delta, \tau]$. Then, applying assumption 2 and relation (4.7), we obtain the inequality

$$0 \leq V(x(\tau)) - V(x(\tau-\delta)) = \int\limits_{\tau-\delta}^{\tau} \frac{d}{dt} V(x(t))dt$$

$$= \int\limits_{\tau-\delta}^{\tau} \left[-(1-\lambda)\|\mathrm{grad}\, V(x(t))\|^2 + \lambda(\mathrm{grad}\, V(x(t)), f(t,x(t)))\right] dt < 0.$$

A contradiction is obtained. Thus, either problem (4.1)-(4.2) has a solution on $\partial\Gamma(\mathcal{V}_0)$ (in this case, the theorem is proved) or problem (4.5)-(4.6) is solvable on $\partial\Gamma(\mathcal{V}_0)$ for no λ from $[0,1]$.

Then define the operator

$$l : \mathrm{dom}\, l := \{x \in C_T : x \text{ is absolutrly continuous}\} \to C_T, \quad lx = x'.$$

Also, for any λ from $[0,T]$, define the Nemytskii operator

$$G(\cdot,\lambda) = G_H(\cdot,\lambda) : C_T \to C_T, \quad (G_H x)(t) = H(t,x,\lambda).$$

It is easy to verify (see, e.g., [10]) that $G(\cdot,\lambda)$ is a family of l-compact operators and l is a linear zero-index Fredholm operator.

The abstract form of (4.5) is

$$lx = G(x,\lambda) \tag{4.8}$$

or

$$lx = -(1-\lambda)\operatorname{grad} V(x) + \lambda f(\cdot,x).$$

Due to the homotopic invariance property of the coincidence topology degree, we have the relation

$$\deg(l, H(\cdot,1), \Gamma(\overline{\mathcal{V}_0}), 0) = \deg(l, H(\cdot,0), \Gamma(\overline{\mathcal{V}_0}), 0).$$

From Conditions 1 and 4, it follows that

$$|\deg(l, H(\cdot,0), \Gamma(\overline{\mathcal{V}_0}), 0)| = |\deg(\operatorname{grad} V, \overline{\mathcal{V}_0}, 0)| \neq 0.$$

Then, from the existence property of the coincidence point, we conclude that

$$l(x) = G(x,\lambda) \quad \text{if } \; x \in \Gamma(\mathcal{V}_0).$$

(b). Now, let Conditions 2 and 3 be satisfied on the set $V^{-1}(0)$.

For this case, the idea of the proof is to construct a sequence of maps $f_m : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ such that

$$\|f_m(t,x)\| \le 2\alpha_\rho(t) + 1 \quad \text{for a. a.} \quad t \in [0,T] \quad \text{and} \quad x \in \overline{\mathcal{V}_0},$$

$$\lim_{\|m\|\to\infty} f_m(t,x) = f(t,x),$$

and

$$(\operatorname{grad} V(x), f_m(t,x)) \le 0 \quad \text{for all} \quad (t,x) \in ([0,T]\setminus N) \times \overline{\mathcal{V}_0},$$

where $N \subset [0,T]$ is a zero measure set. By virtue of the continuity of the maps $\operatorname{grad} V$ and f_m, the last inequality is valid for all (t,x) from $[0,T]\times\overline{\mathcal{V}_0}$.

We consider the periodic problem for the following auxiliary differential equation:

$$x'(t) = f_m(t,x(t)) \quad \text{a. e. in} \quad t \in [0,T], \tag{4.9}$$

$$x(0) = x(T). \tag{4.10}$$

By virtue of the first part (a) of the proof of the theorem, problem (4.9)-(4.10) has at least one T-periodic solution $x_m^*(\cdot)$. Passing to the limit as $m \to \infty$, we obtain the sought solution $x^*(\cdot)$ of problem (4.1)-(4.2) as the limit point of the sequence $x_m^*(\cdot)$ of solutions of (4.9)-(4.10). ☐

4.2. Averaged guiding functions on given sets. As above, let $D \subset \mathbb{R}^n$ be a nonempty set.

Definition 4.2. A continuously differentiable function $V : D \to \mathbb{R}$ is called an *averaged guiding function* on D for Eq. (4.1) if

$$\int_0^T (\operatorname{grad} V(x(s)), f(s,x(s)))ds \le 0 \quad \text{for all} \quad x \in \Gamma(D). \tag{4.11}$$

Remark 4.1. Any generalized guiding function on D for Eq. (4.1) is an averaged guiding function on D for Eq. (4.1). The inverse assertion is not valid.

The following assertion is valid.

Theorem 4.2. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that the following conditions are satisfied*:

(1) $\mathcal{V}_0$ *and* $\mathcal{V}_r$ *are nonempty, open, and bounded sets, where*

$$r = \max\{T \max_{u \in V^{-1}(0)} \|\text{grad}\, V(u)\|^2, \max_{u \in V^{-1}(0)} \int_0^T |(\text{grad}\, V(u), f(t,u))| dt\};$$

(2) V *is an averaged guiding function on the set* $\mathbb{R}^n \backslash \mathcal{V}_0$ *for Eq.* (4.1);
(3) $\text{grad}\, V(x) \neq 0$ *for any* x *from* $\mathbb{R}^n \backslash \mathcal{V}_0$;
(4) $\deg(\text{grad}\, V, \mathcal{V}_0, 0) \neq 0$.

Then Eq. (4.1) *has at least one* T*-periodic solution* x *from* $\Gamma(\mathcal{V}_r \cup V^{-1}(r))$.

Proof. First, consider the homotopy $H : [0,T] \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ defined as follows:

$$H(t,x,\lambda) = -(1-\lambda)\text{grad}\, V(x(t)) + \lambda f(t, x(t)).$$

Let x be a solution of the problem

$$x'(t) = H(t,x,\lambda), \quad \lambda \in [0,1), \tag{4.12}$$

$$x(0) = x(T). \tag{4.13}$$

If we assume that $V(x(t)) \geq 0$ for all t from $[0,T]$, then, taking into account the periodicity of the solution and assumption (2), we see that

$$0 = V(x(T)) - V(x(0)) = \int_0^T V'(x(s))ds = \int_0^T (\text{grad}\, V(x(s)), x'(s))ds$$

$$= -(1-\lambda) \int_0^T \|\text{grad}\, V(x(s))\|^2 ds + \lambda \int_0^T (\text{grad}\, V(x(s)), f(s, x(s)))ds < 0.$$

It follows from the obtained contradiction that there exists t_0 from $[0,T]$ such that

$$V(x(t_0)) < 0, \tag{4.14}$$

i.e., $x(t_0) \in \mathcal{V}_0$. If

$$V(x(t)) \leq 0 \quad \text{for all} \quad t \in [0,T],$$

then assumption (1) yields an a priori estimate for this solution of Eq. (4.12). Otherwise, there exists τ from $[0,T]$ such that $V(x(\tau)) > 0$ and, therefore,

$$\max_{t \in [0,T]} V(x(t)) = V(x(\tau)) > 0. \tag{4.15}$$

From (4.14) and (4.15), it follows that there exists σ from $[0,T]$ such that either

$$\sigma < \tau, \quad V(x(\sigma)) = 0, \quad \text{and} \quad V(x(t)) > 0 \quad \text{for } t \in (\sigma, \tau] \tag{4.16}$$

or

$$\tau < \sigma, \quad V(x(\sigma)) = 0, \quad \text{and} \quad V(x(t)) > 0 \quad \text{for } t \in [\tau, \sigma). \tag{4.17}$$

For definiteness, consider the case where (4.16) is valid (the case where (4.17) is valid is considered in the same way). First, we assume that $\tau \neq 0$ (and $\tau \neq T$); then, for any positive integer n such that $\tau + \frac{1}{n} < T$, define the continuous function $x_n : [0,T] \to \mathbb{R}^n$ as follows:

$$x_n(t) = \begin{cases} x(\sigma) & \text{if } t \in [0, \sigma], \\ x(t) & \text{if } t \in (\sigma, \tau], \\ x(\tau + n(\sigma - \tau)(t - \tau)) & \text{if } t \in (\tau, \tau + \frac{1}{n}], \\ x(\sigma) & \text{if } t \in (\tau + \frac{1}{n}, T]. \end{cases}$$

If $\tau = 0$ and the function $V(x(\cdot))$ attains its maximum at the points 0 and T, then the function $x_n(\cdot)$ is defined for (sufficiently large) values of n such that $0 + \frac{1}{n} < \sigma$:

$$x_n(t) = \begin{cases} x(\tau + n(\sigma - \tau)t) & \text{if } t \in [0, \frac{1}{n}], \\ x(\sigma) & \text{if } t \in (\frac{1}{n}, \sigma], \\ x(t) & \text{if } t \in (\sigma, T]. \end{cases}$$

In any case, $x_n(\cdot)$ is a sequence of continuous and T-periodic functions such that $0 \leq V(x_n(t)) \leq V(x(\tau))$ and the sequence converges on $[0, T]$ to the function $\xi(\cdot)$ defined by the relation

$$\xi(t) = \begin{cases} x(t) & \text{if } t \in [\sigma, \tau], \\ x(\sigma) & \text{if } t \in [0, T] \setminus [\sigma, \tau], \end{cases}$$

and $0 \leq V(\xi(t)) \leq V(x(\tau))$. It follows from assumption (2) that

$$\int_0^T (\operatorname{grad} V(x_n(s), f(s, x_n(s)))ds \leq 0$$

provided that n is sufficiently large. Then

$$\int_0^T (\operatorname{grad} V(\xi(s), f(s, \xi(s))))ds \leq 0$$

and we obtain the relation

$$0 > \left(\int_\sigma^\tau + \int_{[0,T]\setminus[\sigma,\tau]} \right) \left[-(1-\lambda)\|\operatorname{grad} V(\xi(s))\|^2 + \lambda(\operatorname{grad} V(\xi(s)), f(s, \xi(s))) \right] ds$$

$$= \int_\sigma^\tau \frac{d}{dt} V(x(s))ds + \int_{[0,T]\setminus[\sigma,\tau]} \left[-(1-\lambda)\|\operatorname{grad} V(\xi(s))\|^2 + \lambda(\operatorname{grad} V(\xi(s)), f(s, \xi(s))) \right] ds$$

$$= V(x(\tau)) + \int_{[0,T]\setminus[\sigma,\tau]} \left[-(1-\lambda)\|\operatorname{grad} V(x(\sigma))\|^2 + \lambda(\operatorname{grad} V(x(\sigma), f(s, x(\sigma))) \right] ds.$$

Due to assumption (1), this implies the relation

$$V(x(\tau)) < \int_{[0,T]\setminus[\sigma,\tau]} \left[(1-\lambda)\|\operatorname{grad} V(x(\sigma))\|^2 - \lambda(\operatorname{grad} V(x(\sigma), f(s, x(\sigma))) \right] ds$$

$$\leq \max\{T \max_{u \in V^{-1}(0)} \|\operatorname{grad} V(u)\|^2, \max_{u \in V^{-1}(0)} \int_0^T |(\operatorname{grad} V(u), f(t, u))|dt\} = r.$$

Therefore, the inequality $V(x(t)) < r$ holds for possible solutions of Eq. (4.12) and for all $t \in [0, T]$.

Thus, either problem (4.1)-(4.2) has a solution on $\partial\Gamma(\mathcal{V}_r)$ (in this case, the theorem is proved) or problem (4.12)-(4.13) has no solution on $\partial\Gamma(\mathcal{V}_r)$ for λ from $[0, 1]$. Then, considering an equation similar to (4.8) and repeating the reasoning from the proof of Theorem 4.1, we obtain the relation

$$\deg(l, H(\cdot, 1), \Gamma(\overline{\mathcal{V}_r}), 0) = \deg(l, H(\cdot, 0), \Gamma(\overline{\mathcal{V}_r}), 0) = |\deg(\operatorname{grad} V, \overline{\mathcal{V}_r}, 0)| \neq 0.$$

Then the assertion of the theorem follows from the existence property of the coincidence point. □

Corollary 4.1. *Let there exist a positive r such that a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is an averaged guiding function on $\mathbb{R}^n \setminus B(r)$ for Eq.* (4.1) *and*

$$\lim_{\|x\|\to+\infty} V(x) = \pm\infty.$$

Then problem (4.1)-(4.2) *has at least one T-periodic solution.*

4.3. Asymptotically averaged guiding functions on given sets. Results of Sec. 4.2 can be generalized as follows.

Definition 4.3. A continuously differentiable function $V : D \to \mathbb{R}$ is called an *asymptotically averaged guiding function* for Eq. (4.1) if there exists a function $\alpha(\cdot)$ from $L^1_+([0,T],\mathbb{R})$ such that the following conditions are satisfied:

(1) $(\operatorname{grad} V(x), f(t,x)) \le \alpha(t)$ for all x from $\mathbb{R}^n$ and for a. a. t from $[0,T]$;

(2) $\int\limits_0^T \lim\sup\limits_{\|x\|\to\infty} (\operatorname{grad} V(x(s)), f(s,x(s)))ds < 0.$

The following assertion is valid.

Theorem 4.3. *If a positive r is sufficiently large, then any asymptotically averaged guiding function for Eq.* (4.1) *is an averaged guiding function on $\mathbb{R}^n \setminus B(r)$ for the said equation.*

Corollary 4.2. *Let a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ be an asymptotically averaged guiding function for Eq.* (4.1) *and*

$$\lim_{\|x\|\to\infty} V(x) = \pm\infty.$$

Then problem (4.1)-(4.2) *has at least one T-periodic solution.*

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