DOI 10.1007/s10958-018-3909-7
Journal of Mathematical Sciences, Vol. 232, No. 6, August, 2018

ORTHOGONALITY GRAPHS OF MATRICES OVER SKEW FIELDS

A. E. Guterman* and O. V. Markova* UDC 512.643

The paper is devoted to studying the orthogonality graph of the matrix ring over a skew field. It is shown that for $n \geq 3$ and an arbitrary skew field $\mathbb{D}$, the orthogonality graph of the ring $M_n(\mathbb{D})$ of $n \times n$ matrices over a skew field $\mathbb{D}$ is connected and has diameter 4. If $n = 2$, then the graph of the ring $M_n(\mathbb{D})$ is a disjoint union of connected components of diameters 1 and 2. As a corollary, the corresponding results on the orthogonality graphs of simple Artinian rings are obtained. Bibliography: 14 titles.

1. Introduction

Binary relations on associative rings and, in particular, on the matrix ring is an important topic in modern mathematics, which is studied and used in numerous applications. At present, one of efficient approaches to investigating such a relation is to study the so-called *relation graph*, whose vertices are the elements of a certain set, and two vertices are connected by an edge if and only if the corresponding elements are in this relation.

The study of algebraic structures based on their relation graphs has been the focus of attention for the last 20 years. For example, the commuting graph (that is, the graph of the commutativity relation) and the zero-divisor graph are intensively studied, see [1–5] and the references therein. These relations are closely related to the orthogonality relation examined in the present paper. Recall that elements r, s of a ring R are said to be *orthogonal* if $rs = sr = 0$. The orthogonality relation is used in [12–14], where some partial orders on the matrix algebra and matrix mappings monotone with respect to these orders are considered. The notion of orthogonal completeness [8, 9], based on the orthogonality relation, plays an important part in the theory of rings. The orthogonality relation also arises in linear algebra and functional analysis in studying projectors (projection operators).

The graph of the orthogonality relation was introduced by the authors in [7]. Also, in that paper, possible diameters of the orthogonality graphs of commutative Artinian rings were described, the orthogonality graph of the full matrix algebra over an arbitrary field was investigated, the connectedness of the orthogonality graphs of some classical matrix families was established, and their diameters were computed.

Recall some definitions from graph theory. For the notions of graph theory used in this paper, see, for example, [10, Chapter 2].

A *graph* Γ is a nonempty set of vertices $\mathrm{V}(\Gamma)$ and a set of edges $\mathrm{E}(\Gamma)$, that is, a set of unordered pairs of vertices.

If v_1, v_2 are two vertices and $e = (v_1, v_2)$ is the edge connecting them, then the vertex v_1 and the edge e are said to be *incident*; the vertex v_2 and the edge e also are *incident*.

A *loop* is an edge that connects a vertex with itself. Note that in this paper, a graph is understood as a graph without multiple edges, but it is allowed to have loops. A graph free of loops is referred to as a *simple graph*.

A *path* (*walk*) in a graph Γ is a sequence of vertices and edges of the form v_0, e_1, v_1, e_2, $v_2, \ldots, e_k$, v_k in which any two neighbor elements are incident.

*Lomonosov Moscow State University, Moscow, Russia, Moscow Institute of Physics and Technology, Dolgoprudny, Russia, e-mail: guterman@list.ru, ov_markova@mail.ru.

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 463, 2017, pp. 81–93. Original article submitted October 31, 2017.

The *length of a path*, denoted by d, is the number of edges in it, each being counted as many times as it occurs in the path.

A path is said to be *elementary* if all the edges are distinct.

A graph is said to be *connected* if arbitrary two vertices are connected by a path.

A *connected component* of a graph Γ is a maximal (with respect to inclusion) connected subgraph of the graph Γ.

The *distance* $d(u,v)$ between two distinct vertices u and v is the length of the shortest elementary path between them. If u and v are unreachable from each other, then $d(u,v)=\infty$; it is stipulated that $d(u,u)=0$ for any vertex u.

The *diameter* $\operatorname{diam}(\Gamma)$ of a graph Γ is the maximum of the distances between distinct vertices of the graph.

A complete graph is a simple graph in which every two distinct vertices are incident to an edge.

A graph Γ is called a *complete graph with loops* if every two vertices (including coinciding ones) are incident to an edge.

A graph $\Gamma=(V,E)$ is said to be *bipartite* if its vertex set can be subdivided into two nonempty disjoint subsets $V=V_1\cup V_2$ such that the vertices in V_1 are not connected with each other and the vertices in V_2 are not connected with each other. The sets of vertices V_1 and V_2 are called the *parts* of the bipartite graph Γ.

A *complete bipartite graph* is a bipartite graph in which every two vertices $u\in V_1$ and $v\in V_2$ are incident to an edge $(u,v)\in E$.

All rings considered in this paper are assumed to be associative. Recall that an element a of a ring R is called a *left* (*right*) *zero divisor* if there exists a nonzero element $b\in R$ such that $ab=0$ (respectively, $ba=0$). An element a that is both a left and a right zero divisor is called a *two-sided zero divisor*. A *ring without zero divisors* is a ring that contains no zero divisors other than 0, that is, $ab=0$ implies that either $a=0$ or $b=0$.

Recall the main definitions related to the graph of the orthogonality relation of a given ring R (for more details, see [7]).

Definition 1.1. *Two elements $r_1\in R$ and $r_2\in R$ are said to be* orthogonal *if $r_1r_2=r_2r_1=0$.*

For a subset X in R, by $O_R(X)$ we denote the set of elements from R orthogonal to all elements from X; also we denote $O_R^0(X)=O_R(X)\setminus\{0\}$.

Remark 1.2. The zero element $0\in R$ is orthogonal to all elements of the ring. Conversely, if an element $r\in R$ is not a zero divisor, then there is no nonzero element $x\in R$ such that $xr=rx=0$, whence no nonzero element is orthogonal to r in the ring R. For this reason, in studying the orthogonality graph, from the set of vertices we exclude 0 and the elements that are not (at least one-sided) zero divisors.

Definition 1.3 ([7, Definition 2.15])**.** *With every ring R one can associate the orthogonality graph $O(R)$ whose vertex set consists of all nonzero two-sided zero divisors of the ring R and in which two vertices are connected by an edge if and only if the corresponding elements of R are orthogonal.*

Lemma 1.4 ([7, Lemma 2.17])**.** *The vertex set of $O(R)$ is empty if and only if R is a ring without zero divisors.*

The paper is devoted to studying the orthogonality graph of the matrix ring over a skew field. The methods used in this research differ significantly, in some cases, from those used in studying matrix rings over fields in [7], although the ultimate results are similar. It is proved that for $n\geq 3$ and an arbitrary skew field $\mathbb{D}$, the orthogonality graph of the ring $M_n(\mathbb{D})$ of $n\times n$ matrices over $\mathbb{D}$ is connected and has diameter 4. For $n=2$, the graph of the ring

$M_n(\mathbb{D})$ is a disjoint union of connected components of diameters 1 and 2. As implications, the connectedness of the orthogonality graphs of simple Artinian rings is established and the diameters of their connected components are found.

2. The orthogonality graph of the matrix ring $O(M_n(\mathbb{D}))$

We show that the results on the orthogonality graph of the full matrix ring over a field [7, Lemma 4.1, Theorem 4.5] can be extended to the case of the matrix ring over a skew field.

Theorem 2.1. *Let $\mathbb{D}$ be an arbitrary skew field. Then, for $n \geq 3$, the orthogonality graph $O(M_n(\mathbb{D}))$ is connected, and* $\operatorname{diam} O(M_n(\mathbb{D})) = 4$.

Proof. I. In order to establish the connectedness of the graph $O(M_n(\mathbb{D}))$, we demonstrate that arbitrary vertices A and B in $O(M_n(\mathbb{D}))$ are connected by a path of length at most 4. By definition, A and B are two-sided zero divisors in $M_n(\mathbb{D})$, that is, there exist nonzero matrices $X, Y, U, V \in M_n(\mathbb{D})$ such that

$$XA = 0, \quad AY = 0, \quad UB = 0, \quad BV = 0.$$

Let $\mathbf{y}_c$, $\mathbf{v}_c$ be arbitrary nonzero columns of the matrices Y and V, respectively, and let $\mathbf{x}_r$, $\mathbf{u}_r$ be some nonzero rows of the matrices X and U. Then we set

$$R_1 = \mathbf{y}_c\mathbf{x}_r, \quad R_3 = \mathbf{v}_c\mathbf{u}_r.$$

The associativity of the matrix multiplication implies that

$$AR_1 = A(\mathbf{y}_c\mathbf{x}_r) = (A\mathbf{y}_c)\mathbf{x}_r = \mathbf{0}_c\mathbf{x}_r = 0,$$
$$R_1A = (\mathbf{y}_c\mathbf{x}_r)A = \mathbf{y}_c(\mathbf{x}_rA) = \mathbf{y}_c\mathbf{0}_r = 0,$$

and, similarly,

$$R_3B = BR_3 = 0.$$

Consider a system of two linear equations

$$\begin{cases} \mathbf{x}_r\mathbf{z}_c = 0, \\ \mathbf{u}_r\mathbf{z}_c = 0 \end{cases} \tag{1}$$

over $\mathbb{D}$ in n unknowns (with the column vector of unknowns $\mathbf{z}_c$). System (1) over a skew field can be solved by Gaussian elimination [6, Chapter I, Sec. 5] (by performing elementary operations with rows of the coefficient matrix). Since $n \geq 3$, that is, there are more unknowns than equations, system (1) has a nonzero solution $\widetilde{\mathbf{z}}_c$. Therefore,

$$R_1\widetilde{\mathbf{z}}_c = (\mathbf{y}_c\mathbf{x}_r)\widetilde{\mathbf{z}}_c = \mathbf{y}_c(\mathbf{x}_r\widetilde{\mathbf{z}}_c) = 0,$$
$$R_3\widetilde{\mathbf{z}}_c = (\mathbf{v}_c\mathbf{u}_r)\widetilde{\mathbf{z}}_c = \mathbf{v}_c(\mathbf{u}_r\widetilde{\mathbf{z}}_c) = 0.$$

Similarly, consider a system

$$\begin{cases} \mathbf{w}_r\mathbf{y}_c = 0, \\ \mathbf{w}_r\mathbf{v}_c = 0 \end{cases} \tag{2}$$

over $\mathbb{D}$ in n unknowns (with the row vector of unknowns $\mathbf{w}_r$). It can also be solved by Gaussian elimination (by performing elementary operations with columns of the coefficient matrix). Once again, since $n \geq 3$, that is, there are more unknowns than equations, system (2) has a nonzero solution $\widetilde{\mathbf{w}}_r$, whence

$$\widetilde{\mathbf{w}}_rR_1 = \widetilde{\mathbf{w}}_r(\mathbf{y}_c\mathbf{x}_r) = (\widetilde{\mathbf{w}}_r\mathbf{y}_c)\mathbf{x}_r = 0,$$
$$\widetilde{\mathbf{w}}_rR_3 = \widetilde{\mathbf{w}}_r(\mathbf{v}_c\mathbf{u}_r) = (\widetilde{\mathbf{w}}_r\mathbf{v}_c)\mathbf{u}_r = 0.$$

Then the matrix $R_2 = \widetilde{\mathbf{z}}_c \widetilde{\mathbf{w}}_r$ is nonzero and satisfies the relations

$$R_1 R_2 = R_2 R_1 = R_3 R_2 = R_2 R_3 = 0.$$

This yields the desired path

$$A - R_1 - R_2 - R_3 - B$$

of length 4.

II. In order to prove that the diameter of the graph $O(M_n(\mathbb{D}))$ is equal to 4, it remains to provide an elementary path of length 4. We show that as in the case of matrices over a field, the desired path is that between J and J^t, where $J = \sum_{i=1}^{n-1} E_{i,i+1}$ (the Jordan block) and J^t is the transpose of J.

1. Show that $d(J, J^t) > 3$. Let $0 \neq A \in O_{M_n(\mathbb{D})}(J)$. By definition, this means that $JA = AJ = 0$. The equality $JA = 0$ implies that the rows $2, \ldots, n$ of the matrix A are zero because the multiplication by J on the left moves the rows of the matrix A up by one position, whereas the equality $AJ = 0$ similarly implies that the columns $1, \ldots, n-1$ of the matrix A are zero. Thus, only the entry of the matrix A in position $(1, n)$ can be nonzero. Consequently, $O_{M_n(\mathbb{D})}(J) = \{\alpha E_{1n} \mid \alpha \in \mathbb{D}\}$. Similarly, we obtain that $O_{M_n(\mathbb{D})}(J^t) = \{\alpha E_{n1} \mid \alpha \in \mathbb{D}\}$. Since $E_{1n} E_{n1} \neq 0$, it follows that $d(J, J^t) \geq 4$.

2. Present a path of length 4. As a direct computation shows, $J - E_{1n} - E_{22} - E_{n1} - J^t$ is the desired path in $O(M_n(\mathbb{D}))$.

Thus, $d(J, J^t) = 4$. □

In what follows, we consider the remaining small values of n separately.

Lemma 2.2. *Let $\mathbb{D}$ be a noncommutative skew field. For $n = 1$, the orthogonality graph $O(M_n(\mathbb{D}))$ is empty. For $n = 2$, the graph $O(M_n(\mathbb{D}))$ is disconnected, and it is a union of the connected components with the following sets of vertices:*

(1) *the set $V_1 = V_{1a} \cup V_{1b}$, where*

$$V_{1a} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{D}^* \right\}, \quad V_{1b} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \,\middle|\, b \in \mathbb{D}^* \right\};$$

(2) *the set*

$$V_2 = \left\{ \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{D}^* \right\};$$

(3) *the set*

$$V_3 = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{D}^* \right\};$$

(4) *the set $V_{4,\alpha} = V_{4,\alpha,a} \cup V_{4,\alpha,b}$, where $\alpha \in \mathbb{D}^*$ is arbitrary,*

$$V_{4,\alpha,a} = \left\{ \begin{pmatrix} c & c\alpha \\ 0 & 0 \end{pmatrix} \,\middle|\, c \in \mathbb{D}^* \right\}, \quad V_{4,\alpha,b} = \left\{ \begin{pmatrix} 0 & -\alpha d \\ 0 & d \end{pmatrix} \,\middle|\, d \in \mathbb{D}^* \right\};$$

(5) *the set $V_{5,\alpha} = V_{5,\alpha,a} \cup V_{5,\alpha,b}$, where $\alpha \in \mathbb{D}^*$ is arbitrary,*

$$V_{5,\alpha,a} = \left\{ \begin{pmatrix} 0 & 0 \\ c & c\alpha \end{pmatrix} \,\middle|\, c \in \mathbb{D}^* \right\}, \quad V_{5,\alpha,b} = \left\{ \begin{pmatrix} -\alpha d & 0 \\ d & 0 \end{pmatrix} \,\middle|\, d \in \mathbb{D}^* \right\};$$

(6) *the set $V_{6,\alpha,\beta} = V_{6,\alpha,\beta,a} \cup V_{6,\alpha,\beta,b}$, where $\alpha, \beta \in \mathbb{D}^*$ and $\{\alpha, \beta\}$ is an arbitrary pair,*

$$V_{6,\alpha,\beta,a} = \left\{ \begin{pmatrix} -a\alpha & a \\ -\beta a\alpha & \beta a \end{pmatrix} \,\middle|\, a \in \mathbb{D}^* \right\}, \quad V_{6,\alpha,\beta,b} = \left\{ \begin{pmatrix} -b\beta & b \\ -\alpha b\beta & \alpha b \end{pmatrix} \,\middle|\, b \in \mathbb{D}^* \right\}.$$

If $\alpha = \beta$, then $V_{6,\alpha,\alpha} = V_{6,\alpha,\alpha,a} = V_{6,\alpha,\alpha,b}$.

Every connected component Y corresponding to the vertex sets V_1, $V_{4,\alpha}$, $V_{5,\alpha}$, and $V_{6,\alpha,\beta}$ with $\beta \neq \alpha$ is a complete bipartite graph with the parts Y_a and Y_b defined by the partition $Y = Y_a \cup Y_b$ indicated above and with diameter 2.

Every connected component corresponding to the vertex sets V_2, V_3, and $V_{6,\alpha,\alpha}$ is a complete graph with loops and has diameter 1.

Proof. For $n = 1$, the first assertion is obvious because the skew field $\mathbb{D}$ is a ring without zero divisors.

Let $n = 2$.

(1) First we note that every two sets from those listed in the lemma are disjoint. Indeed, the sets V_1, V_2, V_3 consist of matrices with exactly three zero entries; the sets $V_{4,\alpha}, V_{5,\alpha}$ consist of matrices with exactly two zero entries, and the sets $V_{6,\alpha,\beta}$ consist of matrices with no zero entries. Taking into consideration the numbers of zeros in the matrices and their location, we conclude that

 (a) $V_1, V_2, V_3, V_{4,\alpha_1}, V_{5,\alpha_2}$, and $V_{6,\alpha,\beta}$ are pairwise disjoint for all fixed $\alpha_1, \alpha_2, \alpha, \beta \in \mathbb{D}^*$.
 (b) V_{4,α_1} and V_{4,α_2} are pairwise disjoint for all $\alpha_1 \neq \alpha_2$.
 (c) V_{5,α_1} and V_{5,α_2} are pairwise disjoint for all $\alpha_1 \neq \alpha_2$.
 (d) V_{6,α_1,β_1} and V_{6,α_2,β_2} are pairwise disjoint for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{D}^*$, except for the case $\{\alpha_1, \beta_1\} = \{\alpha_2, \beta_2\}$.
 Indeed, assume that $V_{6,\alpha_1,\beta_1,a}$ and $V_{6,\alpha_2,\beta_2,a}$ have a nonempty intersection, that is, for some $a_1, a_2 \in \mathbb{D}^*$,

$$\begin{pmatrix} -a_1\alpha_1 & a_1 \\ -\beta_1 a_1 \alpha_1 & \beta_1 a_1 \end{pmatrix} = \begin{pmatrix} -a_2\alpha_2 & a_2 \\ -\beta_2 a_2 \alpha_2 & \beta_2 a_2 \end{pmatrix}.$$

 Comparing the entries in position $(1,2)$, we conclude that $a_1 = a_2$, which implies that $\beta_1 = \beta_2$ and $\alpha_1 = \alpha_2$. The case where $V_{6,\alpha_1,\beta_1,b}$ and $V_{6,\alpha_2,\beta_2,b}$ have a nonempty intersection is considered similarly.
 Without loss of generality, assume that $V_{6,\alpha_1,\beta_1,a}$ and $V_{6,\alpha_2,\beta_2,b}$ have a nonempty intersection, that is,

$$\begin{pmatrix} -a\alpha_1 & a \\ -\beta_1 a \alpha_1 & \beta_1 a \end{pmatrix} = \begin{pmatrix} -b\beta_2 & b \\ -\alpha_2 b \beta_2 & \alpha_2 b \end{pmatrix},$$

 where $a, b \in \mathbb{D}^*$. Comparing the entries in position $(1,2)$, we conclude that $a = b$, whence $\alpha_1 = \beta_2$ (position $(1,1)$) and $\beta_1 = \alpha_2$ (position $(2,2)$).

(2) The notion of rank is defined for matrices over skew fields; by virtue of the fundamental theorem on the matrix rank, the rank equals, in particular, the left row rank and the right column rank (see, for example, [6, Chapter I, Sec. 5]). If a nonzero matrix $A \in M_2(\mathbb{D})$ is a two-sided zero divisor, then it cannot be of full rank, whence $\operatorname{rank} A = 1$.

 It is clear that the sets indicated in the lemma contain all possible matrices of rank 1 and only them. Prove that for any of the components W indicated above and an arbitrary matrix $A \neq 0$ from W, we have $O^0_{M_2(\mathbb{D})}(A) \subset W$. This will imply that in the graph $O(M_2(\mathbb{D}))$, the distinct components are connected by no edges.

 We subdivide the matrices A of rank 1 into groups depending on the number of zero entries in them and find $O^0_{M_2(\mathbb{D})}(A)$. Observe that by the symmetry of the orthogonality relation, $B \in O^0_{M_2(\mathbb{D})}(A)$ implies $A \in O^0_{M_2(\mathbb{D})}(B)$. Therefore, $\operatorname{rank} B = 1$. Then, by using the row rank, we obtain $\mathbf{r}_i = c_i \mathbf{r}$, where $\mathbf{r}_i$ denotes the ith row of the matrix B, $i = 1, 2$; $\mathbf{r} \neq 0$ is a row of length 2 over $\mathbb{D}$, and $c_1, c_2 \in \mathbb{D}$ are not simultaneously equal to zero. Therefore, $B = \mathbf{c}\mathbf{r}$, $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. Then

$$AB = 0 = A(\mathbf{c}\mathbf{r}) = (A\mathbf{c})\mathbf{r},$$

and the equality to zero is possible for a nonzero row $\mathbf{r}$ if and only if $A\mathbf{c} = 0$. Similarly, from the condition $BA = 0$ we infer that $\mathbf{r}A = 0$.

(a) *The set* V_1. First assume that $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $a \in \mathbb{D}^*$. Then $\mathbf{r} = (0, u)$, $u \in \mathbb{D}^*$; $\mathbf{c} = \begin{pmatrix} 0 \\ y \end{pmatrix}$, $y \in \mathbb{D}^*$. It follows that $B = \begin{pmatrix} 0 & 0 \\ 0 & yu \end{pmatrix}$, where $y, u \in \mathbb{D}^*$; and the product yu can take an arbitrary value from $\mathbb{D}^*$, whence it can be replaced by a single parameter $b \in \mathbb{D}^*$. Similarly, for $A = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$, $a \in \mathbb{D}^*$, we have $\mathbf{r} = (t, 0)$, $t \in \mathbb{D}^*$; $\mathbf{c} = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $x \in \mathbb{D}^*$. Consequently, $B = \begin{pmatrix} xt & 0 \\ 0 & 0 \end{pmatrix}$, $x, t \in \mathbb{D}^*$, and the product xt can take an arbitrary value from $\mathbb{D}^*$, whence it can be replaced by a single parameter $a \in \mathbb{D}^*$. Thus, we have proved that for any matrix $A \in V_1$, $O^0_{M_2(\mathbb{D})}(A) \subset V_1$.

(b) *The set* V_2. Let $A = aE_{21}$, $a \in \mathbb{D}^*$. Then $\mathbf{r} = (t, 0)$, $t \in \mathbb{D}^*$; $\mathbf{c} = \begin{pmatrix} 0 \\ y \end{pmatrix}$, $y \in \mathbb{D}^*$. Consequently, $B = \begin{pmatrix} 0 & 0 \\ yt & 0 \end{pmatrix}$, $y, t \in \mathbb{D}^*$, and the product yt can take an arbitrary value from $\mathbb{D}^*$, whence it can be replaced by a single parameter $b \in \mathbb{D}^*$. Thus, we have proved that $O^0_{M_2(\mathbb{D})}(A) \subset V_2$ for an arbitrary matrix $A \in V_2$.

(c) *The set* V_3. The case $A = aE_{12}$ is analogous to item 2(b) and corresponds to the component V_3.

(d) *The set* $V_{4,\alpha}$. Let $A = a\begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}$, $a, \alpha \in \mathbb{D}^*$. Then $\mathbf{r} = (0, u)$, $u \in \mathbb{D}^*$; $\mathbf{c} = \begin{pmatrix} -\alpha y \\ y \end{pmatrix}$, $y \in \mathbb{D}^*$. It follows that $B = \begin{pmatrix} 0 & -\alpha yu \\ 0 & yu \end{pmatrix}$, $y, u \in \mathbb{D}^*$, and the product yu can take an arbitrary value from $\mathbb{D}^*$, whence it can be replaced by a single parameter $b \in \mathbb{D}^*$.
Similarly, for $A = \begin{pmatrix} 0 & -\alpha \\ 0 & 1 \end{pmatrix} b$, where $b \in \mathbb{D}^*$, we have $\mathbf{r} = (t, t\alpha)$, $t \in \mathbb{D}^*$; $\mathbf{c} = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $x \in \mathbb{D}^*$. Therefore, $B = \begin{pmatrix} xt & xt\alpha \\ 0 & 0 \end{pmatrix}$, $x, t \in \mathbb{D}^*$, and the product xt can take an arbitrary value from $\mathbb{D}^*$, whence it can be replaced by a single parameter $a \in \mathbb{D}^*$. Consequently, we have proved that for any matrix $A \in V_{4,\alpha}$, $O_{M_2(\mathbb{D})}(A) \subset V_{4,\alpha}$.

(e) *The set* $V_{5,\alpha}$. Let $A = a\begin{pmatrix} 0 & 0 \\ 1 & \alpha \end{pmatrix}$, $0 \neq a, \alpha \in \mathbb{D}^*$. Then $\mathbf{r} = (t, 0)$, $t \in \mathbb{D}^*$; $\mathbf{c} = \begin{pmatrix} -\alpha y \\ y \end{pmatrix}$, $y \in \mathbb{D}^*$. It follows that $B = \begin{pmatrix} -\alpha yt & 0 \\ yt & 0 \end{pmatrix}$, $y, t \in \mathbb{D}^*$, and the product yt can take an arbitrary value from $\mathbb{D}^*$, whence it can be replaced by a single parameter $b \in \mathbb{D}^*$.
Similarly, for $A = \begin{pmatrix} -\alpha & 0 \\ 1 & 0 \end{pmatrix} b$, where $b \in \mathbb{D}^*$, we have $\mathbf{r} = (t, t\alpha)$, $t \in \mathbb{D}^*$; $\mathbf{c} = \begin{pmatrix} 0 \\ y \end{pmatrix}$, $y \in \mathbb{D}^*$, implying that $B = \begin{pmatrix} 0 & 0 \\ yt & yt\alpha \end{pmatrix}$, $y, t \in \mathbb{D}^*$, and the product yt can take an arbitrary value from $\mathbb{D}^*$, whence it can be replaced by a single parameter $a \in \mathbb{D}^*$. Thus, we have proved that for any matrix $A \in V_{5,\alpha}$, $O_{M_2(\mathbb{D})}(A) \subset V_{5,\alpha}$.

(f) *The set* $V_{6,\alpha,\beta}$. Let $A=\begin{pmatrix}-a\alpha & a\\ -\beta a\alpha & \beta a\end{pmatrix}a$, where $\alpha,\beta\in\mathbb{D}^*$. Then $\mathbf{r}=(-u\beta,u)$, $u\in\mathbb{D}^*$; $\mathbf{c}=\begin{pmatrix}x\\ \alpha x\end{pmatrix}$, $x\in\mathbb{D}^*$. It follows that $B=\begin{pmatrix}-xu\beta & xu\\ -\alpha xu\beta & \alpha xu\end{pmatrix}$, $x,u\in\mathbb{D}^*$, and the product xu can take an arbitrary value from $\mathbb{D}^*$, whence it can be replaced by a single parameter $b\in\mathbb{D}^*$.
Similarly, for $A=\begin{pmatrix}-b\beta & b\\ -\alpha b\beta & \alpha b\end{pmatrix}$, $b\in\mathbb{D}^*$, we obtain $\mathbf{r}=(-u\alpha,u)$, $u\in\mathbb{D}^*$; $\mathbf{c}=\begin{pmatrix}x\\ \beta x\end{pmatrix}$, $x\in\mathbb{D}^*$. Therefore, $B=\begin{pmatrix}-xu\alpha & xu\\ -\beta xu\alpha & \beta xu\end{pmatrix}$, $x,u\in\mathbb{D}^*$, and the product xu can take an arbitrary value from $\mathbb{D}^*$, whence it can be replaced by a single parameter $a\in\mathbb{D}^*$. Consequently, we have proved that $O^0_{M_2(\mathbb{D})}(A)\subset V_{6,\alpha,\beta}$ for all matrices $A\in V_{6,\alpha,\beta}$.

(3) Prove the connectedness of the components V_i and find their diameters.

(a) Considering the numbers of zeros in matrices and their location, we conclude that $V_{1a}\cap V_{1b}=\varnothing$ and $V_{i,\alpha,a}\cap V_{i,\alpha,b}=\varnothing$, $i=4,5$. The equality

$$\begin{pmatrix}-a\alpha & a\\ -\beta a\alpha & \beta a\end{pmatrix}=\begin{pmatrix}-b\beta & b\\ -\alpha b\beta & \alpha b\end{pmatrix}$$

holds if and only if $\beta=\alpha$ and $b=a$; therefore, $V_{6,\alpha,\beta,a}\cap V_{6,\alpha,\beta,b}=\varnothing$ for all $\beta\in\mathbb{D}^*\setminus\{\alpha\}$ but $V_{6,\alpha,\alpha,a}=V_{6,\alpha,\alpha,b}$.
From the proof of item 2 it follows that each of the components

$$Y\in\{V_1,V_{4,\alpha},V_{5,\alpha},V_{6,\alpha,\beta}|\ \alpha,\beta\in\mathbb{D}^*\}$$

partitioned as $Y=Y_a\cup Y_b$ and arbitrary vertices $C\in Y_a$, $D\in Y_b$ satisfy the relations $O^0_{M_2(\mathbb{D})}(C)=Y_b$ and $O^0_{M_2(\mathbb{D})}(D)=Y_a$. Thus, every component

$$Y\in\{V_1,V_{4,\alpha},V_{5,\alpha},V_{6,\alpha,\beta}|\ \alpha\in\mathbb{D}^*,\ \beta\in\mathbb{D}^*\setminus\{\alpha\}\}$$

is a complete bipartite graph with the parts Y_a and Y_b. Note that the diameter of a complete bipartite graph with at least three vertices is equal to 2. By construction, we have $|Y_a|=|Y_b|=|\mathbb{D}^*|$; consequently, the sets Y_a and Y_b are infinite by Wedderburn's theorem (see, for example, [11, Theorem 3.1.1]).
It remains to consider the components $Y\in\{V_{6,\alpha,\alpha},\alpha\in\mathbb{D}^*\}$. As we have already proved, an arbitrary vertex $C\in Y_a$ satisfies the relation $O^0_{M_2(\mathbb{D})}(C)=Y_b$. On the other hand, $Y_a=Y_b=Y$; therefore, every vertex of the graph $O(Y)$ is incident to a loop, and, on excluding loops, the graph Y is complete, whence it has diameter 1.

(b) The component V_2 is connected and has diameter 1 because the relations $A_1A_2=A_2A_1=0$ hold for the matrices $A_1=a_1E_{2,1}$, $A_2=a_2E_{2,1}\in V_2$ with arbitrary $a_1,a_2\in\mathbb{D}$, $a_1\neq a_2$. Also $A^2=0$ for any matrix $A\in V_2$. Therefore, the graph V_2 is a complete graph with loops. The argument for the component V_3 is similar. □

By the Molin–Wedderburn–Artin theorem (see, for example, [11, Theorem 2.1.6]), for simple Artinian rings the main result can be stated as follows.

Corollary 2.3. *Let R be a simple Artinian ring, n be the cardinality of the maximal set of pairwise orthogonal nonzero idempotents in R (one can also define n as $\dim_{\mathbb{D}}V$, where $\mathbb{D}=\operatorname{End}_R V$, V is a simple left R-module).*

Then

(1) *for $n=1$, the ring R is a ring without zero divisors, and the graph $O(R)$ is empty;*

(2) *for $n=2$, the graph $O(R)$ is disconnected, and the connected components of $O(R)$ have diameters 1 and 2 if $|\mathbb{D}|\geq 2$ or 0 and 1 if $\mathbb{D}=\mathbb{Z}_2$;*

(3) *for* $n \geq 3$, *the graph* $O(R)$ *is connected and has diameter* 4.

This work was supported by the Russian Science Foundation (grant No. 17-11-01124).

Translated by the authors.

REFERENCES

1. S. Akbari, M. Ghandehari, M. Hadian, and A. Mohammadian, "On commuting graphs of semisimple rings," *Linear Algebra Appl.*, **390**, 345–355 (2004).
2. S. Akbari and A. Mohammadian, "On the zero-divisor graph of a commutative ring," *J. Algebra*, **274**, 847–855 (2004).
3. S. Akbari and A. Mohammadian, "Zero-divisor graphs of non-commutative rings." *J. Algebra*, **296**, 462–479 (2006).
4. S. Akbari, A. Mohammadian, H. Radjavi, and P. Raja, "On the diameters of commuting graphs," *Linear Algebra Appl.*, **418**, 161–176 (2006).
5. D. F. Anderson and P. S. Livingston, "The zero-divisor graph of a commutative ring," *J. Algebra*, **217**, 434–447 (1999).
6. E. Artin, *Geometric Algebra* [Russian translation], Nauka, Moscow (1969).
7. B. R. Bakhadly, A. E. Guterman, and O. V. Markova, "Graphs defined by orthogonality," *Zap. Nauchn. Semin. POMI*, **428**, 49–80 (2014).
8. K. I. Beidar and A. V. Mikhalev, "Orthogonal completeness and algebraic systems," *Usp. Mat. Nauk*, **40**, No. 6, 79–115 (1985).
9. K. I. Beidar and A. V. Mikhalev, "The method of orthogonal completeness in the structure theory of rings," *Itogi Nauki Tekhniki*, **4**, 1–44 (1993).
10. F. Harary, *Graph Theory* [Russian translation], Mir, Moscow (1973).
11. I. N. Herstein, *Noncommutative Rings* [Russian translation], Mir, Moscow (1972).
12. A. E. Guterman and M. A. Efimov, "Monotone maps on matrices of index one," *Zap. Nauchn. Semin. POMI*, **405**, 67–96 (2012).
13. P. G. Ovchinnikov, "Automorphisms of the poset of skew projections," *J. Func. Anal.*, **115**, 184–189 (1993).
14. P. Šemrl, "Order-preserving maps on the poset of idempotent matrices," *Acta Sci. Math.* (*Szeged*), **69**, 481–490 (2003).