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REGULARITY OF A BOUNDARY POINT FOR THE $p(x)$-LAPLACIAN

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We study the behavior of solutions to the Dirichlet problem for the $p(x)$-Laplacian with a continuous boundary function. We prove the existence of a weak solution under the assumption that p is separated from 1 and ∞. We present a necessary and sufficient Wiener type condition for regularity of a boundary point provided that the exponent p has the logarithmic modulus of continuity at this point. Bibliography: 24 titles.

Dedicated to the memory of Vasilii Vasil'evich Zhikov

1 Introduction

Let D be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$. This paper is devoted to the behavior of the solutions to the Dirichlet problem in D for the equation

$$Lu = \operatorname{div}\,(|\nabla u|^{p(x)-2}\nabla u) = 0 \tag{1.1}$$

at a boundary point, where the exponent p is measurable and such that

$$1 < \alpha \leqslant p(x) \leqslant \beta \quad \text{for almost all} \quad x \in D. \tag{1.2}$$

Equations of the form (1.1) first arose in works of Zhikov [1, 2] in connection with homogenization of integrands of the form $|\nabla u|^{p(x)}$. Such equation also arise in mathematical modeling of fluids whose properties change under the action of electromagnetic field or temperature [3]–[5]. To define a solution to Equation (1.1), we introduce the class of functions

$$W(D) = \big\{u \in W^{1,1}(D) : \ |\nabla u|^{p(x)} \in L^1(D)\big\},$$

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where $W^{1,1}(D)$ is the Sobolev space of integrable functions in D, together with the first order generalized derivatives. We say that a sequence $u_j \in W(D)$ *converges in* $W(D)$ to a function $u \in W(D)$ if $u_j \to u$ in $L^1(D)$ and

$$\lim_{j\to\infty} \int_D |\nabla u - \nabla u_j|^{p(x)} dx = 0. \tag{1.3}$$

By the Scheffe or Riesz theorem, the last convergence is equivalent to the convergence of ∇u_j to ∇u almost everywhere in D, together with the energy convergence

$$\int_D |\nabla u_j|^{p(x)} dx \to \int_D |\nabla u|^{p(x)} dx.$$

We say that $u \in W(D)$ *belongs to the class* $W_0(D)$ if there exists a sequence of functions $u_j \in W(D)$ compactly supported in D such that (1.3) holds. We say that a sequence $u_j \in W_0(D)$ *converges in* $W_0(D)$ to a function $u \in W_0(D)$ if (1.3) holds.

We are interested in the function classes $H(D)$ and $H_0(D)$ that are the completions in $W(D)$ and $W_0(D)$ of smooth functions in D relative to the above convergences. Namely, we set

$$H(D) = \{u \in W(D) : \ \exists u_j \in C^\infty(D) \cap W(D), u_j \to u \text{ in } W(D)\},$$

$$H_0(D) = \{u \in W(D) : \ \exists u_j \in C_0^\infty(D), u_j \to u \text{ in } W_0(D)\}.$$

From the results of [2] it follows that only the assumption (1.2) is not sufficient for smooth functions to be dense in $W(D)$ and $W_0(D)$. The density of smooth functions in these classes is guaranteed by the known logarithmic condition

$$|p(x) - p(y)| \leqslant k_0 \left(\ln \frac{1}{|x-y|}\right)^{-1}, \quad x, y \in D, \ |x-y| < 1/2, \tag{1.4}$$

which was obtained by Zhikov [6].

If smooth functions are not dense in the set of solutions to Equation (1.1), then the related boundary value problems can be understood in different senses [6].

The most important types of solutions are the so-called H-solutions and W-solutions. We say that a function $u \in H(D)$ ($u \in W(D)$) is an *H-solution* (*W-solution*) to Equation (1.1) if the integral identity

$$\int_D |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \psi dx = 0 \tag{1.5}$$

holds for test functions $\psi \in H_0(D)$ ($\psi \in W_0(D)$).

We consider the Dirichlet problem

$$Lu = 0 \text{ in } D, \quad u \in H(D), \quad f \in H(D), \quad (u - f) \in H_0(D). \tag{1.6}$$

The solution to this problem is connected by the relation $u = w + f$ with the minimizer w of the variational problem

$$\inf_{w \in C_0^\infty(D)} F(w+f) = \min_{w \in H_0(D)} F(w+f), \quad F(v) = \int_D \frac{|\nabla v|^{p(x)}}{p(x)} dx. \tag{1.7}$$

In addition to the problem (1.6), we consider the problem

$$Lu = 0 \text{ in } D, \quad u \in W(D), \quad f \in W(D), \quad (u - f) \in W_0(D). \tag{1.8}$$

In turn, a solution to the problem (1.8) is connected by the relation $u = w + f$ with the minimizer of the variational problem

$$\min_{w \in W_0(D)} F(w + f). \tag{1.9}$$

The unique solvability of the Dirichlet problem (1.6), (1.8) is well known (cf. for example, [7]). In Section 2, we present a simpler proof of this fact. If the boundary function f is only continuous, then it is possible to construct a weak solution to the Dirichlet problem

$$Lu = 0 \text{ in } D, \quad u|_{\partial D} = f \in C(\partial D). \tag{1.10}$$

For this purpose we extend the function f by continuity to the whole space. Let $f_k \in C^\infty(\overline{D})$ be a sequence of functions that uniformly converges to f in $\overline{D}$. We construct solutions u_k to the Dirichlet problem (1.6) (or (1.8)) with the boundary functions f_k. In Section 4 it is shown that the sequence u_k converges to a function u_f, bounded by the maximum of the absolute value of the boundary function, and u_f is an H-solution (a W-solution) to Equation (1.1) in any subdomain $D' \Subset D$. This limit function, called a *weak solution* to the Dirichlet problem (1.10), is uniquely determined independently of the way of extending f and the choice of its smooth approximation Moreover, it is unclear in what sense the constructed weak solution to the Dirichlet problem (1.10) takes the value f on the boundary.

Definition 1.1. A boundary point $x_0 \in \partial D$ is called *regular* if $\operatorname*{ess\,lim}_{D \ni x \to x_0} u_f(x) = f(x_0)$ for any continuous function f on ∂D.

In 1913, Lebesgue [8] published an example of a boundary point for the Laplace equation that is not regular. The notion of the regularity of a boundary point is also due to Lebesgue. The criterion for regularity of a boundary point for the Laplace equation was obtained by Wiener, [9, 10]. For linear divergence form uniformly elliptic equations of the second order a similar result was obtained in [11]. A sufficient Wiener type condition for regularity of a boundary point and an estimate for the modulus of continuity of solutions near the boundary for the p-Laplacian were proved by Maz'ya [12]. These estimates were extended in [13] to a large class of quasilinear elliptic equations of p-Laplacian type. The necessity of the Maz'ya condition for regularity of a boundary point for p-Laplacian type equations was established in [14] in the case $n - 1 < p \leqslant n$ and in [15] in the general case. The criterion for regularity of a boundary point for Equation (1.1) with the condition (1.4) was obtained in [7].

The goal of this paper is to study the behavior at a boundary point of a solution u_f to the generalized Dirichlet problem with continuous boundary function f. This solution can be a W-solution, as well as an H-solution. In what follows, we simply call it a solution to the Dirichlet problem.

Assume that, in addition to (1.2), the following condition holds:

$$|p(x) - p(x_0)| \leqslant k_0 \left(\ln \frac{1}{|x - x_0|} \right)^{-1}, \quad x \in D, \ |x - x_0| < 1/2. \tag{1.11}$$

We note that for a point x_0 inside D the condition (1.11) guarantees the Hölder continuity of the solution at this point [16].

To formulate the results, we introduce the notion of a capacity. First, we extend the exponent p to the space $\mathbb{R}^n$ with preserving the properties (1.2), (1.11) and denote by $B_R^{x_0}$ the open ball with radius R and center x_0. The *H-capacity* of a compact set $K \subset B_R$ with respect to B_R is the number

$$C_p(K, B_R^{x_0}) = \inf \int\limits_{B_R} \frac{|\nabla \varphi|^{p(x)}}{p(x)} dx, \tag{1.12}$$

where the greatest lower bound is taken over the set of functions $\varphi \in H_0(B_R^{x_0})$ that are larger than or equal to 1 almost everywhere on K. The *W-capacity* of a compact set is defined by the equality (1.12), where the greatest lower bound is taken over the set of functions $\varphi \in W_0(B_R^{x_0})$ that are larger than or equal to 1 almost everywhere on K. By a capacity we will mean the H-capacity in the case of H-solutions or the W-capacity in the case of W-solutions.

From Lemmas 3.1 and 3.2 below it follows that in the definition of H-solutions one can take functions $\varphi \in C_0^\infty(B_R^{x_0})$ that are equal to 1 in a neighborhood of K.

For $x_0 \in \partial D$ we set

$$p_0 = p(x_0), \gamma(t) = (C_p(\overline{B}_t^{x_0} \setminus D, B_{2t}^{x_0}) t^{p_0 - n})^{1/(p_0-1)}, \tag{1.13}$$

where $B_t^{x_0}$ denotes the ball with radius t and center x_0.

The main result of this paper is formulated in the following theorem.

Theorem 1.1. *If the conditions* (1.2), (1.11) *hold and*

$$\int\limits_0 \gamma(t) t^{-1} dt = \infty, \tag{1.14}$$

then the boundary point $x_0 \in \partial D$ is regular.

As is shown in [7], if the exponent p satisfies the logarithmic condition (1.4), then (1.14) is a necessary condition for regularity of a point x_0. In Section 7, we describe the slightly modified proof in [7] to obtain the necessity of the condition (1.14) for regularity of a boundary point in our case.

Theorem 1.2. *If the conditions* (1.2) *and* (1.11) *hold, then the condition* (1.14) *is necessary for regularity of a boundary point $x_0 \in \partial D$.*

In the case $p_0 > n$, the condition (1.14) always holds, which follows from the capacity estimate

$$C_p(\{x_0\}, B_t^{x_0}) \geqslant C(n,p) t^{n-p_0}.$$

Any point $x_0 \in \partial D$, where $p(x_0) > n$ and the condition (1.11) holds, is regular. Using the Sobolev embedding theorem, for sufficiently small ρ and $r \leqslant \rho/4$ we can prove the estimate

$$\operatorname*{ess\,sup}_{D \cap B_r^{x_0}} |u_f(x) - f(x_0)| \leqslant \operatorname*{osc}_{\partial D \cap B_\rho^{x_0}} f + C(n,p) \operatorname*{osc}_{\partial D} f (r/\rho)^{1-n/p_0}$$

which implies the continuity u_f at the point x_0. The proof is the same as in the case of constant exponent (cf. details in [7]).

In what follows, we assume that $p_0 \leqslant n$. In this case, the key role in the proof of Theorem 1.1 is played by the estimate for oscillation of the solution u to the problem (1.6) with smooth in $\overline{D}$ boundary function f in balls of sufficiently small radius. In the following assertion, u is a solution to the Dirichlet problem (1.6) with a smooth function f.

Lemma 1.1. *In the ball $B^{x_0}_{4r}$, $r < 1/16$, the following inequality holds*:

$$\operatorname*{ess\,osc}_{D\cap B^{x_0}_r} u \leqslant (1-\delta\gamma(r))\operatorname*{ess\,osc}_{D\cap B^{x_0}_{4r}} u + \delta\gamma(r)\operatorname*{osc}_{\partial D\cap B^{x_0}_{4r}} f + r, \tag{1.15}$$

where δ depends only on n, $\max\limits_{\partial D}|f|$, and the constant in the conditions (1.2) *and* (1.11).

We prove this lemma below. The following assertion is obtained in a standard way.

Theorem 1.3. *Let u be a solution to the generalized Dirichlet problem* (1.10). *Then there exist positive constants θ and C that depend only on n, p, $\max\limits_{\partial D}|f|$ and for $\rho \leqslant \rho_0(n,p)$, $r \leqslant \rho/4$*

$$\operatorname*{ess\,sup}_{D\cap B^{x_0}_r}|u - f(x_0)| \leqslant C\left(\operatorname*{osc}_{\partial D\cap B^{x_0}_\rho} f + \rho + \exp\left(-\theta\int\limits_r^\rho \gamma(t)t^{-1}dt\right)\right). \tag{1.16}$$

Here, the dependence on p is determined by the constant in the conditions (1.2) *and* (1.11).

We first note that for smooth boundary functions, iterating (1.15), we obtain the estimate

$$\operatorname*{ess\,osc}_{D\cap B^{x_0}_r} u \leqslant C\left(\operatorname*{osc}_{\partial D\cap B^{x_0}_\rho} f + \rho + \exp\left(-\theta\int\limits_r^\rho \gamma(t)t^{-1}dt\right)\right).$$

By this estimate, the limit of $u(x)$ as $x \to x_0$ exists if the condition (1.14) holds. Then it is proved that this limit coincides with $f(x_0)$. Indeed, assume the contrary. Then there are positive numbers ρ and ε such that $|u(x) - f(x_0)| > \varepsilon$ for all $x \in D\cap B^{x_0}_\rho$. We consider the function $w = (u-f)/\varepsilon$. It is obvious that it takes the zero value on ∂D in the sense of the corresponding spaces, i.e., belongs to $W_0(D)$ or $H_0(D)$ depending on what solutions we consider. We extend w by zero outside D and consider the function $v = \min(w, 1)$. It is obvious that $v = 1$ in $D\cap B^{x_0}_\rho$ and $v = 0$ on ∂D and in $B^{x_0}_\rho \setminus D$. Since $\nabla v = 0$ in $B^{x_0}_\rho \setminus D$ and almost everywhere in $D\cap B^{x_0}_\rho$, we have $\nabla v = 0$ almost everywhere in $B^{x_0}_\rho$. Since $1 - v = 0$ in $D\cap B^{x_0}_\rho$, $|D\cap B^{x_0}_\rho| > 0$ and $\nabla(1-v) = 0$ almost everywhere in $B^{x_0}_\rho$, according to the De Giorgi–Poincaré estimate, we have

$$|B^{x_0}_\rho \setminus D| = \int\limits_{B^{x_0}_\rho}(1-v)dx \leqslant C(n)\rho^{n+1}|D\cap B^{x_0}_\rho|^{-1}\int\limits_{B^{x_0}_\rho}|\nabla(1-v)|dx = 0.$$

Let $\varphi \in C^\infty_0(B^{x_0}_\rho)$ be such that $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ in a neighborhood of $B^{x_0}_\rho/2$. The function $(1-v)\varphi$ is admissible in the definition of a capacity. Hence

$$\begin{aligned}C_p(\overline{B}^{x_0}_{\rho/2}\setminus D, B^{x_0}_\rho) &\leqslant \int\limits_{B^{x_0}_\rho}\frac{|\nabla(1-v)\varphi|^{p(x)}}{p(x)}dx\\ &= \int\limits_{B^{x_0}_\rho}\frac{|(1-v)\nabla\varphi|^{p(x)}}{p(x)}dx = \int\limits_{B^{x_0}_\rho\setminus D}\frac{|\nabla\varphi|^{p(x)}}{p(x)}dx = 0\end{aligned}$$

because the measure of the integration set is equal to zero. This contradict the divergence of the integral (1.14) in the formulation of Theorem 1.1. This implies (1.16) in the case of smooth boundary functions. Passing to the limit along the sequence of solutions u_k to the Dirichlet problem (4.1) with smooth f_k, we conclude that (1.16) also holds for solutions to the generalized Dirichlet problem (1.10) (cf. details in [7]).

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of a solution to the Dirichlet problem (1.6), (1.8) with variational boundary data. Section 3 contains auxiliary results about functions of the spaces $W(D)$ and $H(D)$. In Section 4, we prove the existence and uniqueness of a weak solution to the Dirichlet problem (1.10). In Section 5, we formulate and prove the weak type Harnack inequality. The results of this section serve as the keystone of the proof of Lemma 1.1. In Section 6, we prove Lemma 1.1. At the end of the paper, we formulate rather standard geometric conditions for regularity of a boundary point for H-solutions.

2 The Dirichlet Problem with Variational Boundary Data

The existence and uniqueness of minimizers of the variational problem (1.7), (1.9) in an arbitrary bounded domain D is established in [7, Theorem 3.1]. This implies the existence and uniqueness of solutions to the Dirichlet problem (1.6), (1.8) (cf. [7, Theorem 3.2]). We give a simpler proof. We first establish the strong convergence of a minimizing sequence $\{u_j\}$ in $W(D)$. For this purpose we need the Clarkson inequality in the form (cf. [17])

$$|b|^p - |a|^p \geqslant p|a|^{p-2}a(b-a) + \frac{|b-a|^p}{2^{p-1}-1}, \quad p \geqslant 2,$$

$$|b|^p - |a|^p \geqslant p|a|^{p-2}a(b-a) + \frac{3p(p-1)}{16}\frac{|b-a|^2}{(|a|+|b|)^{2-p}}, \quad 1 < p < 2,$$

for arbitrary vectors $a, b \in \mathbb{R}^n$ that do not vanish simultaneously. We apply this inequality to $b = u_j$, $a = (u_j + u_k)/2$ and $b = u_k$, $a = (u_j + u_k)/2$ and add the results. Denoting $c_0 = 3\alpha(\alpha-1)/16$. we get

$$\int\limits_{\{x\in D: p(x)\geqslant 2\}} \frac{|\nabla(u_j - u_k)|^{p(x)}}{2^{p(x)}p(x)}dx + \frac{c_0}{2}\int\limits_{\{x\in D: p(x)<2\}} \frac{|\nabla(u_j-u_k)|^2}{(3(|\nabla u_j| + |\nabla u_k|)/2)^{2-p(x)}p(x)}dx$$

$$\leqslant \int\limits_D \frac{|\nabla u_j|^{p(x)}}{p(x)}dx + \int\limits_D \frac{|\nabla u_k|^{p(x)}}{p(x)}dx - 2\int\limits_D \frac{|\nabla(u_j+u_k)/2|^{p(x)}}{p(x)}dx.$$

By the definition of a minimizing sequence, the right-hand side of the last inequality can be made less than any positive number $\varepsilon \in (0,1)$ by taking sufficiently large j and k. Then

$$\int\limits_{\{x\in D: p(x)\geqslant 2\}} |\nabla(u_j-u_k)|^{p(x)}dx < C\varepsilon.$$

By the Young inequality,

$$\int\limits_{\{x\in D: p(x)<2\}} |\nabla(u_j-u_k)|^{p(x)}dx \leqslant \varepsilon^{-1/2}\int\limits_{\{x\in D: p(x)<2\}} \frac{|\nabla(u_j-u_k)|^2}{(|\nabla u_j| + |\nabla u_k|)^{2-p(x)}}dx$$

$$+\,\varepsilon^{1/2}\int\limits_D (|\nabla u_j| + |\nabla u_k|)^p(x)dx \leqslant C\sqrt{\varepsilon},$$

where the constant C is independent of j, k. Thus, the sequence $\{u_j\}$ is fundamental in D. We denote by u the limit of this sequence. Using first the mean value theorem and then the Young inequality, for any $\varepsilon \in (0,1)$ we find

$$\int\limits_D \frac{|\nabla u|^{p(x)}}{p(x)}dx \leqslant \int\limits_D \frac{|\nabla u_j|^{p(x)}}{p(x)}dx + \int\limits_D (|\nabla u_j|^{p(x)-1} + |\nabla u|^{p(x)-1})|\nabla u_j - \nabla u|dx$$

$$\leqslant \int\limits_D \frac{|\nabla u_j|^{p(x)}}{p(x)}dx + \varepsilon^{1/(\beta-1)} \int\limits_D (|\nabla u_j|^{p(x)} + |\nabla u|^{p(x)})dx$$

$$+ \varepsilon^{-1} \int\limits_D |\nabla u_j - \nabla u|^{p(x)}dx.$$

By the arbitrariness of ε and j,

$$\int\limits_D \frac{|\nabla u|^{p(x)}}{p(x)}dx \leqslant \liminf_{j\to\infty} \int\limits_D \frac{|\nabla u_j|^{p(x)}}{p(x)}dx.$$

Thus, u is a minimizer. The uniqueness of a minimizer is obtained by the same arguments as those made in derivation of the fundamental sequence $\{u_j\}$: for two minimizers u, v and $w = (u-v)/2$ we get

$$\int\limits_{\{x\in D: p(x)\geqslant 2\}} \frac{|\nabla w|^{p(x)}}{p(x)}dx + 2c_0 \int\limits_{\{x\in D: p(x)<2\}} \frac{|\nabla w|^2}{(3(|\nabla u| + |\nabla v|)/2)^{2-p(x)}p(x)}dx$$

$$\leqslant F(u) + F(v) - 2F((u+v)/2) \leqslant 0 \Rightarrow w = 0.$$

It is easy to verify (the integrand satisfies the assumptions of the Lebesgue dominated convergence theorem) that the Gateaux derivative of the functional F is expressed by

$$\langle F'(u), h\rangle = \lim_{t\to 0} \frac{F(u+th) - F(u)}{t}$$

$$= \lim_{t\to 0} \int\limits_D \frac{|\nabla(u+th)|^{p(x)} - |\nabla u|^{p(x)}}{t}dx = \int\limits_D |\nabla u|^{p(x)-2}\nabla u \nabla h dx$$

if $h \in H_0(D)$ (or $h \in W_0(D)$). Thus, the constructed minimizer (1.7) ((1.9)) is a solution to the Dirichlet problem (1.6) ((1.8)).

3 Cut-Off Functions, Inequality on the Boundary, and Maximum Principle

In this section, we recall some facts concerning cut-off functions in Sobolev spaces. For the Sobolev spaces with constant exponents these facts are well known, but should be proved in the case under consideration. We begin with the following assertion.

Lemma 3.1. *Let $m \in \mathbb{R}$. If a sequence of functions u_j converges to u in the space $W(D)$, then there is a sequence of functions $v_j \in W(D)$, $v_j \leqslant m$, converging to* $\min(u, m)$ *in the same space. If $u_j \in C^\infty(D)$, then $v_j \in C^\infty(D)$. If $m \geqslant 0$ and the functions u_j have compact support in D, then v_j have compact support in D. If $u_j \geqslant m$ on the set E, then $v_j = m$ on E.*

Proof. Denote $f(u)=\min(u,m)$. We construct smooth approximations of f as follows. Let

$$\omega\in C_0^\infty(-1/2,1/2),\quad \omega\geqslant 0,\quad \omega(-s)=\omega(s),\quad \int\limits_{-1/2}^{1/2}\omega(s)ds=1,\quad \omega_\varepsilon(s)=\varepsilon^{-1}\omega(s\varepsilon^{-1}).$$

We define the function

$$h_\varepsilon(t)=1-\int\limits_{-\infty}^{t}\omega_\varepsilon(s-m+\varepsilon/2)ds=\int\limits_{t}^{\infty}\omega_\varepsilon(s-m+\varepsilon/2)ds=\frac{1}{2}+\int\limits_{t}^{m-\varepsilon/2}\omega_\varepsilon(s-m+\varepsilon/2)ds.$$

This function is smooth, is equal to 1 for $t\leqslant m-\varepsilon$, is equal to 0 for $t\geqslant m$, and monotonically decreases in $(m-\varepsilon,m)$. It is easy to see that

$$\int\limits_{m-\varepsilon}^{m}h_\varepsilon(t)dt=\int\limits_{-\varepsilon/2}^{\varepsilon/2}dt\left(\frac{1}{2}+\int\limits_{t}^{0}\omega_\varepsilon(s)ds\right)=\varepsilon/2.$$

Let

$$g_\varepsilon(t)=h_\varepsilon(t)+\frac{\varepsilon}{2}\omega_\varepsilon(t-m+\varepsilon/2),\quad f_\varepsilon(t)=m-\int\limits_{t}^{\infty}g_\varepsilon(s)ds.$$

It is easy to see that $|g_\varepsilon|\leqslant 3/2$, $g_\varepsilon(t)=1$ for $t\leqslant m-\varepsilon$ and $g_\varepsilon(t)=0$ for $t\geqslant m$. Since

$$\int\limits_{m-\varepsilon}^{m}g_\varepsilon(t)dt=\varepsilon,$$

we have $f_\varepsilon(t)=f(t)$ for $t\geqslant m$ and $t\leqslant m-\varepsilon$.

Let u_j be a sequence of functions in $W(D)$ (or $C^\infty(D)$ in the case of $H(D)$) approximating u in D. We consider functions $f_\varepsilon(u_j)$ possessing the properties indicated in the formulation of the lemma. We show that, choosing ε and j in a suitable way, it is possible to make the difference $f_\varepsilon(u_j)-f(u)$ arbitrarily small in $W(D)$. By the triangle inequality,

$$\begin{aligned}I&=\int\limits_D|f_\varepsilon(u_j)-u|+|\nabla f_\varepsilon(u_j)-\nabla f(u)|^{p(x)}dx\\&\leqslant\int\limits_D|f_\varepsilon(u_j)-f_\varepsilon(u)|dx+\int\limits_D|f_\varepsilon(u)-f(u)|dx+\int\limits_D|f'_\varepsilon(u_j)|^{p(x)}|\nabla u_j-\nabla u|^{p(x)}dx\\&+\int\limits_D|f'_\varepsilon(u_j)-f'_\varepsilon(u)|^{p(x)}|\nabla u|^{p(x)}dx+\int\limits_D|f'_\varepsilon(u)-f'(u)|^{p(x)}|\nabla u|^{p(x)}dx=\sum_{k=1}^{5}I_k.\end{aligned}$$

Let us estimate terms on the right-hand side of this inequality. It is easy to see that

$$I_1\leqslant\frac{3}{2}\int\limits_D|u_j-u|dx,\quad I_2\leqslant C\varepsilon|\{m-\varepsilon<u<m\}|.$$

Further,

$$I_3 \leqslant (3/2)^{\beta} \int\limits_D |\nabla u_j - \nabla u|^{p(x)} dx, \quad I_5 \leqslant \int\limits_{\{m-\varepsilon<u<m\}} |\nabla u|^{p(x)} dx.$$

It remains to estimate I_4. We set $F_\varepsilon = \{m - 2\varepsilon \leqslant u \leqslant m\}$. Then $f'_\varepsilon(u_j) - f'_\varepsilon(u) = 0$ on the complement to the set $F_\varepsilon \cup \{|u_j - u| > \varepsilon\}$. Therefore,

$$I_4 \leqslant C \int\limits_{F_\varepsilon} |\nabla u|^{p(x)} dx + C \int\limits_{\{|u_j-u|>\varepsilon\}} |\nabla u|^{p(x)} dx.$$

The first integral on the right-hand side converges to zero as $\varepsilon \to 0$, and the second integral converges to zero as $j \to \infty$ for any fixed $\varepsilon > 0$. Therefore, taking sufficiently small ε and then sufficiently large j, we can made all I_k and, respectively, I as small as desired. □

From Lemma 3.1 we obtain the following assertion.

Lemma 3.2. *If* $u \in H(D)$ $(u \in W(D))$, *then* $\min(u, m) \in H(D)$ $(\min(u, m) \in W(D))$. *For* $m \geqslant 0$, *if* $u \in H_0(D)$ $(u \in W_0(D))$, *then* $\min(u, m) \in H_0(D)$ $(\min(u, m) \in W_0(D))$.

Thus, the set $W(D)$, $H(D)$, $W_0(D)$, $H_0(D)$ forms a lattice, i.e., for any $u, v \in H(D)$ $(W(D)$, $H_0(D)$, $W_0(D))$

$$\min(u, v) = u + \min(v - u, 0) \in H(D) \quad (W(D), H_0(D), W_0(D)),$$
$$\max(u, v) = u - \min(u - v, 0) \in H(D) \quad (W(D), H_0(D), W_0(D)).$$

We recall that $\nabla \min(u, v) = \chi_{\{u\leqslant v\}} \nabla u + \chi_{\{u>v\}} \nabla v$ almost everywhere in D. Hereinafter, χ_A denotes the characteristic function of a set A. We define by $u_+(x) = \max(u(x), 0) = -\min(-u, 0)$ and $u_-(x) = \max(-u(x), 0) = -\min(u, 0)$ the positive and negative parts of a function. By Lemma 3.2, for $u \in H(D)$ $(W(D)$, $H_0(D)$, $W_0(D))$ we have $u_\pm \in H(D)$ $(W(D)$, $H_0(D)$, $W_0(D))$. Furthermore, $u \in W_0(D)$ $(u \in H_0(D))$ implies $(u - k)_+ \in W_0(D)$ $((u - k)_+ \in H_0(D))$ for all $k \geqslant 0$. Indeed, let u_j be a sequence of functions with compact support in $W(D)$ $(C_0^\infty(D))$ converging to u in $W(D)$. Applying Lemma 3.1 with $m = 0$ to the function $k - u$ and the sequence $k - u_j$ converging to this function, we obtain the sequence of functions v_j converging to $\min(k - u, 0)$ with compact support in D.

We say that a function $u \in W(D)$ $(u \in H(D))$ satisfies the inequality $u \geqslant 0$ on the boundary of D if $u_- \in W_0(D)$ $(u_- \in H_0(D))$. Respectively, $u \geqslant v$ on the boundary of D if $u - v \geqslant 0$ on ∂D in the sense of this definition and $u \leqslant v$ if $v - u \geqslant 0$ on ∂D. If it is additionally known that the function $u \in W(D)$ $(u \in H(D))$ is continuous in the closure of the domain, then the inequality $u \geqslant 0$ on ∂D in the usual sense implies the same inequality in the sense of the above definition. This fact is true because the functions $(u + \varepsilon)_-$ vanish in a neighborhood of ∂D and converge to u_- in $W(D)$.

The natural transitivity property holds: if $u \geqslant v$ and $v \geqslant w$ on ∂D in the sense of the above definition, then $u \geqslant w$ on ∂D. Indeed, it is obvious that $(u - w)_- \leqslant (u - v)_- + (v - w)_-$. We consider a sequence of functions $\psi_j \geqslant 0$ in $H(D)$ $(W(D))$ with compact support in D converging to $(u - v)_- + (v - w)_-$ in $W(D)$. Then it is easy to verify that the sequence $\min((u - v)_-, \psi_j)$ converges to $(u - v)_-$ in $W(D)$. Therefore, $(u - v)_- \in H_0(D)$ $(W_0(D))$.

For solutions to the Dirichlet problem (1.6) the maximum principle holds (cf. [7]).

Lemma 3.3 (maximum principle)**.** *If $u_1, u_2 \in H(D)$ $(u_1, u_2 \in W(D))$ are solutions to the Dirichlet problem* (1.6) *with the boundary functions f_1, f_2 and $f_1 \leqslant f_2$ on ∂D, then $u_1 \leqslant u_2$ almost everywhere in D.*

We need another definition generalizing the above definition for inequalities on the boundary. A set E of class $W_0(D,E)$ $(H_0(D,E))$ is introduced as a set of functions in $W(D)$ $(H(D))$ such that there exists a sequence of functions $u_j \in W(D)$ $(u_j \in H(D))$ vanishing in a neighborhood of $\overline{E} \cap D$ and converging to u in $W(D)$. We say that $u \geqslant 0$ on the set E if $u_- \in W_0(D,E)$ $(u_- \in H_0(D,E))$. Other inequalities are defined in a similar way. In this case, the transitivity property also takes place: if $u \geqslant v$ and $v \geqslant w$ on E, then $u \geqslant w$ on E in the sense of this definition.

We also use the following simple assertion.

Lemma 3.4. *Assume that $u \in H(D)$ $(u \in W(D))$, $m \leqslant u \leqslant M$, and f is a Lipschitz function on $[m, M]$. Then $f(u) \in W(D)$ $(f(u) \in H(D))$.*

Proof. For $u \in W(D)$ the assertion is obvious. Let $u \in H(D)$. Assume that $u_j \in C^\infty(D)$, $u_j \to u$ in $W(D)$. We can assume that $m \leqslant u_j \leqslant M$, which is proved as in Lemma 3.1. We extend f by a constant outside $[m, M]$. Let f_ε be a smoothing of f, so that $f_\varepsilon \to f$, $f'_\varepsilon \to f'$ almost everywhere, $|f'_\varepsilon| \leqslant \sup |f'|$. Using the triangle inequality, we obtain the estimate

$$\int\limits_D |\nabla f_\varepsilon(u_j) - \nabla f(u)|^{p(x)} dx \leqslant C \int\limits_D |\nabla f_\varepsilon(u) - \nabla f(u)|^{p(x)} dx$$

$$+ C \int\limits_D |\nabla f_\varepsilon(u_j) - f'_\varepsilon(u_j)\nabla u|^{p(x)} dx + C \int\limits_D |(f'_\varepsilon(u_j) - f'_\varepsilon(u))\nabla u|^{p(x)} dx.$$

The convergence of the first integral on the right-hand side to zero as $\varepsilon \to 0$ follows from the Lebesgue dominated convergence theorem. The convergence of the second integral to zero as $j \to \infty$ follows from the choice of $\{u_j\}$. For fixed ε the third integral on the right-hand side can be made small as desired by the suitable choice of j in view the Egorov theorem, the absolute continuity of the Lebesgue integral, the continuity of f'_ε, and the Lebesgue dominated convergence theorem. □

4 The Generalized Dirichlet Problem

If the boundary function is only continuous, it is possible to construct a solution to the generalized Dirichlet problem. The construction can be performed in different ways. For example, one can approximate the domain by smooth ones as in [9, 10] or by using the Poincaré–Perron method [18, 19]. For nonlinear equations the last method is presented in [20]. We will use a simper construction described for linear equations in [21].

In this section, it is convenient to use the Lebesgue–Orlicz spaces of functions with variable integrability exponent $L^{p(x)}(\Omega) = \{v : |v|^{p(x)} \in L^1(\Omega)\}$ equipped with the Luxemburg norm

$$\|v\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int\limits_\Omega (f/\lambda)^{p(x)} dx \leqslant 1 \right\}.$$

Under the assumption (1.2), the space $L^{p(x)}(\Omega)$ is a reflexive separable Banach space. The dual is the space $L^{p'(x)}(\Omega)$, $p'(x) = p(x)/(p(x)-1)$. We need the following assertion which is not different from the case of a constant exponent p (cf. [22]) and is proved in the same way.

Lemma 4.1. *Let $f_j \to f$ almost everywhere in Ω, and let*

$$\sup_j \int\limits_{\Omega} |f_j|^{p(x)} dx < \infty.$$

Then the sequence f_j weakly converges to f in $L^{p(x)}(\Omega)$.

Let us show how to construct a solution to the generalized Dirichlet problem (1.10).

We extend the boundary function $f \in C(\partial D)$ by continuity to $\overline{D}$, keeping the notation for the extended function, and consider the sequence of infinitely differentiable functions f_k in $\mathbb{R}^n$ that uniformly converges to f on $\overline{D}$. We solve the problem Dirichlet

$$Lu_k = 0 \text{ in } D, \quad u_k \in H(D), \quad (u_k - f_k) \in H_0(D). \tag{4.1}$$

Denote $M = \max\limits_{\partial D} |f|$. We can assume that $|f| \leqslant M$ on $\overline{D}$ and $|f_k| \leqslant M$ on $\overline{D}$. By the maximum principle, $|u| \leqslant M$ almost everywhere in the domain D. Let $\xi \in C_0^\infty(D)$. Taking the test function $u\xi^\beta$ in the integral identity (1.5), we get

$$\int\limits_D |\nabla u_k|^{p(x)} \xi^\beta dx = -\int\limits_D \beta \xi^{\beta-1} u_k |\nabla u_k|^{p(x)-2} \nabla u \nabla \xi dx.$$

Applying the Young inequality, we find

$$\int\limits_D |\nabla u_k|^{p(x)} \xi^\beta dx \leqslant C(\alpha, \beta) \int\limits_D |u_k|^{p(x)} |\nabla \xi|^{p(x)} \xi^{\beta - p(x)} dx.$$

Hence for any subdomain $D' \subset D$

$$\int\limits_{D'} |\nabla u_k|^{p(x)} dx \leqslant C(\alpha, \beta, n, D')(1 + M^\beta).$$

Furthermore, by the maximum principle,

$$\sup_D |u_k - u_m| \leqslant \sup_{\partial D} |f_k - f_m|.$$

Thus, the sequence $\{u_k\}$ uniformly converges to some function $u \in W^{1,1}_{\text{loc}}(D)$ in D; moreover, $|u| \leqslant M$. From the properties of the Lebesgue space with variable integrability exponent it follows that ∇u_k weakly converges to ∇u in $L^{p(x)}_{\text{loc}}(D)$. Therefore, for any subdomain $D' \Subset D$

$$\int\limits_{D'} |\nabla u|^{p(x)} dx \leqslant C(D').$$

Consequently, $u \in W(D')$ for all $D' \Subset D$. If u_k are H-solutions, then, by the Mazur theorem, a sequence of convex combinations of u_k that strongly converges in $W(D)'$ for any subdomain $D' \Subset D$, which implies $u \in H(D')$. We also assume that $|\nabla u_k|^{p(x)-2} \nabla u_k$ weakly converges in

$L^{p'(x)}_{\text{loc}}(D)$ to some element $\xi \in L^{p'(x)}_{\text{loc}}(D)$. We show that $\xi \equiv |\nabla u|^{p(x)-2}\nabla u$ by using the methods of [22]. Let $0 \leqslant \eta \in C_0^\infty(D)$. Applying the integral identity (1.5) for u_k with the test function $(u_k - u)\eta$, we have

$$\int\limits_D (|\nabla u_k|^{p(x)-2}\nabla u_k - |\nabla u|^{p(x)-2}\nabla u)\cdot(\nabla u_k - \nabla u)\eta dx$$

$$= -\int\limits_D (u_k - u)|\nabla u_k|^{p(x)-2}\nabla u_k \cdot \nabla \eta dx - \int\limits_D \eta |\nabla u|^{p(x)-2}\nabla u \cdot (\nabla u_k - \nabla u) dx \to 0$$

as $k \to \infty$. Consequently, $(|\nabla u_k|^{p(x)-2}\nabla u_k - |\nabla u|^{p(x)-2}\nabla u)\cdot(\nabla u_k - \nabla u) \to 0$ almost everywhere in D as $k \to \infty$. By the monotonicity property,

$$(|\xi|^{p(x)-2}\xi - |\eta|^{p(x)-2}\eta)\cdot(\xi - \eta) > 0 \quad \forall\, \xi, \eta \in \mathbb{R}^n,\ \xi \neq \eta,$$

we find that $\nabla u_k(x)$ converges to ∇u almost everywhere in D.

By Lemma 4.1, $|\nabla u_k|^{p(x)-2}\nabla u_k$ weakly converges to $|\nabla u|^{p(x)-2}\nabla u$ in $L^{p'(x)}(D')$ for any $D' \Subset D$. Passing to the limit in the integral identity (1.5) for u_k, we see that the limit function u satisfies the integral identity (1.5) with test functions $\psi \in H_0(D)$. The limit function, called a weak solution to the Dirichlet problem (1.10), is bounded in D and is independent of the method of extending and approximating the boundary function f. In this construction, for p the only condition (1.2) is required.

We show that u_k also converge to the limit function u in the sense of $W(D')$ for any $D' \Subset D$. We recall the Scheffe theorem.

Theorem (Scheffe)**.** *If a sequence of nonnegative functions f_j converges to a function f almost everywhere in D and*

$$\int\limits_D f_j dx \to \int\limits_D f dx,$$

then $f_j \to f$ in $L^1(D)$.

For $\eta \in C_0^\infty(D)$, using the integral identity (1.5) for u_k and then for u, we write

$$\int\limits_D |\nabla u_k|^{p(x)}\eta dx = -\int\limits_D u_k |\nabla u_k|^{p(x)-2}\nabla u_k \nabla \eta dx$$

$$\to -\int\limits_D u|\nabla u|^{p(x)-2}\nabla u \nabla \eta dx = \int\limits_D |\nabla u|^{p(x)}\eta dx.$$

By the Scheffe theorem, $|\nabla u_k|^{p(x)}\eta \to |\nabla u|^{p(x)}\eta$ in $L^1(D)$. Then the sequence $|\nabla u_k|^{p(x)}\eta$ is equicontinuous. Since

$$|\nabla u_k - \nabla u|^{p(x)}\eta \leqslant C|\nabla u_k|^{p(x)}\eta + C|\nabla u|^{p(x)}\eta,$$

the convergence

$$\int\limits_D |\nabla u_k - \nabla u|^{p(x)}\eta dx \to 0$$

follows from the convergence of integrands to zero almost everywhere in D and the Lebesgue theorem. Thus, with an arbitrary continuous boundary function on ∂D we associate a solution to Equation (1.1) in D. This solution is called a *weak solution* to the Dirichlet problem (1.10). Depending on how the solutions u_k to the original problem were regarded, as W-solutions or H-solutions, the obtained solution is a W-solution or an H-solution respectively. However, it is not known whether the solution takes the required value on the boundary.

We note that it is questionable if the Wiener method can be applied to the case under consideration since the continuity of a solution is not guaranteed in any smooth domain if the logarithmic condition on the exponent p fails.

5 The Weak Type Harnack Inequality

The proof of the estimate (1.15) is based on the weak type Harnack inequality of a special form for the solution u to the problem (1.6) with a smooth boundary function f in $\overline{D}$ such that $0 \leqslant f \leqslant M$ on ∂D. We set

$$m = \inf_{\partial D \cap B_{4R}^{x_0}} f, \quad R < 1/16,$$

$$u_m(x) = \begin{cases} \min(u(x), m), & x \in D \cap B_{4R}^{x_0}, \\ m, & x \in B_{4R}^{x_0} \setminus D, \end{cases}$$

$$v_m(x) = u_m(x) + R.$$

We extend p to $\mathbb{R}^n \setminus D$ by a constant p_0. We show that v_m is a supersolution to Equation (1.1) in $\mathbb{R}^n$.

Lemma 5.1. *For any nonnegative* $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\int_D |\nabla v_m|^{p(x)-2} \nabla v_m \nabla \varphi dx \geqslant 0. \tag{5.1}$$

Proof. Let a nonnegative Lipschitz function Φ_ε on the real axis be such that $\Phi_\varepsilon(s) = 0$ for $s \leqslant 0$ and $\Phi'_\varepsilon \geqslant 0$, $\Phi_\varepsilon(s) = 1$ for $s \geqslant \varepsilon$. Choosing a test function $\varphi \Phi_\varepsilon(m + R - v_m) \in H_0(D)$ in the definition of a solution, we get

$$\int_D \Phi_\varepsilon(m + R - v_m)|\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx = \int_D \varphi \Phi'_\varepsilon(m + R - v_m)|\nabla u|^{p(x)} dx \geqslant 0.$$

Letting $\varepsilon \to 0$, we obtain (5.1). □

In what follows, for a measurable set $E \subset \mathbb{R}^n$ and $f \in L^1(E)$ we introduce the notation

$$\fint_E f dx = |E|^{-1} \int_E f dx,$$

where $|E|$ is the Lebesgue measure of E. We will assume that $R \leqslant 1$.

Theorem 5.1. *For any* $0 < q < n(p_0 - 1)/(n-1)$ *and* $0 < s < t < 3$

$$\left(\fint_{B^{x_0}_{tR}} v_m^q dx \right)^{1/q} \leqslant C(n,p,q,M,s,t) \operatorname*{ess\,inf}_{B^{x_0}_{sR}} v_m. \tag{5.2}$$

The proof of the estimate (5.2) in [7, Theorem 6.1] essentially uses the logarithmic condition (1.4) in a neighborhood of the point x_0. Since v_m is a supersolution, the proof is a slightly more complicated from the technical point of view version of the proof in [23] of the weak type Harnack inequality. In the case under consideration, we use the John–Nirenberg lemma so that we need the following estimate for the solution in balls with an arbitrary small radius and centers in a neighborhood of the sought point:

$$\int_{B^z_r} |\nabla \ln v_m| dx \leqslant C(n,p,M) r^{n-1}.$$

These estimate essentially depends on the validity of the condition (1.4). If only the condition (1.11) holds, then such estimates are available only in balls with center x_0.

We propose a new method, which allows us to prove Theorem 5.1 under a weaker assumption than (1.11). The proof is based on a modified technique of [24], where an original method for proving the weak type Harnack inequality was proposed for nonnegative solutions to nonuniformly degenerate linear elliptic equations of divergence form. We note that the assertion of Theorem 5.1 remains valid if v_m is replaced with an arbitrary nonnegative supersolution to Equation (1.1) whose the greatest lower bound is not less than R, whereas the upper bound does not exceed $M + R$.

We proceed by proving Theorem 5.1. The proof consists of two parts. Let σ be the greatest lower bound of p over the ball $B^{x_0}_{4R}$.

Lemma 5.2. *For any* $0 < s < t < 4$

$$\inf_{B^{x_0}_{sR}} v_m \geqslant \exp\left(C + \fint_{B^{x_0}_{tR}} \ln v_m dx \right) \tag{5.3}$$

where the constant C *depends only on* s*,* t*,* M*, exponent* p*, and dimension* n*.*

Proof. We set

$$\ln k = \fint_{B^{x_0}_{tR}} \ln v_m dx. \tag{5.4}$$

By the Jensen inequality,

$$k = \exp \fint_{B^{x_0}_{tR}} \ln v_m dx \leqslant \fint_{B^{x_0}_{tR}} v_m dx \leqslant m + R.$$

In the integral identities (1.5), we take the test function

$$\psi = \left(\ln \frac{k}{v_m} \right)_+^{\gamma} v_m^{1-\sigma} \eta^{\beta},$$

where $\gamma \geqslant 1$, $\eta \in C_0^\infty(B_R)$, $0 \leqslant \eta \leqslant 1$. It is clear that $\psi \in W_0(D)$ and $\psi = 0$ in $B_R^{x_0} \setminus D$. By the integral identity, we have

$$\gamma \int\limits_{B_R^{x_0}} |\nabla v_m|^{p(x)} v_m^{-\sigma} \left(\ln \frac{k}{v_m}\right)_+^{\gamma-1} \eta^\beta dx + (\sigma - 1) \int\limits_{B_R^{x_0}} |\nabla v_m|^{p(x)} v_m^{-\sigma} \left(\ln \frac{k}{v_m}\right)_+^{\gamma} \eta^\beta dx$$
$$\leqslant \beta \int\limits_{B_R^{x_0}} |\nabla v_m|^{p(x)-1} v_m^{1-\sigma} \left(\ln \frac{k}{v_m}\right)_+^{\gamma} \eta^{\beta-1} |\nabla \eta| dx.$$

Applying the Young inequality to the integrand on the right-hand side of this estimate, we find

$$\int\limits_{B_R^{x_0}} |\nabla v_m|^{p(x)} v_m^{-\sigma} \left(\ln \frac{k}{v_m}\right)_+^{\gamma-1} \eta^\beta dx \leqslant C(\beta) \int\limits_{B_R^{x_0}} v_m^{p(x)-\sigma} \left(\ln \frac{k}{v_m}\right)_+^{\gamma+p(x)-1} \eta^{\beta-p(x)} |\nabla \eta|^{p(x)} dx.$$

Setting

$$w = \left(\ln \frac{k}{v_m}\right)_+,$$

From the last inequality it follows that

$$\int\limits_{B_R^{x_0}} |\nabla w|^{p(x)} v_m^{p(x)-\sigma} w^{\gamma-1} \eta^l dx \leqslant C(\beta) \int\limits_{B_R^{x_0}} v_m^{p(x)-\sigma} w^{\gamma+p(x)-1} |\nabla \eta|^{p(x)} dx. \tag{5.5}$$

By the Young inequality,

$$|\nabla w|^\sigma \leqslant |\nabla w|^{p(x)} v^{p(x)-\sigma} + v_m^{-\sigma}.$$

It is clear that

$$v_m^{p(x)-\sigma} \leqslant (M+1)^\beta,$$
$$w^{p(x)-\sigma} \leqslant \left(\ln \frac{m+R}{R}\right)^{p(x)-\sigma} \leqslant C(k_0)(M+1)^\beta.$$

Using these inequalities, from (5.5) we find

$$\int\limits_{B_R^{x_0}} |\nabla w|^\sigma w^{\gamma-1} \eta^\beta dx \leqslant C(p, M) \int\limits_{B_R^{x_0}} (w^{\gamma+\sigma-1} |\nabla \eta|^{p(x)} + w^{\gamma-1} R^{-\sigma} \eta^\beta) dx.$$

Hence

$$\int\limits_{B_R^{x_0}} |\nabla \max(w,1)|^\sigma \max(w,1)^{\gamma-1} \eta^\beta dx \leqslant C(p, M) \int\limits_{B_R^{x_0}} \max(w,1)^{\gamma+\sigma-1} (|\nabla \eta|^{p(x)} + R^{-\sigma} \eta^\beta) dx.$$

Using the Moser iteration, we obtain the estimate

$$\sup_{B_{sR}^{x_0}} w \leqslant C(p, M, n, t, s) \left(1 + \fint_{B_{tR}^{x_0}} w^\sigma dx\right)^{1/\sigma}. \tag{5.6}$$

To estimate the integral

$$\fint_{B^{x_0}_{tR}} w^\sigma \leqslant \fint_{B^{x_0}_{tR}} \left|\ln\frac{k}{v_m}\right|^\sigma dx,$$

we take the test function $\psi = (v_m^{1-\sigma} - (m+R)^{1-\sigma})\eta^\beta$ such that $\eta \in C_0^\infty(B^{x_0}_{4R})$, $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in $B^{x_0}_{tR}$, $|\nabla\eta| \leqslant C(n,t)(tR)^{-1}$ in the integral identity (1.5). From the integral identity (1.5) and the inequality $v_m \leqslant m+R$ it follows that

$$\int_{B^{x_0}_{4tR/3}} |\nabla u|^{p(x)} v_m^{-\sigma}\eta^\beta dx \leqslant \beta \int_{B^{x_0}_{4tR/3}} |\nabla u|^{p(x)-1} v_m^{1-\sigma}\eta^{\beta-1}|\nabla\eta| dx.$$

By the Young inequality,

$$\int_{B^{x_0}_{4R}} |\nabla v_m|^{p(x)} v_m^{-\sigma}\eta^\beta dx \leqslant C(\beta) \int_{B^{x_0}_{4R}} v_m^{p(x)-\sigma}\eta^{\beta-p(x)}|\nabla\eta|^{p(x)} dx.$$

Since $|\nabla v_m|^\sigma \leqslant |\nabla v_m|^{p(x)} + 1$, we obtain the estimate

$$\int_{B^{x_0}_{tR}} \left|\nabla \ln\frac{k}{v_m}\right|^\sigma \eta^\beta dx \leqslant C(n,p,M,t)(tR)^{n-\sigma}.$$

By the choice of the constant k in (5.4), we have

$$\int_{B^{x_0}_{tR}} \ln\frac{k}{v_m} dx = 0.$$

Therefore, by the Poincaré inequality,

$$\int_{B^{x_0}_{tR}} \left|\ln\frac{k}{v_m}\right|^\sigma dx \leqslant C(n,\sigma)(tR)^\sigma \int_{B^{x_0}_{tR}} \left|\nabla \ln\frac{k}{w_m}\right|^\sigma dx \leqslant C(n,p,M,t)(tR)^n. \tag{5.7}$$

The last estimate and (5.6) imply $\sup\limits_{B^{x_0}_{sR}} w \leqslant C(n,p,M,t,s)$. Recalling the definition of w, we arrive at the required assertion. □

We pass to the proof of the second assertion, where $p_0 = p(x_0)$.

Lemma 5.3. *For any $t/64 < \tau < t < 4$ and $0 < q < n(p_0-1)/(n-1)$*

$$\left(\fint_{B^{x_0}_{\tau R}} v_m^q dx\right)^{1/q} \leqslant \exp\left(C + \fint_{B^{x_0}_{tR}} \ln v_m dx\right), \tag{5.8}$$

where the constant C depends only on τ, t, M, q, p, n.

Proof. We set $w = \left(\ln \frac{v_m}{k}\right)_+$, where k has the same sense as in the previous assertion. In the integral identity (1.5), we take the test function $\psi = \max(w, c_0(\gamma+\sigma))^\gamma (v_m^{1-\sigma} - (m+R)^{1-\sigma})\eta^\beta$, where $\gamma \geqslant 0$, $\eta \in C_0^\infty(B_{tR}^{x_0})$, $0 \leqslant \eta \leqslant 1$, and $c_0 = 2/(\sigma - 1)$. We have

$$(\sigma - 1) \int\limits_{B_{tR}^{x_0}} |\nabla v_m|^{p(x)} v_m^{-\sigma} \max(w, c_0(\gamma+\sigma))^\gamma \eta^\beta dx$$

$$= \gamma \int\limits_{B_{tR}^{x_0}} |\nabla v_m|^{p(x)} v_m^{-\sigma} \chi_{\{w > c_0(\gamma+\sigma)\}} \max(w, c_0(\gamma+\sigma))^{\gamma-1} \eta^\beta dx$$

$$+ \beta \int\limits_{B_{tR}^{x_0}} |\nabla v_m|^{p(x)-2} (v_m^{1-\sigma} - (m+R)^{1-\sigma}) \max(w, c_0(\gamma+\sigma))^\gamma \eta^{\beta-1} \nabla v_m \nabla \eta dx,$$

where we used the fact that $\psi = 0$ on the set, where $u + R \neq v_m$. Since

$$\gamma / \max(w, c_0(\gamma+\sigma)) \leqslant 1/c_0 = (\sigma-1)/2,$$

we find

$$\int\limits_{B_{tR}^{x_0}} |\nabla v_m|^{p(x)} v_m^{-\sigma} \max(w, c_0(\gamma+\sigma))^\gamma \eta^\beta dx$$

$$\leqslant \frac{2\beta}{\sigma - 1} \int\limits_{B_{tR}^{x_0}} |\nabla v_m|^{p(x)-2} (v_m^{1-\sigma} - (m+R)^{1-\sigma}) \max(w, c_0(\gamma+\sigma))^\gamma \eta^{\beta-1} \nabla u \nabla \eta dx.$$

By the Young inequality,

$$\int\limits_{B_{tR}^{x_0}} |\nabla v_m|^{p(x)} v_m^{-\sigma} \max(w, c_0(\gamma+\sigma))^\gamma \eta^\beta dx$$

$$\leqslant C(\alpha, \beta) \int\limits_{B_{tR}^{x_0}} v_m^{p(x)-\sigma} \max(w, c_0(\gamma+\sigma))^\gamma \eta^{\beta - p(x)} |\nabla \eta|^{p(x)} dx.$$

Since

$$|\nabla u|^\sigma \leqslant |\nabla u|^{p(x)} + 1, \quad v_m^{p(x)-\sigma} \leqslant (M+1)^\beta,$$

we obtain the inequality

$$\int\limits_{B_{tR}^{x_0}} |\nabla v_m|^\sigma v_m^{-\sigma} \max(w, c_0(\gamma+\sigma))^\gamma \eta^\beta dx$$

$$\leqslant C(\alpha, \beta)(M+1)^\beta \int\limits_{B_{tR}^{x_0}} \max(w, c_0(\gamma+\sigma))^\gamma (v_m^{-\sigma} \eta^\beta + |\nabla \eta|^{p(x)}) dx.$$

Hence

$$(tR)^{\sigma} \fint_{B_{tR}^{x_0}} |\nabla(\max(w, c_0(\gamma+\sigma))^{1+\gamma/\sigma}\eta^{\beta/\sigma})|^{\sigma} dx$$

$$\leqslant C(\alpha,\beta)(M+1)^{\beta} \fint_{B_{tR}^{x_0}} \max(w, c_0(\gamma+\sigma))^{\gamma+\sigma}(tR|\nabla\eta|)^{\sigma} + (v_m/R)^{-\sigma}\eta^{\beta} + (tR|\nabla\eta|)^{p(x)})dx.$$

We set $\varkappa = n/(n-1)$. Applying the Sobolev inequality, we find

$$\fint_{B_{tR}^{x_0}} \max(w, c_0(\gamma+\sigma))^{(\gamma+\sigma)\varkappa}\eta^{\beta\varkappa} dx$$

$$\leqslant C(\alpha,\beta,M)\Bigg(\fint_{B_{tR}^{x_0}\cap \operatorname{supp}\eta} \max(w, c_0(\gamma+\sigma))^{\gamma+\sigma}(1+(tR|\nabla\eta|)^{p(x)})dx\Bigg)^{\varkappa}. \tag{5.9}$$

We define cut-off functions $\eta = \eta_j$ as follows. Assume that $\tau < s < t$, $r_j = s + (t-s)2^{-j}$, $j = 0, 1, 2, \dots$, $\eta_j \in C_0^{\infty}(B_{r_j}^{x_0})$, $0 \leqslant \eta_j \leqslant 1$, $\eta_j = 1$ in $B_{r_{j+1}}^{x_0}$, and $|\nabla\eta_j| \leqslant 2^{j+3}(t-s)^{-1}$. We define the sequence γ_j by the relation $\gamma_{j+1} + \sigma = (\gamma_j + \sigma)\varkappa$, $\gamma_0 = 0$. Then from (5.9) with $\gamma = \gamma_j$, $\eta = \eta_j$ we have

$$\begin{aligned}\fint_{B_{r_{j+1}}^{x_0}} \max(w, c_0(\gamma_{j+1}+\sigma))^{\gamma_{j+1}+\sigma} dx &\leqslant \varkappa^{\gamma_{j+1}+\sigma} \fint_{B_{r_{j+1}}^{x_0}} \max(w, c_0(\gamma_j+\sigma))^{\gamma_{j+1}+\sigma} dx \\ &\leqslant C(\alpha,\beta,M)\varkappa^{\gamma_{j+1}+\sigma}(t-s)^{-\beta\varkappa}2^{j\beta\varkappa}\Bigg(\fint_{B_{r_j}^{x_0}} \max(w, c_0(\gamma_j+\sigma))^{\gamma_j+\sigma} dx\Bigg)^{\varkappa}.\end{aligned} \tag{5.10}$$

From (5.7) it follows that

$$\fint_{B_{tR}^{x_0}} w^{\gamma_0+\sigma} dx \leqslant C(p,n,M).$$

Consequently,

$$\fint_{B_{tR}^{x_0}} \max(w, c_0(\gamma_0+\sigma))^{\gamma_0+\sigma} dx \leqslant C(n,p,M). \tag{5.11}$$

Iterating (5.10) and taking into account (5.11), we obtain the estimate

$$\fint_{B_{r_k}^{x_0}} \max(w, c_0(\gamma_k+\sigma))^{\gamma_k+\sigma} dx \leqslant (C(t-s)^{-\beta\varkappa})^{\sum\limits_{j=1}^{k-1}\varkappa^j} 2^{\beta\varkappa\sum\limits_{j=1}^{k-1} j\varkappa^{k-j-1}} \varkappa^{\sum\limits_{j=1}^{k-1}(\gamma_{j+1}+\sigma)\varkappa^{k-1-j}}. \tag{5.12}$$

We have

$$\sum_{j=1}^{k-1}\varkappa^j = \frac{\varkappa^k - \varkappa}{\varkappa - 1}, \quad \sum_{j=1}^{k-1} j\varkappa^{k-j-1} \leqslant \frac{\varkappa^k}{(\varkappa-1)^2}$$

and (since $k = \log_\varkappa \dfrac{\gamma_k+\sigma}{\gamma_0+\sigma}$)

$$\varkappa^{\sum\limits_{j=1}^{k-1}(\gamma_{j+1}+\sigma)\varkappa^{k-1-j}} = \varkappa^{k(\gamma_k+\sigma)} = \left(\frac{\gamma_k+\sigma}{\gamma_0+\sigma}\right)^{\gamma_k+\sigma}$$

Thus,

$$\fint_{B^{x_0}_{sR}} \max(w, c_0(\gamma_k+\sigma))^{\gamma_k+\sigma} dx \leqslant (C(t-s)^{-\beta\varkappa})^{c(\gamma_k+\sigma)} b^{\gamma_k+\sigma}(\gamma_k+\sigma)^{\gamma_k+\sigma}, \tag{5.13}$$

where $\ln_2 b = \beta\varkappa(\gamma_0+\sigma)^{-1}(\varkappa-1)^{-2}$. Let r be an integer in $[\gamma_{k-1}+\sigma, \gamma_k+\sigma]$. Using the Hölder inequality and the Stirling formula for factorial, we get

$$\begin{aligned}\fint_{B^{x_0}_{sR}} \frac{w^r}{r!} dx &\leqslant \frac{1}{r!}\left(\fint_{B^{x_0}_{sR}} w^{\gamma_k+\sigma} dx\right)^{r/(\gamma_k+\sigma)} \\ &\leqslant C((C(t-s)^{-\beta\varkappa})^c b\varkappa r)^r (e/r)^r = C((C(t-s)^{-\beta\varkappa})^c b\varkappa e)^r. \end{aligned}\tag{5.14}$$

Consequently, taking sufficiently small $\delta_0 = \delta_0(n,p,M,s,t)$, we find

$$\fint_{B^{x_0}_{sR}} \frac{(w\delta_0)^r}{r!} dx \leqslant C2^{-r}.$$

Using the Hölder inequality, for $1 \leqslant r \leqslant \gamma_0+\sigma$ we find

$$\fint_{B^{x_0}_{sR}} \frac{w^r}{r!} dx \leqslant \frac{1}{r!}\left(\fint_{B^{x_0}_{tR}} w^{\gamma_0+\sigma} dx\right)^{r/(\gamma_0+\sigma)} \leqslant \frac{1}{r!}.$$

Thus,

$$\fint_{B^{x_0}_{sR}} e^{\delta_0 w} dx = \sum_{k=0}^{\infty} \fint_{B^{x_0}_{sR}} \frac{(\delta_0 w)^r}{r!} dx \leqslant C(n,p,M,t,s).$$

By the definition of w, we find

$$\left(\fint_{B^{x_0}_{sR}} v_m^{\delta_0} dx\right)^{1/\delta_0} \leqslant C(n,p,M,t,s)k. \tag{5.15}$$

We show that for any $q < n(p_0-1)/(n-1)$

$$\left(\fint_{B^{x_0}_{\tau R}} v_m^{q} dx\right)^{1/q} \leqslant C(n,p,M,\tau,s,q,\delta_0)\left(\fint_{B^{x_0}_{sR}} v_m^{\delta_0} dx\right)^{1/\delta_0}.$$

For this purpose, in the integral identity (1.5), we take the test function $\psi = (v_m^{1-\sigma+\gamma} - (m+R)^{1-\sigma+\gamma})\eta^\beta$, $0 < \gamma < \sigma - 1$, $0 \leqslant \eta \leqslant 1$, $\eta \in C_0^\infty(B_{sR}^{x_0})$. Then

$$(\sigma - 1 - \gamma) \int\limits_{B_{sR}^{x_0}} |\nabla v_m|^{p(x)} v_m^{\gamma-\sigma} \eta^\beta dx$$

$$= \beta \int\limits_{B_{sR}^{x_0}} |\nabla u|^{p(x)-2} (v_m^{1-\sigma+\gamma} - (m+R)^{1-\sigma+\gamma}) \eta^{\beta-1} \nabla u \nabla \eta dx$$

$$\leqslant \beta \int\limits_{B_{sR}^{x_0}} |\nabla v_m|^{p(x)-1} v_m^{1-\sigma+\gamma} \eta^{\beta-1} |\nabla \beta| dx.$$

Applying the Young inequality, we obtain the estimate

$$\int\limits_{B_{sR}^{x_0}} |\nabla v_m|^{p(x)} v_m^{\gamma-\sigma} \eta^\beta dx \leqslant C \int\limits_{B_{sR}^{x_0}} v_m^{p(x)-\sigma+\gamma} |\nabla \eta|^{p(x)} dx \tag{5.16}$$

which, as above, implies

$$\int\limits_{B_{sR}^{x_0}} |\nabla (v_m^{\gamma/\sigma} \eta^{\beta/\sigma})|^\sigma dx \leqslant C(M+1)^\beta \int\limits_{B_{sR}^{x_0}} v_m^\gamma (|\nabla \eta|^{p(x)} + v_m^{-\sigma} \eta^\beta + |\nabla \eta|^\sigma) dx. \tag{5.17}$$

We set $\varkappa = n/(n-1)$. Using the Sobolev inequality, from (5.17) we find

$$\fint\limits_{B_{sR}^{x_0}} (v_m^\gamma \eta^\beta)^\varkappa dx \leqslant C \left((M+1)^\beta \fint\limits_{B_{sR}^{x_0} \cap \operatorname{supp} \eta} v_m^\gamma ((R|\nabla \eta|)^{p(x)} + 1) dx \right)^\varkappa. \tag{5.18}$$

Let j_0 be the minimal natural number such that $q \leqslant \delta_0 \varkappa^{j_0}$. We set $\delta_1 = q\varkappa^{-j_0}$ and $r_j = s - j(s-\tau)/(j_0)$. Assume that $\eta_j \in C_0^\infty(B_{r_j}^{x_0})$, $\eta_j = 1$ on $B_{r_{j+1}}^{x_0}$, $0 \leqslant \eta_j \leqslant 1$, $|\nabla \eta_j| \leqslant 2(j_0+2)/(s-\tau)$. We write the inequality (5.18) for η_j, $\gamma = \gamma_j = \delta_1 \varkappa^j$, $j = 0, 1, \ldots, j_0$. Then

$$\fint\limits_{B_{\tau R}^{x_0}} v_m^q dx = \fint\limits_{B_{r_{j_0}}^{x_0}} v_m^{\gamma_{j_0}} dx \leqslant C(n, p, M, t, s, q, \delta_0, j_0) \left(\fint\limits_{B_{sR}^{x_0}} v_m^{\delta_1} dx \right)^{\gamma_{j_0}/\delta_1}$$

$$\leqslant C(n, p, M, t, s, q, \delta_0, j_0) \left(\fint\limits_{B_{sR}^{x_0}} v_m^{\delta_0} dx \right)^{q/\delta_0},$$

where we applied the Hölder inequality at the last step. Combining the last inequality with (5.15), where $s = (t+\tau)/2$, we obtain the estimate

$$\left(\fint\limits_{B_{\tau R}^{x_0}} v_m^q dx \right)^{1/q} \leqslant C(M, p, n, t, \tau, q) k,$$

which leads to the assertion of the lemma. □

Combining the estimates of Lemmas 5.2 and 5.3, where $s = 1$, $t = 3$, $\tau = 2$, we obtain the assertion of Theorem 5.1. We note that the upper bound for the exponent q in the assertions of Theorem 5.1 and Lemma 5.3 is less than that in the case of a constant exponent p: $0 < q < n(p-1)/(n-p)$ for $1 < p < n$ and $0 < q < +\infty$ if $p = n$. This is done only for the sake of convenience. Formulations remain valid for all $0 < q < n(p_0-1)/(n-p_0)$ in the case $1 < p_0 < n$ and $0 < q < +\infty$ in the case $p_0 = n$ provided that $R < R_0(n,p)$. The difference in the proof of Lemma 5.3 consists in the following. Let $\sigma < n$. For sufficiently small R we have $q < n(\sigma-1)/(n-\sigma)$. Using the Sobolev inequality, from (5.17) we obtain (5.18) with $\varkappa = n/(n-\sigma)$. Then we repeat the above arguments. If $\sigma = n$, then for $\varkappa$ we can take q/δ_0, $\gamma_0 = \delta_0$, and the required estimate is obtained at one step.

To conclude the section, we formulate the weak type Harnack inequality for nonnegative supersolutions.

Theorem 5.2. *Assume that $p_0 = p(x_0)$, $1 < p_0 \leqslant n$, $R \leqslant 1$, v is a nonnegative bounded supersolution to Equation* (1.1) *in $B^{x_0}_{4R}$ such that $v \geqslant R$. Then for any $0 < q < n(p_0-1)/(n-1)$ and $0 < s < t < 3$*

$$\Bigg(\ \fint\limits_{B^{x_0}_{tR}} v^q dx\Bigg)^{1/q} \leqslant C(n,p,q,\max_{B^{x_0}_{4R}} v, s, t)\operatorname*{ess\,inf}_{B^{x_0}_{sR}} v. \tag{5.19}$$

This estimate is also valid if $p_0 < n$ for all $0 < q < n(p_0-1)/(n-p_0)$ or $p_0 = n$ for all $0 < q < +\infty$ with sufficiently small $R < R_0(n,p,q)$.

The proof repeats that of Theorem 5.1. In Lemmas 5.2 and 5.3, in test functions containing expressions of the form $v_m^{1-\sigma} - (m+R)^{1-\sigma}$ the last difference is replaced with $v^{1-\sigma}$ and v_m is replaced with v in other places. Estimates of the form $v_m \leqslant M + R$ are replaced with the estimate $v \leqslant \max\limits_{B^{x_0}_{4R}} v$.

6 Proof of Lemma 1.1

We begin with an important consequence of the weak type Harnack inequality.

Lemma 6.1.

$$\fint\limits_{B^{x_0}_{2R}} |\nabla v_m|^{p(x)-1} dx \leqslant C(n,p,M) R^{1-p_0}\big(\inf_{B^{x_0}_{R}} v_m\big)^{p_0-1}.$$

Proof. From the inequality (5.16) and Theorem 5.1 for $0 < \gamma < \sigma - 1$ we get

$$\int\limits_{B^{x_0}_{2R}} |\nabla v_m|^{p(x)} v_m^{\gamma-\sigma} dx \leqslant C \int\limits_{B^{x_0}_{2R}} v_m^{p(x)-\sigma+\gamma} R^{-p(x)} dx \leqslant C R^{n-p_0} \int\limits_{B^{x_0}_{2R}} v_m^{p_0-\sigma+\gamma} v_m^{p(x)-p_0} dx$$

$$\leqslant C(n,p,M,\sigma,\gamma) R^{n-p_0}\big(\inf_{B^{x_0}_{R}} v_m\big)^{p_0-\sigma+\gamma}.$$

By the Young inequality, for any $t > 0$

$$\fint\limits_{B^{x_0}_{2R}} |\nabla v_m|^{p(x)-1} dx \leqslant \fint\limits_{B^{x_0}_{2R}} t|\nabla v_m|^{p(x)} v_m^{\gamma-\sigma} dx + \fint\limits_{B^{x_0}_{2R}} t^{1-p(x)} v_m^{(\sigma-\gamma)(\sigma-1)} v_m^{(\sigma-\gamma)(p(x)-\sigma)} dx. \tag{6.1}$$

We set $t = \big(\inf_{B_R^{x_0}} v_m\big)^{\sigma-\gamma-1}R$. Using the inequality $v_m \geqslant R$ and the logarithmic condition (1.11), we obtain the estimate

$$t^{1-p(x)} = R^{1-\sigma}\big(\inf_{B_R^{x_0}} v_m\big)^{(\sigma-1-\gamma)(1-\sigma)}R^{\sigma-p(x)}\big(\inf_{B_R^{x_0}} v_m\big)^{(\sigma-1-\gamma)(\sigma-p(x))}$$

$$\leqslant R^{1-\sigma}\big(\inf_{B_R^{x_0}} v_m\big)^{(\sigma-1-\gamma)(1-\sigma)}R^{(\sigma-\gamma)(\sigma-p(x))} \leqslant CR^{1-\sigma}\big(\inf_{B_R^{x_0}} v_m\big)^{(\sigma-1-\gamma)(1-\sigma)}.$$

Let γ be such that $(\sigma-\gamma)(\sigma-1) < n(p_0-1)/(n-1)$. It suffices that $\gamma > \sigma - n/(n-1)$. Thus, we can take any $\gamma \in (\sigma - n/(n-1), \sigma - 1)$. By Theorem 5.1, from (6.1) it follows that

$$\fint_{B_{2R}^{x_0}} |\nabla v_m|^{p(x)-1}dx \leqslant CR^{1-p_0}\big(\inf_{B_R^{x_0}} v_m\big)^{p_0-1} + CR^{1-\sigma}\big(\inf_{B_R^{x_0}} v_m\big)^{\sigma-1}$$

$$\leqslant C(n,p,M)R^{1-p_0}\big(\inf_{B_R^{x_0}} v_m\big)^{p_0-1}.$$

Here, we used the inequality $R^{p_0-\sigma}\big(\inf_{B_R^{x_0}} v_m\big)^{\sigma-p_0} \leqslant 1$. □

In the following assertion, $\gamma(R)$ has the same sense as in (1.13)

Lemma 6.2.

$$(m+R)\gamma(R) \leqslant C(n,p,M)\inf_{B_R^{x_0}} v_m.$$

Proof. Assume that $\eta \in C_0^\infty(B_{2R}^{x_0})$, $\eta = 1$ in $B_R^{x_0}$, $|\nabla\eta| \leqslant CR^{-1}$. Taking $\psi = (m+R-v_m)\eta^\sigma$ for a test function in the integral identity (1.5), we get

$$\int_{B_{2R}^{x_0}} |\nabla v_m|^{p(x)}\eta^\sigma dx = \sigma \int_{B_{2R}^{x_0}} |\nabla v_m|^{p(x)-2}(m+R-v_m)\eta^{\sigma-1}\nabla v_m \nabla\eta dx$$

$$\leqslant C(p)(m+R)R^{-1}\int_{B_{2R}^{x_0}} |\nabla v_m|^{p(x)-1}dx.$$

Setting $w_m = v_m/(m+R)$, we find

$$\int_{B_R^{x_0}} (m+R)^{p(x)}|\nabla(w_m\eta)|^{p(x)}dx$$

$$\leqslant C(p)(m+R)R^{-1}\int_{B_{2R}^{x_0}} |\nabla v_m|^{p(x)-1} + C(p)(m+R)\int_{B_{2R}^{x_0}} v_m^{p(x)-1}R^{-p(x)}. \qquad (6.2)$$

Since $v_m \geqslant R$, we have

$$(m+R)^{p(x)} \leqslant C(p)(m+R)^{p_0}, \quad R^{-p(x)} \leqslant C(p)R^{-p_0}, \quad v_m^{p(x)-1} \leqslant C(p,M)v_m^{p_0-1}$$

in view of the logarithmic condition (1.11). Applying the last inequalities and assertions of Theorem 5.1 and Lemma 6.1 to the estimate (6.2), we arrive at the inequality

$$(m+R)^{p_0-1}\int\limits_{B_{2R}^{x_0}}|\nabla(w_m\eta)|^{p(x)}dx\leqslant C(n,p,M)R^{n-p_0}\big(\inf_{B_R^{x_0}}v_m\big)^{p_0-1}.$$

Let us estimate from below the left-hand side of the last expression by using the definition of capacity:

$$(m+R)^{p_0-1}\alpha C_p(B_R^{x_0}\setminus D,B_{2R}^{x_0})\leqslant C(n,p,M)R^{n-p_0}\big(\inf_{B_R^{x_0}}v_m\big)^{p_0-1}.$$

Raising both sides of the last inequality to the power $1/(p_0-1)$, we get the required estimate. □

By the definition of v_m, from Lemma 6.2 it follows that

$$\gamma(R)\big(\inf_{\partial D\cap B_{4R}^{x_0}}f+R\big)\leqslant C_0(n,p,M)\big(\inf_{D\cap B_R^{x_0}}u+R\big).\tag{6.3}$$

Proof of Lemma 1.1. We set

$$F_R=\sup_{\partial D\cap B_R^{x_0}}f,\quad f_R=\inf_{\partial D\cap B_R^{x_0}}f,\quad M_R=\sup_{D\cap B_R^{x_0}}u,\quad m_R=\inf_{D\cap B_R^{x_0}}u.$$

Applying the estimate (6.3) to the functions $M_{4R}-u$ and $u-m_{4R}$, we find

$$(M_{4R}-F_{4R}+R)\gamma(R)\leqslant C_0(M_{4R}-M_R+R),$$

$$(f_{4R}-m_{4R}+R)\gamma(R)\leqslant C_0(m_R-m_{4R}+R).$$

Adding these inequalities, we obtain the inequality

$$M_R-m_R\leqslant(1-C_0^{-1}\gamma(R))(M_{4R}-m_{4R})+C_0^{-1}\gamma(R)(F_{4R}-f_{4R})+R.$$

Thus, we obtain the estimate from Lemma 1.1. □

7 Necessary Condition for Regularity of a Boundary Point

In this section, we comment Theorem 1.2. We show that the condition (1.14) in Theorem 1.1 is also necessary for regularity of a boundary point. We argue in the same way as in [7]. Let the condition (1.14) fail. Then for some $0<r<\rho$ we consider a solution to the generalized Dirichlet problem (1.10) with a smooth boundary function f that is equal to 3/2 on $\partial D\cap B_{r/2}^{x_0}$, vanishes outside $B_r^{x_0}$, and satisfies the inequality $0\leqslant f\leqslant 3/2$ in $\partial D\cap B_r^{x_0}\setminus B_r^{x_0}$. Then we consider the U_ρ–capacitory potential of the compact set $\overline{B}_\rho^{x_0}\setminus D$ with respect to $B_{8\rho}^{x_0}$. We recall that the capacitory potential of a compact set K with respect to a ball B is a function at which the inifimum in the definition of a capacity is attained Such a function exists and is a solution to Equation (1.1) in $B\setminus K$ with the value 0 on ∂B and 1 on K in the sense of the corresponding Sobolev space.

By the results of [7, Theorem 8.1 and Lemma 8.3], if the condition (1.14) fails, then there is a sufficiently small ρ and a nondecreasing sequence of numbers $a_j\geqslant 0$ such that $\lim_{j\to\infty}a_j<1$ and for all $j\geqslant 1$

$$|\{U_\rho<a_j\}\cap B_{2^{-j}\rho}^{x_0}|>\frac{1}{2}|B_{2^{-j}\rho}^{x_0}|.\tag{7.1}$$

To prove this property, only the logarithmic continuity of the exponent p at the point x_0, i.e., the condition (1.11), is used, Therefore, the estimate (7.1) remains valid in our case.

If r is sufficiently small in comparison with ρ, then

$$u_f \leqslant \frac{1}{2} + U_\rho \quad \text{almost everywhere in} \quad D \cap B^{x_0}_{8\rho}. \tag{7.2}$$

One can verify that the proof of this estimate given in [7] remains valid under the condition (1.11). It suffices to verify that

$$\sup_{B^z_{\rho/2}} u_f(x) \leqslant C(n,p)\left(\fint_{B^z_\rho} u_f(x)dx + \rho \right),$$

where $z \in \partial B^{x_0}_{8\rho} \cap D$. It is easy to see that the proof of Lemma 6.6 in [7], where the required estimate is contained as a particular case, remains valid if the estimate (1.4) is replaced with (1.11). As in [7], the estimates (7.1) and (7.2) imply

$$\operatorname*{ess\,inf}_{D \ni x \to x_0} u_f \leqslant \frac{1}{2} + \lim_{j\to\infty} a_j < \frac{3}{2}.$$

Thus, if the condition (1.14) fails, then the boundary point x_0 is not regular.

8 Geometric Regularity Conditions

We present geometric condition of regularity of a boundary point x_0 for H-solutions to the generalized Dirichlet problem (1.10). Assume that x_0 coincides with the origin O. Let $\mathbb{R}^n \setminus \overline{D}$ in a neighborhood of O has the form

$$\left\{ x : 0 < x_n < a, \sum_{i=1}^{n-1} x_i^2 < g^2(x_n) \right\},$$

where $g(t)$ is a continuous nondecreasing function such that $t^b \leqslant g(t) \leqslant t$ for $t \in [0,a]$, where $b > 1$.

Theorem 8.1. *If $n-1 \leqslant p_0 \leqslant n$, then the boundary point O is regular. For $p_0 < n-1$ the boundary point O is regular if and only if the following integral is divergent at the origin*:

$$\int_0 \left(\frac{g(t)}{t} \right)^{\frac{n-1-p_0}{p_0-1}} \frac{dt}{t} = \infty.$$

The proof is the same as in [7]. In the case $g(t) \geqslant Ct$, the domain satisfies the Poincaré exterior cone condition and the condition (1.14) for any p_0.

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