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LOWER BOUNDS ON THE NUMBER OF LEAVES IN SPANNING TREES

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Let G be a connected graph on $n \geq 2$ vertices with girth at least g such that the length of a maximal chain of successively adjacent vertices of degree 2 in G does not exceed $k \geq 1$. Denote by $u(G)$ the maximum number of leaves in a spanning tree of G. We prove that $u(G) \geq \alpha_{g,k}(v(G) - k - 2) + 2$ where $\alpha_{g,1} = \frac{\left[\frac{g+1}{2}\right]}{4\left[\frac{g+1}{2}\right]+1}$ and $\alpha_{g,k} = \frac{1}{2k+2}$ for $k \geq 2$. We present an infinite series of examples showing that all these bounds are tight. Bibliography: 14 titles.

1. Introduction

We consider finite undirected graphs without loops and multiple edges and use the standard notation. For a graph G, we denote the set of its vertices by $V(G)$ and the set of its edges by $E(G)$. We use the notation $v(G)$ and $e(G)$ for the number of vertices and edges of G, respectively.

We denote the *degree* of a vertex x in G by $d_G(x)$ and, as usual, denote the minimum vertex degree of the graph G by $\delta(G)$.

Let $\mathrm{N}_G(w)$ denote the *neighborhood* of a vertex $w \in V(G)$ (i.e., the set of all vertices adjacent to w).

We denote the girth of the graph G (i.e., the length of a minimal cycle of G) by $g(G)$. If G is a forest, then we set $g(G) = \infty$.

Let $R \subset V(G) \cup E(G)$. We denote by $G - R$ the graph obtained from G by deleting all vertices and edges from the set R and all edges incident to vertices from R.

For a connected graph G, we denote by $u(G)$ the maximum number of leaves in a spanning tree of G.

Remark 1. Obviously, if F is a tree, then $u(F)$ is the number of its leaves.

There are several papers about lower bounds on $u(G)$. In 1981, Storer [1] conjectured that $u(G) > \frac{1}{4}v(G)$ for $\delta(G) \geq 3$. Linial formulated a stronger conjecture: $u(G) \geq \frac{\delta(G)-2}{\delta(G)+1}v(G)+c$ for $\delta(G) \geq 3$, where $c > 0$ is a constant that depends only on $\delta(G)$. This conjecture is suggested by the fact that for every $d \geq 3$ one can easily construct an infinite series of graphs with minimum degree d for which $\frac{u(G)}{v(G)}$ tends to $\frac{d-2}{d+1}$. Thus Linial's conjecture is asymptotically tight in the cases where it holds.

For $\delta(G) = 3$ and $\delta(G) = 4$, Linial's conjecture was proved by Kleitman and West [3] in 1991, for $\delta(G) = 5$ it was proved by Griggs and Wu [4] in 1992. In both papers, the proofs are based on the method of *dead vertices*. There are considerable difficulties with extending this method to the case $\delta(G) \geq 6$, and no further results in this direction are obtained. It follows from [5–7] that Linial's conjecture fails for sufficiently large $\delta(G)$. However, for small $\delta(G) > 5$, the question remains open.

A number of papers consider spanning trees in classes of graphs with various constraints. First, in 1989, Griggs, Kleitman, and Shastri [2] proved that $u(G) \geq \frac{v(G)+4}{3}$ for a connected cubic graph without K_4^- (a complete subgraph on four vertices without one edge). Later, in

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2008, Bonsma [8] proved two bounds for a connected graph with $\delta(G) \geq 3$; namely, $u(G) \geq \frac{v(G)+4}{3}$ for a graph without triangles (that is, with $g(G) \geq 4$) and $u(G) \geq \frac{2v(G)+12}{7}$ for a graph without K_4^-.

These results do not answer the question of how to estimate the number of leaves in a spanning tree for a connected graph with vertices of degrees 1 and 2. Recently, some papers were published in which vertices of degree 1 and 2 do not affect the construction of a spanning tree with many leaves. It is proved in [9] that $u(G) \geq \frac{v_3+4}{3}$ for a connected graph G with $g(G) \geq 4$ and v_3 vertices of degree at least 3 (in fact, in [9, Theorem 1] this bound was stated and proved for a more general class of graphs). In [11], for a connected graph G with v_3 vertices of degree 3 and v_4 vertices of degree at least 4, the bound $u(G) \geq \frac{2v_4}{5} + \frac{2v_3}{15}$ is proved.

Let G be a connected graph with $v(G) \geq 2$ and s vertices of degree different from 2. Karpov and Bankevich [13] proved that $u(G) \geq \frac{1}{4}(s-2)+2$. In the same notation, for a triangle-free graph (i.e., a graph of girth at least 4), Bankevich [14] proved the bound $u(G) \geq \frac{1}{3}(s-2)+2$. Both bounds are tight, infinite series of graphs are constructed for which these bounds are attained. Looking at these results, one may conjecture that $u(G) \geq \frac{g-2}{2g-2}(s-2)+2$ for a connected graph of girth at least g. However, it is shown in [14] that this conjecture fails for $g \geq 10$.

Denote by $\ell(G)$ the number of vertices in a maximal chain of successively adjacent vertices of degree 2 in a graph G.

For a connected graph G with $\ell(G) \leq k$ (where $k \geq 1$), the bound $u(G) > \frac{1}{2k+4}v(G) + \frac{3}{2}$ is proved in [12]. It is also tight. In this paper, we prove two new bounds, taking in account both the girth of a graph and the length of a maximal chain of successively adjacent vertices of degree 2.

Theorem 1. *Let G be a connected graph such that $v(G) \geq 2$, $g(G) \geq g \geq 4$, $\ell(G) \leq k$, where g and $k \geq 1$ are positive integers. Then $u(G) \geq \alpha_{g,k}(v(G)-k-2)+2$ where*

$$\alpha_{g,k} = \begin{cases} \frac{\lceil \frac{g}{2} \rceil - 1}{4(\lceil \frac{g}{2} \rceil - 1)+1} & \text{if } k = 1, \\ \frac{1}{2k+2} & \text{if } k \geq 2. \end{cases}$$

2. Auxiliary lemmas

In this section, we state necessary lemmas, proved in [13] and [12].

Definition 1. 1. *Let G_1 and G_2 be two graphs with marked vertices $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$, respectively, and $V(G_1) \cap V(G_2) = \varnothing$. By* gluing *the graphs G_1 and G_2 at the vertices x_1 and x_2 we mean contracting the vertices x_1 and x_2 into one vertex x, making it incident to all edges incident to x_1 or x_2 in both graphs G_1 and G_2. All the other vertices and edges of the graphs G_1 and G_2 become vertices and edges of the resulting graph* (*see Fig.* 1).

2. *For any edge $e \in E(G)$, we denote by $G \cdot e$ the graph obtained by merging the endpoints of the edge $e = xy$ into one vertex and making it incident to all edges incident to x or y. We say that the graph $G \cdot e$ is obtained from G by contracting the edge e.*

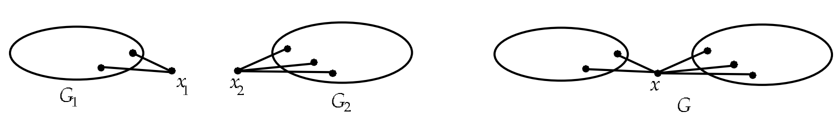

Fig. 1. Gluing graphs.

Remark 2. 1. Contracting a bridge does not lead to the appearance of loops and multiple edges.

2. Assume that a graph H is obtained from a graph H' by contracting several bridges not incident to pendant vertices. Then, obviously, $u(H) = u(H')$.

Lemma 1 ([13, Lemma 1]). *Let G_1 and G_2 be connected graphs with $V(G_1) \cap V(G_2) = \varnothing$, $v(G_1) \geq 2$, $v(G_2) \geq 2$. Let $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$ be pendant vertices. Denote by G the graph obtained by gluing G_1 and G_2 at the vertices x_1 and x_2 and subsequently contracting $m' - 1$ bridges not incident to pendant vertices. Then the following assertions hold.*

(i) $u(G) = u(G'_1) + u(G'_2) - 2$.

(ii) *Let*

$$u(G_1) \geq \alpha(v(G_1) - m) + 2, \quad u(G_2) \geq \alpha(v(G_2) - m) + 2, \quad \text{and} \quad m' \geq m. \tag{1}$$

Then $u(G) \geq \alpha(v(G) - m) + 2$. If all three inequalities in (1) become equalities, then $u(G) = \alpha(v(G) - m) + 2$.

Our method uses the theory of blocks and cutpoints. Recall its basic notions. Proofs of the classical facts on blocks and cutpoints can be found in [10].

Definition 2. *A* cutpoint *of a connected graph G is a vertex $x \in V(G)$ such that the graph $G - x$ is disconnected.*

A graph is biconnected *if it is connected, has at least three vertices, and has no cutpoints.*

A block *of a graph G is a maximal (by inclusion) subgraph of G without cutpoints.*

A bridge *of a graph G is an edge that does not belong to any cycle.*

Definition 3. *Let B be a block of a graph G.*

The boundary *of B (denoted by* Bound(B)*) is the set of all cutpoints of G contained in B. The* interior *of B is the set* $\mathrm{Int}(B) = V(B) \setminus \mathrm{Bound}(B)$. *Vertices of* $\mathrm{Int}(B)$ *are called* internal *vertices of the block B.*

A block B is called empty *if it has no internal vertices (i.e.,* $\mathrm{Int}(B) = \varnothing$*). Otherwise, it is called* nonempty.

A block B is called large *if* $|\mathrm{Int}(B)| > |\mathrm{Bound}(B)|$.

Lemma 2 ([12, Lemma 2]). *Let G be a connected graph with more than 2 vertices. Then there exists a set of edges $F \subset E(G)$ satisfying the following conditions*:

(1°) *the graph $G - F$ is connected*;

(2°) *the graph $G - F$ has no large blocks*;

(3°) *if vertices x and y are adjacent in $G - F$ and $d_{G-F}(x) = d_{G-F}(y) = 2$, then $d_G(x) = d_G(y) = 2$.*

3. Proof of Theorem 1

1. Descent. We say that a graph G' is *smaller* than G if either $u(G') < u(G)$, or $u(G') = u(G)$ and $e(G') < e(G)$. In the first part of the proof, we analyze cases where the conclusion of Theorem 1 for our graph G follows from this conclusion for all smaller graphs.

Let a *spine* be a tree that has no vertices of degree more than 2 and is joined by an edge incident to one of its leaves to a cutpoint a. The cutpoint a will be called the *base* of the spine.

We say that a cutpoint a of a graph G is *inessential* if the graph $G - a$ has exactly two connected components and one of these components is a spine with base a. Otherwise, we call a an *essential* cutpoint.

In some cases (if the condition of case 1.1 or 1.2 holds), we perform a descent from G to smaller graphs.

1.1. The graph G has an essential cutpoint a. We will prove that G has an essential cutpoint of degree at least 3. Let $d_G(a) = 2$. Then the vertex a belongs to a chain of successively adjacent vertices of degree 2; let the endpoints of this chain be adjacent to vertices b and b' of degree different from 2. Since a is an essential cutpoint, $d_G(b) > 2$ and $d_G(b') > 2$. Clearly, both b and b' are essential cutpoints.

Hence it suffices to consider the case $d_G(a) \geq 3$. The vertex a is an essential cutpoint of the graph G, hence there exist connected graphs G_1 and G_2 such that $V(G_1) \cup V(G_2) = V(G)$ and $V(G_1) \cap V(G_2) = \{a\}$. Moreover, none of the graphs G_1 and G_2 is a spine with base a.

Let us construct a graph G_1' from the graph G_1. If $d_{G_1}(a) = 1$, then $G_1' = G_1$. If $d_{G_1}(a) \geq 2$, then attach a spine with $k+1$ vertices to the vertex a. Thus $\ell(G_1') \leq k$, $g(G_1') \geq g(G)$. We also construct a graph G_2' in a similar way.

Since $3 \leq d_G(a) = d_{G_1}(a) + d_{G_2}(a)$, we have $d_{G_1}(a) \geq 2$ or $d_{G_2}(a) \geq 2$. Hence, when constructing at least one of the graphs G_1' or G_2', we have added a spine with $k+1$ vertices. Since a is a vertex of both graphs G_1' and G_2', we obtain the inequality

$$v(G_1') + v(G_2') \geq v(G) + k + 2.$$

The graph G is obtained by gluing the graphs G_1' and G_2' at two pendant vertices (these vertices are copies of a or endpoints of attached spines) and contracting at least $k+1$ bridges (since at least one spine was attached when constructing G_1' and G_2'). As a result, two copies of the vertex a in the graphs G_1' and G_2' are contracted into the vertex a of the graph G. By claim (i) of Lemma 1, we have $u(G) = u(G_1') + u(G_2') - 2$. Since the graphs G_1 and G_2 are not spines with base a, we have $u(G_1'), u(G_2') \geq 3$ and, consequently, $u(G_1') < u(G)$ and $u(G_2') < u(G)$. Then, by the inductive assumption, we have

$$u(G_1') \geq \alpha_{g,k}(v(G_1') - k - 2) + 2, \quad u(G_2') \geq \alpha_{g,k}(v(G_2') - k - 2) + 2.$$

By claim (ii) of Lemma 1, we obtain the inequality $u(G) \geq \alpha_{g,k}(v(G) - k - 2) + 2$, as required.

1.2. The graph G has large blocks. By Lemma 2, we can choose a set of edges $F \subset E(G)$ such that the graph $G' = G - F$ is connected, has no large blocks, and for any two vertices x and y adjacent in G' with $d_{G'}(x) = d_{G'}(y) = 2$ we have $d_G(x) = d_G(y) = 2$. Hence $\ell(G') = \max(\ell(G), 1) \leq k$. Obviously, $g(G') \geq g(G) = g$. Thus we can apply the inductive assumption to the smaller graph G'. Since any spanning tree of G' is a spanning tree of G, we obtain $u(G) \geq u(G') \geq \alpha_{g,k}(v(G) - k - 2) + 2$, as required.

2. The base. Let us reduce our graph by performing steps 1.1 and 1.2 while it is possible. It remains to verify the conclusion of the theorem only for graphs G without essential cutpoints and large blocks. Every cutpoint a of such a graph G splits G into two connected components, and one of these components is a spine with base a.

Let us consider several cases.

2.1. The graph G is a tree. Recall that the tree G has no essential cutpoints. Let a and b be two vertices of degree at least 3. We will prove that a is an essential cutpoint. Let G_1 contain a and exactly one connected component of $G - a$, namely, the component that contains b (see Fig. 2a). Clearly, G_1 is not a spine. Let G_2 contain a and all other connected components of $G - a$. Then a is not a pendant vertex of G_2, i.e., G_2 is not a spine with base a. We obtain a contradiction.

Let G have a vertex a of degree at least 4. Then, clearly, a is an essential cutpoint (see Fig. 2b).

If $\Delta(G) \leq 2$, then G has two leaves and at most $\ell(G) = k$ vertices of degree 2. Then for any $\alpha_{g,k}$ we have

$$u(G) = 2 \geq \alpha_{g,k}(v(G) - k - 2) + 2.$$

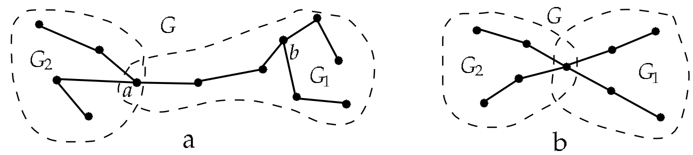

Fig. 2. An essential cutpoint of a tree.

The only remaining case is where the tree G has one vertex of degree 3 and all other vertices have degree at most 2. Then G has three leaves and at most $3 \cdot \ell(G) = 3k$ vertices of degree 2. Hence $v(G) \leq 3k + 4$. It is easy to see that in this case for $\delta_k = \frac{1}{2k+2}$ the following inequality holds:

$$u(G) = 3 \geq \delta_k(2k + 2) + 2 \geq \delta_k(v(G) - k - 2) + 2.$$

2.2. The graph G has a cycle. Since G has no essential cutpoints, each cutpoint of G splits it into two connected components and one of them is a spine with base a. Let H be the graph obtained from G by deleting all vertices of all these spines. It is easy to see that the graph H is biconnected (any cutpoint of H would be an essential cutpoint of G). Then H is a block of G. Since G has a cycle, H also has a cycle.

Let $h = v(H)$ and m be the number of cutpoints of the graph G. Since H is not a large block of G, we have $m \geq \frac{v(H)}{2}$. Every cutpoint separates from the graph a spine with at most $\ell(G) + 1 \leq k + 1$ vertices. Hence $v(G) \leq h + (k + 1)m$.

A biconnected graph H has a cycle with at most h vertices. Hence $h \geq g(G) = g$. Consider two cases.

a. $m = h$**.** Then $v(G) \leq (k + 2)h$ and $u(G) \geq h$. A straightforward calculation shows that in this case

$$u(G) \geq \beta_{h,k}(v(G) - k - 2) + 2 \quad \text{for} \quad \beta_{h,k} = \frac{h-2}{(h-1)(k+2)}.$$

It is easy to see that $\beta_{h,k}$ increases as h increases. Since $h \geq g$, we have $\beta_{h,k} \geq \beta_{g,k}$. Clearly, $g \geq 4$. Then for $k \geq 2$,

$$\beta_{g,k} \geq \beta_{4,k} \geq \frac{4-2}{(4-1)(k+2)} = \frac{2}{3k+6} \geq \frac{1}{2k+2} = \delta_k. \tag{1}$$

For $g \geq 5$ and $k = 1$,

$$\beta_{g,1} \geq \frac{5-2}{(5-1)(1+2)} = \frac{3}{12} = \frac{1}{4} = \delta_1. \tag{2}$$

b. $m < h$**.** In the case under consideration, H is a nonempty block. Choose a vertex $u \in \mathrm{Int}(H)$. It is easy to pick in G a spanning tree whose leaves are m endpoints of spines and the vertex u. Thus $u(G) \geq m + 1$. A straightforward calculation shows that

$$u(G) \geq \gamma_{h,m,k}(v(G) - k - 2) + 2 \quad \text{for} \quad \gamma_{h,m,k} = \frac{m-1}{(k+1)(m-1) + h - 1}. \tag{3}$$

Note that $\gamma_{h,m,k}$ increases as m increases. Since $m \geq \lceil \frac{h}{2} \rceil$, we obtain

$$\gamma_{h,m,k} \geq \varepsilon_{h,k} = \frac{\lceil \frac{h}{2} \rceil - 1}{(\lceil \frac{h}{2} \rceil - 1)(k+1) + h - 1}.$$

Note that

$$\varepsilon_{2t,k} = \frac{t-1}{(t-1)(k+1) + 2t - 1} < \frac{t-1}{(t-1)(k+1) + 2t - 2} = \varepsilon_{2t-1,k}.$$

Moreover, $\varepsilon_{2t,k} = \frac{t-1}{(k+3)(t-1)+1}$ increases as t increases. In view of the above discussion and since $h \geq g$, we conclude that the minimum coefficient in inequality (3) is equal to $\varepsilon_{2t,k}$, where $t = \lceil \frac{g}{2} \rceil \geq 2$. Note that

$$\varepsilon_{2t,k} = \frac{t-1}{(k+3)(t-1)+1} < \frac{1}{2k+2} = \delta_k \iff (1-k)(t-1) > -1 \iff k = 1. \quad (4)$$

Let us summarize the cases analyzed above.

For $k \geq 2$, inequalities (1) and (4) imply that

$$u(G) \geq \frac{1}{2k+2}(v(G) - k - 2) + 2.$$

One of the assertions of the theorem is proved.

For $k = 1$, we obtain the following inequality:

$$u(G) \geq \alpha_{g,1}(v(G) - 3) + 2,$$

where $\alpha_{g,1}$ is the minimum coefficient obtained when analyzing cases **a** and **b**. For $g \geq 5$, inequality (2) implies

$$\alpha_{g,1} \geq \min\left(\varepsilon_{2\lceil \frac{g}{2} \rceil,1}, \beta_1\right) = \min\left(\frac{\lceil \frac{g}{2} \rceil - 1}{4(\lceil \frac{g}{2} \rceil - 1) + 1}, \frac{1}{4}\right) = \frac{\lceil \frac{g}{2} \rceil - 1}{4(\lceil \frac{g}{2} \rceil - 1) + 1}.$$

In this case, the theorem is proved.

For $g = 4$, we have

$$\alpha_{4,1} \geq \min(\beta_{4,1}, \varepsilon_{4,1}) = \min\left(\frac{2}{9}, \frac{1}{5}\right) = \frac{1}{5} = \varepsilon_{4,1}.$$

Now the proof of Theorem 1 is complete.

4. Extremal examples

We describe an infinite series of examples showing that all bounds of Theorem 1 are tight. Our reasoning is quite simple: we construct a graph for which all inequalities proved in the theorem become equalities. Let $\ell(G) = k$, $g(G) = g$. Consider two cases.

1. $k = 1$. Let $n = \lceil \frac{g+1}{2} \rceil - 1$. In the case under consideration, $\alpha_{g,1} = \frac{n}{4n+1}$. Let $B_{g,1}$ be a cycle of length $2n+2$ with $n+1$ marked vertices (no two marked vertices are adjacent) and a spine with two vertices attached to each marked vertex. All marked vertices are cutpoints of our graph. Unmarked vertices are vertices of degree 2. Thus no two vertices of degree 2 are adjacent in our graph. We have $\ell(B_{g,1}) = 1$, $g(B_{g,1}) = 2n+2 \geq g$, $v(B_{g,1}) = 4n+4$. An example of such a graph for $n = 2$ (i.e., $g = 5$ or $g = 6$) is shown in Fig. 3a.

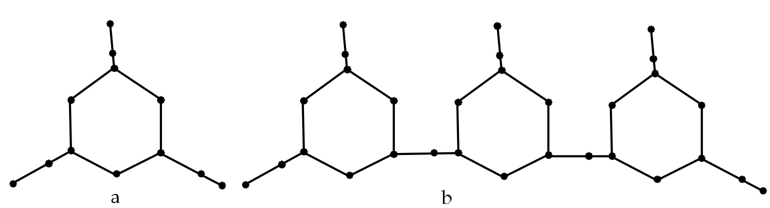

Fig. 3. An extremal example for $k = 1$.

Let us compute $u(B_{g,1})$. All $n+1$ pendant vertices of $B_{g,1}$ (endpoints of spines) are leaves of any spanning tree of $B_{g,1}$. Since deleting leaves of a spanning tree cannot make the graph disconnected, only one nonpendant vertex of $B_{g,1}$ can be a leaf of a spanning tree (of course, this vertex will not be the base of a spine). Thus $u(B_{g,1}) = n + 2$. It is easy to check that

$$u(B_{g,1}) = n + 2 = 2 + \frac{n}{4n+1} \cdot \big(v(B_{g,1}) - 1 - 2\big).$$

Therefore, the bound of Theorem 1 is tight for the graph $B_{g,1}$.

2. $k \geq 2$. In this case, $\alpha_{g,k} = \delta_k = \frac{1}{2k+2}$. Let $B_{g,k}$ be the following tree: a vertex of degree 3 with three spines (with $k+1$ vertices each) attached to it. Then $g(B_{g,k}) = \infty$, $v(B_{g,k}) = 3k+4$, $\ell(B_{g,k}) = k$, $u(B_{g,k}) = 3$. An example of such a graph for $k = 3$ is shown in Fig. 4a. It can be easily checked that

$$u(B_{g,k}) = 3 = 2 + \frac{1}{2k+2} \cdot \big(v(B_{g,k}) - k - 2\big).$$

Therefore, the bound of Theorem 1 is tight for the graph $B_{g,k}$.

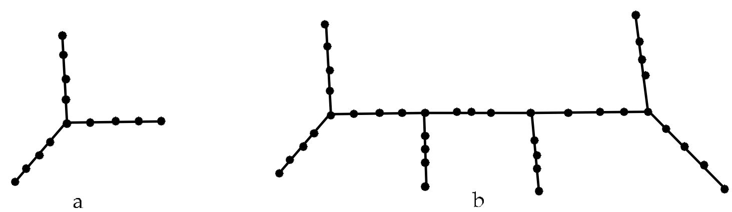

Fig. 4. An extremal example for $k \geq 2$.

3. *Now we show how one can construct extremal examples from the details $B_{g,k}$ in both cases.* Let G be a graph satisfying the conditions

$$u(G) = \alpha_{g,k} \cdot (v(G) - k - 2) + 2, \quad g(G) \geq g, \quad \ell(G) \leq k$$

and having at least one pendant vertex a. We construct a graph G' as follows: merge the vertex a of G with an endpoint of a spine of $B_{g,k}$, and then contract $k+1$ bridges (edges of the attached spine of $B_{g,k}$). As a result, we obtain a graph G' satisfying the following conditions:

$$v(G') = v(G) + v(B_{g,k}) - k - 2, \quad g(G') \geq g, \quad \ell(G') \leq k.$$

By claim (ii) of Lemma 1, we have $u(G') = \alpha_{g,k} \cdot (v(G') - k - 2) + 2$, i.e., the graph G' is also an extremal example, which confirms that the bound of Theorem 1 is tight. First we take $G = B_{g,k}$, and then we can construct arbitrarily large extremal examples by gluing each time a new graph $B_{g,k}$ to the graph we already have. Two such examples are shown in Figs. 3b and 4b.

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