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BOUNDS ON THE DYNAMIC CHROMATIC NUMBER OF A GRAPH IN TERMS OF ITS CHROMATIC NUMBER

N. Y. Vlasova* **and D. V. Karpov**† UDC 519.174.7

A vertex coloring of a graph is called dynamic if the neighborhood of any vertex of degree at least 2 *contains at least two vertices of distinct colors. Similarly to the chromatic number* $\chi(G)$ *of a graph* G*, one can define its dynamic number* $\chi_d(G)$ (*the minimum number of colors in a dynamic coloring*) *and dynamic chromatic number* $\chi_2(G)$ (*the minimum number of colors in a proper dynamic coloring*). *We prove that* $\chi_2(G) \le \chi(G) \cdot \chi_d(G)$ *and construct an infinite series of graphs for which this bound on* $\chi_2(G)$ *is tight.*

For a graph G*, set* $k = \lceil \frac{2\Delta(G)}{\delta(G)} \rceil$. *We prove that* $\chi_2(G) \le (k+1)c$. *Moreover, in the case where* $k \ge 3$ *and* $\Delta(G) \ge 3$*, we prove the stronger bound* $\chi_2(G) \le kc$. *Bibliography:* 9 *titles.*

1. Notation and main results

In this note, we consider finite undirected graphs without loops and multiple edges and their proper colorings.

We use the standard notation. The vertex set of a graph G is denoted by $V(G)$.

We denote the *degree* of a vertex x in a graph G by $d_G(x)$. We denote the maximum and minimum vertex degree of a graph G by $\Delta(G)$ and $\delta(G)$, respectively.

Let $\mathrm{N}_G(w)$ denote the *neighborhood* of a vertex $w \in V(G)$ (i.e., the set of all vertices of a graph G adjacent to w).

When considering a vertex coloring ρ of a graph G, we denote by $\rho(v)$ the color of a vertex v.

Definition 1. (1) *A vertex coloring is proper if any two adjacent vertices have distinct colors.*

(2) *A vertex coloring of a graph* G *is dynamic if for every vertex* $v \in V(G)$ *with* $d_G(x) \ge 2$*, its neighborhood* $\mathrm{N}_G(v)$ *contains vertices of at least two distinct colors.*

(3) *A vertex coloring of a hypergraph is proper if every hyperedge contains at least two vertices of distinct colors.*

Given a graph G, consider the hypergraph $\mathcal{G}$ on the vertex set of G whose hyperedges are the neighborhoods of vertices of G. Then a proper dynamic vertex coloring of the graph G is a proper coloring of G and, simultaneously, a proper vertex coloring of the hypergraph $\mathcal{G}$.

Definition 2. (1) *The chromatic number of a graph* (*or hypergraph*) G (*denoted by* $\chi(G)$) *is the smallest positive integer such that there is a proper vertex coloring of* G *with* $\chi(G)$ *colors.*

(2) *The dynamic number of a graph* G (*denoted by* $\chi_d(G)$) *is the smallest positive integer such that there is a dynamic vertex coloring of* G *with* $\chi_d(G)$ *colors.*

(3) *The dynamic chromatic number of a graph* G (*denoted by* $\chi_2(G)$) *is the smallest positive integer such that there is a proper dynamic vertex coloring of* G *with* $\chi_2(G)$ *colors.*

Let G be a connected graph with $\Delta(G) \ge 3$ different from a complete graph K_{d+1} on $d+1$ vertices. Brooks' theorem [1] tells us that $\chi(G) \le \Delta(G)$.

*St.Petersburg State University, St.Petersburg, Russia, e-mail: `evropa2100@mail.ru`.

†St.Petersburg Department of Steklov Institute of Mathematics and St.Petersburg State University, St.Petersburg, Russia, e-mail: `dvk0@yandex.ru`.

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In [2], it is proved that $\chi_2(G) \le \Delta(G)+1$. Moreover, if $\Delta(G) \le 3$, then the inequality $\chi_2(G) \le 4$ holds with the only exception, the case where G is the cycle on 5 vertices.

In [5] and [7], an analog of Brooks' theorem for proper dynamic colorings is proved: for any connected graph G with $\Delta(G) \le d$ and $d \ge 6$, with several exceptions, the inequality $\chi_2(G) \le \Delta(G)$ holds.

In [6], the following result for dynamic colorings is proved.

Theorem 1. *Let G be a connected graph and $k = \left\lceil \frac{2\Delta(G)}{\delta(G)} \right\rceil$. Then*

(1) $\chi_d(G) \le k+1$;

(2) *if $\delta(G) \ge 3$ and $k \ge 3$, then $\chi_d(G) \le k$.*

Recently, several attempts to estimate the dynamic chromatic number of a graph in terms of its chromatic number have appeared. In [8], it is proved that a connected graph G with $\chi(G) \ge 6$ has a proper coloring with $\chi(G)$ colors such that the set of bad vertices is independent (a vertex x is *bad* if $d_G(x) \ge 2$ and all vertices in $\mathrm{N}_G(x)$ have the same color).

In [4], it is proved that any regular bipartite graph G has a proper dynamic coloring with 4 colors. Moreover, in this coloring each of the two parts of G is divided into two new colors.

In this paper, we estimate the dynamic chromatic number of a graph in terms of its dynamic and chromatic numbers.

Theorem 2. *For any graph G, the inequality $\chi_2(G) \le \chi_d(G) \cdot \chi(G)$ holds.*

In what follows, we prove this theorem (the proof is quite easy) and construct a series of graphs for which the bound of Theorem 2 is attained.

An immediate corollary of Theorems 2 and 1 is the following result, generalizing that proved in [4] for an arbitrary chromatic number and a graph that is not necessarily regular.

Corollary 1. *Let $k = \left\lceil \frac{2\Delta(G)}{\delta(G)} \right\rceil$, $c = \chi(G)$. Then*

(1) $\chi_2(G) \le (k+1)c$;

(2) *if $\delta(G) \ge 3$ and $k \ge 3$, then $\chi_2(G) \le kc$.*

The bound from Corollary 1 is not tight. For a regular bipartite graph G, Corollary 1 gives $\chi_2(G) \le 6$, while the result of [4] gives $\chi_2(G) \le 4$. However, for a bipartite graph with a small difference between $\delta(G)$ and $\Delta(G)$, Corollary 1 also provides the bound $\chi_2(G) \le 6$, which is much more interesting. Note that Corollary 1 provides rather good bounds on $\chi_2(G)$ for graphs G with small $\frac{\Delta(G)}{\delta(G)}$.

2. Proof of Theorem 2 and a series of examples

Proof of Theorem 2. Consider a dynamic coloring ρ_d of G with $\chi_d(G)$ colors and a proper vertex coloring ρ_c of G with $\chi(G)$ colors. Define a new coloring ρ as follows:

$$\rho(v) = (\rho_c(v), \rho_d(v)).$$

Clearly, the coloring ρ uses $\chi(G) \cdot \chi_d(G)$ colors. Since ρ_c is a proper coloring, ρ is also proper. Let v be a vertex with $d_G(v) \ge 2$. Then there are two vertices in $\mathrm{N}_G(v)$ having different colors in the dynamic coloring ρ_d. Clearly, these two vertices have different colors in ρ. Hence ρ is a proper dynamic vertex coloring of G.

Thus $\chi_2(G) \le \chi(G) \cdot \chi_d(G)$. □

Let us construct a series of graphs for which $\chi_2(G) = \chi(G) \cdot \chi_d(G)$.

For any positive integers $k \ge 2$ and c, we construct a graph G with

$$\chi_d(G) = k, \quad \chi(G) = c, \quad \text{and} \quad \chi_2(G) = kc.$$

Let $\mathcal{H}$ be a hypergraph with $\chi(\mathcal{H}) = k$. Clearly, such a hypergraph exists. Indeed, for any positive integer $n \geq 2$, we can construct a hypergraph $\mathcal{H}$ such that $|V(\mathcal{H})| = k(n-1)$ and any n vertices of $V(\mathcal{H})$ form a hyperedge. If we color the vertices of $\mathcal{H}$ with at most $k-1$ colors, then we can find n vertices of the same color. Since these vertices form a hyperedge, the coloring is not proper. But if we color the vertices with k colors so that the number of vertices of each color is exactly $n-1$, then, clearly, we obtain a proper coloring of $\mathcal{H}$. Hence $\chi(\mathcal{H}) = k$.

Consider c copies $\mathcal{H}_1, \ldots, \mathcal{H}_c$ of the hypergraph $\mathcal{H}$ with $V(\mathcal{H}_i) = A_i$ and $E(\mathcal{H}_i) = B_i$. Set $B = \bigcup\limits_{i=1}^{c} B_i$. We will construct a graph G on the vertex set

$$V = \left(\bigcup_{i=1}^{c} A_i\right) \cup B$$

(i.e., the vertices of G correspond to the vertices and edges of the c copies of $\mathcal{H}$). The sets $A_1, \ldots, A_c$ are independent in G, and any two vertices of different sets A_i and A_j are adjacent in G. Any vertex $b \in B_i$ is adjacent in G to all vertices of the corresponding hyperedge of $\mathcal{H}_i$.

Let us find the dynamic, chromatic, and dynamic chromatic numbers of G.

1. $\chi(G) = c$.
Clearly, for distinct $i, j \in \{1, \ldots, c\}$, the sets A_i and A_j cannot contain vertices of the same color. Hence $\chi(G) \geq c$. On the other hand, one can color each set A_i with color i, and then color the vertices of each set B_i with any color different from i. Clearly, we obtain a proper coloring. Hence $\chi(G) = c$.

2. $\chi_d(G) = k$.
Since $\chi(\mathcal{H}) = k$, for any vertex coloring of G with less than k colors, the hypergraph $\mathcal{H}_1$ has a hyperedge $b \in B_i$ whose all vertices have the same color. Hence $\mathrm{N}_G(b)$ is colored with one color, and the coloring is not dynamic.

On the other hand, there is a proper coloring of $\mathcal{H}$ with k colors. Let us color all copies of $\mathcal{H}$ as in this coloring. After that, color all vertices of B with color 1. Let us prove that the obtained coloring of G is dynamic. A vertex $b \in B$ is not bad, since the corresponding hyperedge contains two vertices of distinct colors (recall that the coloring of each copy of $\mathcal{H}$ is proper). A vertex $a \in A_i$ is adjacent to all vertices of A_j (where $j \neq i$), and these vertices are colored with $\chi(\mathcal{H}) = k \geq 2$ colors.

Thus $\chi_d(G) = k$.

3. $\chi_2(G) \geq kc$.
Consider a proper dynamic coloring of G. As proved in Claim 2 above, in a dynamic coloring the vertices of each of the sets $A_1, \ldots, A_c$ must be colored with at least k colors. Recall that for $i \neq j$, any vertex of A_i is adjacent to any vertex of A_j. Since the coloring is proper, A_i and A_j cannot contain vertices of the same color. Hence $\chi_2(G) \geq kc$.

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