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SINGULAR CAUCHY PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION UNSOLVED WITH RESPECT TO THE DERIVATIVE OF THE UNKNOWN FUNCTION

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For a singular Cauchy problem

$$\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} a_{ijk}t^i(x(t))^j(x'(t))^k + \varphi(t,x(t),x'(t)) = 0, \quad x(0)=0,$$

where $N \geq 2$ and a_{ijk} are constants, $a_{00k} = 0$, $k \in \{0,1,\ldots,N\}$, $a_{100} \neq 0$, $a_{010} \neq 0$, $a_{ijk} = 0$, $1 \leq i+j < m$, $k \in \{1,\ldots,N\}$, $2 \leq m \leq N$, and φ is a function small in a certain sense, we find a nonempty set of continuously differentiable solutions $x\colon (0,\rho] \to \mathbb{R}$, where ρ is sufficiently small, such that

$$x(t) = \sum_{k=1}^{m} c_k t^k + o(t^m), \quad t \to +0,$$

where $c_1,\ldots,c_m$ are known constants.

At present, a singular Cauchy problem of the form

$$x'(t) = f(t,x(t)), \quad x(0) = 0,$$

is fairly well studied. Important results are obtained on the solvability of the problem [7, 15, 17, 18], the number of solutions, and the asymptotic behavior of these solutions [3, 7] . Moreover, for a Cauchy problem of the form

$$F(t,x(t),x'(t)) = 0, \quad x(0) = 0,$$

the problems of existence and uniqueness of solutions [1, 2, 5, 17, 20] and the problems of convergence of different sequences of approximations to the solution [4, 16, 19, 21] are thoroughly investigated. However, the asymptotic properties of the solutions of this problem are poorly investigated even in the regular case. In the present paper, we consider a class of singular problems of this type and establish sufficient conditions for the existence of continuously differentiable solutions defined in a certain (sufficiently small) right half neighborhood of the initial point and possessing the required properties in this half neighborhood. In the analysis of the posed problem, we use methods of the qualitative theory of differential equations [6, 7]; see also [8]. The present paper is a continuation of a series of our works [9–13]. For a more detailed survey of the available literature, the subsequent investigations of the analyzed problem, and numerous examples, see [14].

Consider a Cauchy problem

$$P(t,x(t),x'(t)) + \varphi(t,x(t),x'(t)) = 0, \tag{1}$$

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$$x(0) = 0, \tag{2}$$

where t is a real variable, $t \in (0, \tau)$, $x\colon (0, \tau) \to \mathbb{R}$ is an unknown function, and $P\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a polynomial given by the equality

$$P(t, y_1, y_2) = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=0}^{N} a_{ijk} t^i y_1^j y_2^k. \tag{3}$$

Here, N is a natural number, $N \geq 2$, all i, j, and k are nonnegative integer numbers, all a_{ijk} are constants and, in addition, $a_{00k} = 0$, $k \in \{0, 1, \dots, N\}$. Assume that $\varphi\colon D_\varphi \to \mathbb{R}$ is a continuous function, $D_\varphi \subset (0, \tau) \times \mathbb{R} \times \mathbb{R}$, with some properties indicated in what follows. Here, we only note that the analyzed values of φ are, in a certain sense, small as compared with the values of P.

Definition. *We define the solution of problem (1), (2) as a continuously differentiable function* $x\colon (0, \rho] \to \mathbb{R}$ *(*ρ *is a constant,* $\rho \in (0, \tau)$*) with the following properties:*

(i) $(t, x(t), x'(t)) \in D_\varphi$, $t \in (0, \rho]$;

(ii) x *identically satisfies Eq. (1) for all* $t \in (0, \rho]$;

(iii) $\lim_{t\to+0} x(t) = 0$.

We study the problem of existence of the solutions $x\colon (0, \rho] \to \mathbb{R}$ of problem (1), (2) such that

$$x(t) = S_m(t) + o(t^m), \quad t \to +0.$$

Here, $\rho \in (0, \tau)$ is sufficiently small, m is a natural number, $2 \leq m \leq N$, and $S_m\colon \mathbb{R} \to \mathbb{R}$ is a polynomial given by the equality

$$S_m(t) = \sum_{k=1}^{m} c_k t^k, \tag{4}$$

where all c_k are constants uniquely expressed via the coefficients a_{ijk}.

Assume that the polynomial P in Eq. (1) has a special form, namely, for (3), the following conditions are satisfied:

$$a_{100} \neq 0, \quad a_{010} \neq 0, \tag{5}$$

$$a_{ijk} = 0, \quad 1 \leq i + j < m, \quad k \in \{1, \dots, N\}, \tag{6}$$

for some natural $m, 2 \leq m \leq N$. Thus, we can rewrite Eq. (1) in the form

$$-\sum_{\substack{i=0 \\ i+j=m}}^{m}\sum_{j=0}^{m}\sum_{k=1}^{N} a_{ijk} t^i (x(t))^j (x'(t))^k = \sum_{\substack{i=0 \\ l\leq i+j\leq m}}^{m}\sum_{j=0}^{m} a_{ij0} t^i (x(t))^j$$

$$+\sum_{\substack{i=0\\ i+j\geq m+1}}^{N}\sum_{j=0}^{N}\sum_{k=1}^{N}a_{ijk}t^i(x(t))^j(x'(t))^k+\varphi(t,x(t),x'(t)).$$

In what follows, we consider this special class of Eqs. (1) with the initial condition (2).

Assume that the following condition is satisfied:

$$P\left(t, S_m(t), S_m'(t)\right) = O(t^{m+1}), \quad t \to +0, \tag{7}$$

where P and S_m are given by equalities (3) and (4), respectively. Thus, all coefficients $c_1, \dots, c_m$ in (4) can be uniquely determined; in particular, we get

$$c_1 = -\frac{a_{100}}{a_{010}}.$$

Indeed, for any $n \in \{1, \dots, m\}$, by b_n we denote the coefficient of t^n in the sum $P\left(t, S_m(t), S_m'(t)\right)$. This yields

$$b_1 = a_{100} + a_{010}c_1,$$

$$b_k = a_{010}c_k + g_k(c_1, \dots, c_{k-1}), \quad k \in \{2, \dots, m\},$$

where $g_2, \dots, g_m$ are known polynomials. If we set

$$b_n = 0, \quad n \in \{1, \dots, m\}, \tag{8}$$

then condition (7) is satisfied. According to (5),we conclude that $a_{010} \neq 0$. Hence, we can uniquely determine all coefficients $c_1, \dots, c_m$ from the system of equations (8).

It follows from (7) that

$$\left|P(t, S_m(t), S_m'(t))\right| \leq Kt^{m+1}, \quad t \in (0, \tau), \tag{9}$$

where K is a positive constant.

Let

$$D = \left\{(t, y_1, y_2) : t \in (0, \tau), |y_1 - S_m(t)| < t^m\gamma(t), |y_2 - S_m'(t)| < \gamma(t)\right\},$$

where $\gamma\colon (0, \tau) \to (0, +\infty)$ is a continuous function, $\lim_{t\to+0}\gamma(t) = 0$, and $S_m : \mathbb{R} \to \mathbb{R}$ is the polynomial given by equality (4) and satisfying condition (7). Assume that $D \subset D_\varphi$.

The name *Conditions (A)* is used for the following set of conditions:

(i) $\alpha \neq 0$, where

$$\alpha = -\sum_{\substack{i=0\\ i+j=m}}^{m}\sum_{j=0}^{m}\sum_{k=1}^{N}ka_{ijk}c_1^{j+k-1}, \quad c_1 = -\frac{a_{100}}{a_{010}}; \tag{10}$$

(ii) $|\varphi(t, S_m(t), S'_m(t))| \le t^m \xi(t)$, $t \in (0, \tau)$, where $\xi\colon (0,\tau) \to (0, +\infty)$ is a continuously differentiable function and, in addition,

$$\lim_{t\to+0} \xi(t)(\gamma(t))^{-1} = 0, \quad \lim_{t\to+0} t(\xi(t))^{-1} = L_1,$$
$$\lim_{t\to+0} t\xi'(t)(\xi(t))^{-1} = L_2, \quad 0 \le L_i < +\infty, \quad i \in \{1,2\} \tag{11}$$

[the second and third conditions in (11) imply that $0 \le L_2 \le 1$; moreover, if $L_2 < 1$, then definitely $L_1 = 0$ and, hence, the second condition in (11) is essential only in the case where $L_2 = 1$];

(iii) $|\varphi(t_1, x, y) - \varphi(t_2, x, y)| \le l_1(\mu)|t_1 - t_2|$, $(t_i, x, y) \in D$, $0 < \mu \le t_i$, $i \in \{1,2\}$, where $l_1\colon (0,\tau) \to (0, +\infty)$ is a continuous nonincreasing function;

(iv) $|\varphi(t, x_1, y_2) - \varphi(t, x_2, y_2)| \le l_2(t)|x_1 - x_2| + l_3 t^m |y_1 - y_2|$, $(t, x_i, y_i) \in D$, $i \in \{1,2\}$, where $l_2\colon (0,\tau) \to (0, +\infty)$ is a continuous function, $\lim_{t\to+0} l_2(t) = 0$, l_3 is a constant, and $0 < l_3 < |\alpha|/2$.

By $U(\rho, M)$ we denote the set of all continuously differentiable functions $u\colon (0,\rho] \to \mathbb{R}$ each of which satisfies the conditions

$$|u(t) - S_m(t)| \le M t^m \xi(t), \quad |u'(t) - S'_m(t)| \le |a_{010}| l_3^{-1} M \xi(t), \quad t \in (0, \rho]. \tag{12}$$

Here, ρ and M are constants, $\rho \in (0,\tau)$, and $M > 0$.

Theorem 1. *Suppose that conditions (A) are satisfied. The following assertions are true:*

(a) if $\alpha a_{010} > 0$, then there exist ρ and M such that problem (1), (2) has an infinite set of solutions from the set $U(\rho, M)$; moreover, if β is a constant satisfying the condition

$$|\beta - S_m(\rho)| < M \rho^m \xi(\rho), \tag{13}$$

then there exists a solution $x_\beta \in U(\rho, M)$ of problem (1), (2) such that $x_\beta(\rho) = \beta$;

(b) if $\alpha a_{010} < 0$, then there exist ρ and M such that problem (1), (2) has at least one solution from the set $U(\rho, M)$.

Proof. We first choose the constants ρ, M, and q. Let

$$2|a_{010}||\alpha|^{-1} < q < |a_{010}| l_3^{-1}, \tag{14}$$

$$M > (KL_1 + 1)(|a_{010}| - q l_3)^{-1}, \tag{15}$$

where K is the constant from condition (9). Here, we do not present conditions specifying the choice of ρ and only note that ρ is sufficiently small and the choice of ρ guarantees the validity of all subsequent reasoning in which it is assumed that the value of ρ is small. In what follows, in each case where the smallness of ρ is required, this is explicitly indicated.

Let B be a space of continuously differentiable functions $x\colon [0,\rho] \to \mathbb{R}$ with the norm

$$\|x\|_B = \max_{t\in[0,\rho]} (|x(t)| + |x'(t)|). \tag{16}$$

By U we denote a subset of B each element of which $u\colon [0,\rho] \to \mathbb{R}$ satisfies the conditions

$$|u(t) - S_m(t)| \le Mt^m \xi(t), \quad |u'(t) - S'_m(t)| \le qM\xi(t), \quad t \in (0,\rho]. \tag{17}$$

Moreover,

$$u(0) = 0, \quad u'(0) = c_1. \tag{18}$$

In addition, the following condition is satisfied:

$$\forall u \in U \ \ \forall \varepsilon > 0 \ \ \forall t_i \in [0,\rho], \quad i \in \{1,2\}\colon |t_1 - t_2| \le \delta(\varepsilon) \Rightarrow \left|u'(t_1) - u'(t_2)\right| \le \varepsilon, \tag{19}$$

where

$$\delta(\varepsilon) = \varepsilon/(8B(t_\varepsilon)) \quad \text{and} \quad B(t_\varepsilon) = t_\varepsilon^{-(m+1)}(l_1(t_\varepsilon) + 1).$$

Furthermore, the constant $t_\varepsilon \in (0,\rho)$ is chosen so that, for $t \in (0, t_\varepsilon]$, the conditions

$$qM\xi(t) \le \varepsilon/37 \quad \text{and} \quad (2|c_1| + 1)t \le \varepsilon/37 \tag{20}$$

are simultaneously satisfied. The set U is closed, bounded, convex, and (by the Arzela theorem) compact.

We transform Eq. (1) as follows:

$$-\sum_{\substack{i=0\\ i+j=m}}^{m}\sum_{j=0}^{m}\sum_{k=1}^{N} k a_{ijk} t^i (S_m(t))^j (S'_m(t))^{k-1} (x'(t) - S'_m(t))$$

$$= \sum_{\substack{i=0\\ l\le i+j\le m}}^{m}\sum_{j=0}^{m} a_{ij0} t^i (x(t))^j + \sum_{\substack{i=0\\ i+j\ge m+1}}^{N}\sum_{j=0}^{N}\sum_{k=1}^{N} a_{ijk} t^i (x(t))^j (x'(t))^k$$

$$+ \varphi(t, x(t), x'(t)) + \sum_{\substack{i=0\\ i+j=m}}^{m}\sum_{j=0}^{m}\sum_{k=1}^{N} a_{ijk} t^i (S_m(t))^j (S'_m(t))^k$$

$$+ \sum_{\substack{i=0\\ i+j=m}}^{m}\sum_{j=0}^{m}\sum_{k=1}^{N} a_{ijk} t^i (S_m(t))^j \sum_{r=2}^{k} C_k^r (S'_m(t))^{k-r} (x'(t) - S'_m(t))^r$$

$$+ \sum_{\substack{i=0\\ i+j=m}}^{m}\sum_{j=0}^{m}\sum_{k=1}^{N} a_{ijk} t^i \left((x(t))^j - (S_m(t))^j\right) (x'(t))^k \tag{21}$$

[it is clear that the right-hand side of (21) contains the penultimate sum only for $k \geq 2$; in what follows, this fact is not specially indicated]. Further, we consider a differential equation

$$
\begin{aligned}
x'(t) = S'_m(t) + (\lambda(t))^{-1} \Bigg(& \sum_{i=0}^{m} \sum_{j=0}^{m} a_{ij0} t^i (x(t))^j \\
& + \sum_{\substack{i=0 \\ i+j \geqslant m+1}}^{N} \sum_{j=0}^{N} \sum_{k=1}^{N} a_{ijk} t^i (u(t))^j (u'(t))^k + \varphi(t, u(t), u'(t)) \\
& + \sum_{\substack{i=0 \\ i+j=m}}^{m} \sum_{j=0}^{m} \sum_{k=1}^{N} a_{ijk} t^i (S_m(t))^j (S'_m(t))^k \\
& + \sum_{\substack{i=0 \\ i+j=m}}^{m} \sum_{j=0}^{m} \sum_{k=1}^{N} a_{ijk} t^i (S_m(t))^j \sum_{r=2}^{k} C_k^r \left(S'_m(t)\right)^{k-r} \left(u'(t) - S'_m(t)\right)^r \\
& + \sum_{\substack{i=0 \\ i+j=m}}^{m} \sum_{j=0}^{m} \sum_{k=1}^{N} a_{ijk} t^i \left((x(t))^j - (S_m(t))^j\right) \left(u'(t)\right)^k \Bigg),
\end{aligned}
\tag{22}
$$

where the function $\lambda: \mathbb{R} \to \mathbb{R}$ is given by the equality

$$
\lambda(t) = -\sum_{\substack{i=0 \\ i+j=m}}^{m} \sum_{j=0}^{m} \sum_{k=1}^{N} k a_{ijk} t^i (S_m(t))^j (S'_m(t))^{k-1} \tag{23}
$$

and $u \in U$ is an arbitrary fixed function. Since ρ is sufficiently small, $(t, u(t), u'(t)) \in D$, $t \in (0, \rho]$, for all $u \in U$. Note that

$$
\lambda(t) = (\alpha + o(1)) t^m, \quad t \to +0. \tag{24}
$$

Let

$$
D_0 = \{(t, x) : t \in (0, \rho], \quad x \in \mathbb{R}\}. \tag{25}
$$

If $(t, x) \in D_0$, then the conditions of the theorem on existence and uniqueness of the solution and continuous dependence of the solutions on the initial data are satisfied for Eq. (22) because, for any given $r \in (0, \rho)$, in a closed subdomain

$$
D_0(r) = \{(t, x) : t \in [r, \rho], x \in\in \mathbb{R}\}
$$

of the domain D_0, the right-hand side of Eq. (22) is continuous and satisfies the Lipschitz condition with respect to the variable x. We set

$$\Phi_1 = \{(t,x)\colon t \in (0,\rho], |x - S_m(t)| = Mt^m \xi(t)\},$$

$$D_1 = \{(t,x)\colon t \in (0,\rho], |x - S_m(t)| < Mt^m \xi(t)\},$$

$$H = \{(t,x)\colon t = \rho, |x - S_m(\rho)| < M\rho^m \xi(\rho)\}.$$

Assume that the auxiliary function $A_1\colon D_0 \to [0,+\infty)$ is given by the equality

$$A_1(t,x) = (x - S_m(t))^2 (t^m \xi(t))^{-2}.$$

By $a_1\colon D_0 \to \mathbb{R}$ we denote the derivative of this function in a sense of Eq. (22). If $(t,x) \in \Phi_1$, then we conclude that

$$a_1(t,x) = 2(t^m \xi(t))^{-2} (\lambda(t))^{-1} \left((a_{010} + o(1))(x - S_m(t))^2 + (x - S_m(t))\Lambda_1(t)\right), \quad t \to +0,$$

where

$$|\Lambda_1(t)| \le Mt^m \xi(t)(ql_3 + M^{-1} + KL_1 M^{-1} + o(1)), \quad t \to +0.$$

Since ρ is sufficiently small and conditions (14) and (15) are satisfied, then, in view of (24) and the equality

$$Mt^m \xi(t) = |x - S_m(t)|$$

valid for $(t,x) \in \Phi_1$, we get

$$\operatorname{sign} a_1(t,x) = \operatorname{sign} \alpha a_{010} \quad \text{for} \quad (t,x) \in \Phi_1.$$

Further, we successively consider two cases.

I. Let $\alpha a_{010} > 0$. Then $a_1(t,x) > 0$ for $(t,x) \in \Phi_1$. Taking an arbitrary point $(t_0,x_0) \in \Phi_1$ and denoting the integral curve of Eq. (22) passing through this point by $J_0\colon (t, x_0(t))$, for a sufficiently small $\delta > 0$, we obtain $(t, x_0(t)) \notin \overline{D_1}$ for $t \in (t_0, t_0 + \delta)$ (here, $t \le \rho$) and $(t, x_0(t)) \in D_1$ for $t \in (t_0 - \delta, t_0)$.

Indeed, since

$$A_1(t_0, x_0(t_0)) = A_1(t_0, x_0) = M^2, \quad a_1(t_0, x_0(t_0)) = a_1(t_0, x_0) > 0,$$

for $t_0 \in (0,\rho)$, there exists $\delta > 0$ such that

$$\operatorname{sign}(A_1(t, x_0(t)) - A_1(t_0, x_0(t_0))) = \operatorname{sign}(t - t_0), \quad |t - t_0| < \delta,$$

or

$$\operatorname{sign}\left(|x_0(t) - S_m(t)| \left(t^m \xi(t)\right)^{-1} - M\right) = \operatorname{sign}(t - t_0), \quad |t - t_0| < \delta,$$

or

$$\operatorname{sign}\left(|x_0(t) - S_m(t)| - Mt^m\xi(t)\right) = \operatorname{sign}(t - t_0), \quad |t - t_0| < \delta.$$

If $t_0 = \rho$, then there exists $\delta > 0$ such that

$$A_1(t, x_0(t)) < A_1(t_0, x_0(t_0)), \quad t \in (\rho - \delta, \rho),$$

or

$$|x_0(t) - S_m(t)| \left(t^m\xi(t)\right)^{-1} < M, \quad t \in (\rho - \delta, \rho),$$

or

$$|x_0(t) - S_m(t)| < Mt^m\xi(t), \quad t \in (\rho - \delta, \rho).$$

The assertion is proved.

This implies that every integral curve of Eq. (22) crossing the set H is defined for $t \in (0, \rho]$ and lies in D_1 for all $t \in (0, \rho]$. Indeed, any integral curve of this type cannot have common points with Φ_1 as t decreases because, otherwise, we arrive at a contradiction with the already proved assertion.

Let $G(\rho, x_G) \in H$ be an arbitrary fixed point. By $J_u\colon (t, x_u(t))$ we denote the integral curve of Eq. (22) passing through the point G. As indicated above, the integral curve $J_u\colon (t, x_u(t))$ lies in D_1 for $t \in (0, \rho]$. Hence,

$$|x_u(t) - S_m(t)| \le Mt^m\xi(t), \quad t \in (0, \rho]. \tag{26}$$

It is easy to see that

$$\left|x_u'(t) - S_m'(t)\right| \le qM\xi(t), \quad t \in (0, \rho]. \tag{27}$$

In the proof of (27), we use the estimates obtained in the proof of (26), condition (14), and the fact that ρ is sufficiently small. By definition, we set

$$x_u(0) = 0, \quad x_u'(0) = c_1. \tag{28}$$

We now show that, for any $\varepsilon > 0$, any $u \in U$, and any $t_i \in [0, \rho]$, $i \in \{1, 2\}$, such that $|t_1 - t_2| \le \delta(\varepsilon)$, the inequality

$$\left|x_u'(t_1) - x_u'(t_2)\right| \le \varepsilon \tag{29}$$

is true, i.e., we prove that the function $x_u\colon [0, \rho] \to \mathbb{R}$ satisfies condition (19). To this end, we successively consider three possible cases.

1. If $t_i \in [0, t_\varepsilon]$, $i \in \{1, 2\}$, then, by virtue of (20) and (26), we find

$$\left|x_u'(t_1) - x_u'(t_2)\right| \le \left|x_u'(t_1) - S_m'(t_1)\right| + \left|S_m'(t_1) - c_1\right|$$
$$+ \left|x_u'(t_2) - S_m'(t_2)\right| + \left|S_m'(t_2) - c_1\right| \le qM\xi(t_1) + (2|c_2| + 1)t_1$$

$$+ qM\xi(t_2) + (2|c_2| + 1)t_2 \le \frac{4\varepsilon}{37} < \frac{\varepsilon}{9} < \varepsilon. \tag{30}$$

2. If $t_i \in [t_\varepsilon, \rho]$, $i \in \{1, 2\}$, and $|t_1 - t_2| \le \delta(\varepsilon)$, then, in view of the sufficient smallness of ρ, we obtain

$$\left|x'_u(t_1) - x'_u(t_2)\right| \le \left(l_3|\alpha|^{-1} + \frac{1}{4}\right) \left|u'(t_1) - u'(t_2)\right| + (l_1(t_\varepsilon) + 1)t_\varepsilon^{-(m+1)}|t_1 - t_2|.$$

Since, by the assumption,

$$l_3|\alpha|^{-1} < \frac{1}{2},$$

we get

$$\left|x'_u(t_1) - x'_u(t_2)\right| \le \frac{3}{4} \left|u'(t_1) - u'(t_2)\right| + B(t_\varepsilon)|t_1 - t_2|.$$

According to the definition of the set U, it follows from the condition $|t_1 - t_2| \le \delta(\varepsilon)$ that $|u'(t_1) - u'(t_2)| \le \varepsilon$. This yields

$$\left|x'_u(t_1) - x'_u(t_2)\right| \leqslant \frac{3}{4}\,\varepsilon + B(t_\varepsilon)\delta(\varepsilon) = \frac{7}{8}\,\varepsilon < \varepsilon. \tag{31}$$

3. If $t_1 \in [0, t_\varepsilon]$, $t_2 \in [t_\varepsilon, \rho]$, and in addition, $|t_1 - t_2| \le \delta(\varepsilon)$ (the case where $t_2 \in [0, t_\varepsilon]$ and $t_1 \in [t_\varepsilon, \rho]$ but $|t_1 - t_2| \le \delta(\varepsilon)$ is considered in a similar way), then it is clear that t_1 and t_ε belong to the segment $[0, t_\varepsilon]$, while t_ε and t_2 belong to the segment $[t_\varepsilon, \rho]$ and, moreover, $|t_\varepsilon - t_2| \le |t_1 - t_2| \le \delta(\varepsilon)$. Hence, we can use the estimates obtained in the previous two cases. According to (30) and (31), we get

$$\left|x'_u(t_1) - x'_u(t_2)\right| \le \left|x'_u(t_1) - x'_u(t_\varepsilon)\right| + \left|x'_u(t_\varepsilon) - x'_u(t_2)\right| \le \frac{\varepsilon}{9} + \frac{7\varepsilon}{8} < \varepsilon.$$

Inequality (29) is proved. This means that $x_u \in U$.

We define the operator $T\colon U \to U$ by setting $Tu = x_u$.

It is worth noting that the point $G(\rho, x_G)$ of the set H fixed earlier does not change for any choice of the function $u \in U$ on the right-hand side of Eq. (22). Therefore, the condition $x_u(\rho) = x_G$ is always satisfied.

II. Let $\alpha a_{010} < 0$. Then $a_1(t, x) < 0$ for $(t, x) \in \Phi_1$. Taking an arbitrary point $(t_0, x_0) \in \Phi_1$ and denoting the integral curve of Eq. (22) passing through this point by $J_0\colon (t, x_0(t))$, for sufficiently small $\delta > 0$, we get $(t, x_0(t)) \in D_1$ for $t \in (t_0, t_0 + \delta)$ (here, $t \le \rho$) and $(t, x_0(t)) \notin \overline{D_1}$ for $t \in (t_0 - \delta, t_0)$. The proof of this statement is similar to the proof in the case $\alpha a_{010} > 0$.

We now prove that at least one integral curve (among the integral curves of Eq. (22) crossing H) is defined for $t \in (0, \rho]$ and lies in D_1 for all $t \in (0, \rho]$. Indeed, assume that an integral curve of Eq. (22) crosses Φ_1. Thus, if t increases further, then this curve cannot have common points with Φ_1 [otherwise, we arrive at a contradiction with the above-mentioned assertion about the properties of the integral curves of Eq. (22) crossing Φ_1]. Hence, this integral curve crosses $\overline{H}$. We define a mapping $\psi\colon \Phi_1 \to \overline{H}$ as follows: We associate each point $P \in \Phi_1$ with a point $\psi(P) \in \overline{H}$ lying on the same integral curve of Eq. (22) as the point P. By $\psi(\Phi_1)$ we denote the set of images of all points of the set Φ_1 under the mapping ψ. Since the set Φ_1 in not closed [it does not contain its limit point $(0, 0)$], its image $\psi(\Phi_1)$ is also a nonclosed set. At the same time, the set $\overline{H}$ is closed. Therefore, the set $\Omega = \overline{H} \setminus \psi(\Phi_1)$ is nonempty. Let $J_u\colon (t, x_u(t))$ be an integral curve of Eq. (22) such that $(\rho, x_u(\rho)) \in \Omega$. It is clear that, in the case where t decreases from $t = \rho$, this integral curve cannot have common points with Φ_1. Therefore, the indicated integral curve is defined for all $t \in (0, \rho]$ and lies in D_1 for $t \in (0, \rho]$. As in the case $\alpha a_{010} > 0$, we can easily show that estimates (26) and (27) are true. Assume that, by definition, equalities (28) are true. As in the case $\alpha a_{010} > 0$, for any $\varepsilon > 0$, $u \in U$, $t_i \in [0, \rho]$, $i \in \{1, 2\}$, we establish condition (29) for $|t_1 - t_2| \leq \delta(\varepsilon)$. Therefore, we have proved that $x_u \in U$.

We now show that Eq. (22) has a unique integral curve with the indicated properties, namely, the integral curve $J_u \colon (t, x_u(t))$. Indeed, consider one-parameter families of sets

$$\Phi_2(\nu) = \{(t, x) \colon t \in (0, \rho], |x - x_u(t)| = \nu t^m \xi(t)(-\ln t)\},$$

$$D_2(\nu) = \{(t, x) \colon t \in (0, \rho], |x - x_u(t)| < \nu t^m \xi(t)(-\ln t)\},$$

where ν, $\nu \in (0, 1]$, is a parameter. Assume that the auxiliary function $A_2 \colon D_0 \to [0, +\infty)$ is defined by the equality

$$A_2(t, x) = (x - x_u(t))^2 \left(t^m \xi(t)(-\ln t)\right)^{-2}.$$

By $a_2 \colon D_0 \to \mathbb{R}$ we denote the derivative of this function in a sense of Eq. (22). It is easy to see that if $(t, x) \in D_0$, $x \neq x_u(t)$, then

$$a_2(t, x) = 2\left(t^m \xi(t)(-\ln t)\right)^{-2} (\lambda(t))^{-1}(a_{010} + o(1))(x - x_u(t))^2, \quad t \to +0.$$

Since ρ is sufficiently small, according to (24), we get

$$\operatorname{sign} a_2(t, x) = \operatorname{sign} \alpha a_{010}$$

for $(t, x) \in D_0$, $x \neq x_u(t)$, i.e., $a_2(t, x) < 0$ for $(t, x) \in D_0$, $x \neq x_u(t)$. In particular, $a_2(t, x) < 0$ at any point of each curve $\Phi_2(\nu)$ of the constructed family. Thus, if we take an arbitrary point (t_0, x_0) of any curve $\Phi_2(\nu)$, $\nu \in (0, 1]$, and consider the integral curve $J_0(t, x_0(t))$ of Eq. (22) passing through this point, then, for sufficiently small $\delta > 0$, we obtain $(t, x_0(t)) \in D_2(\nu)$ for $t \in (t_0, t_0 + \delta)$ (here, $t \leq \rho$) and $(t, x_0(t)) \notin \overline{D_2(\nu)}$ for $t \in (t_0 - \delta, t_0)$. This statement is proved by analogy with the similar statement for Φ_1 for $\alpha a_{010} < 0$. Further, let $P_*(t_*, x_*) \in \overline{D_1} \setminus \{(0, 0)\}$ be any point satisfying the condition $x_* \neq x_u(t_*)$. There exists $\nu_* \in (0, 1]$ such that $P_* \in \Phi_2(\nu_*)$. By $x^* \colon (t_* - \varepsilon, t_*) \to \mathbb{R}$ we denote a (unique) solution of the differential equation (22) satisfying the initial condition $x(t_*) = x_*$. Here, $\varepsilon > 0$ is sufficiently small. Let (t_-, t_*) be the left maximum interval of existence of this solution (here, $t_- \geq 0$). As indicated above, the integral curve $J_* \colon (t, x^*(t))$ of Eq. (22) passing through the point P_* lies outside $\overline{D_2(\nu_*)}$ for all $t \in (t_-, t_*)$. At the same time, if t_{**} is sufficiently small, $t_{**} \in (0, \rho)$ and (t, x) is any point of the set $\overline{D_1} \setminus \{(0, 0)\}$ satisfying the condition $t \in (0, t_{**}]$, then

$$|x - x_u(t)| \leq |x - S_m(t)| + |x_u(t) - S_m(t)| \leq 2Mt^m \xi(t) < \nu_* t^m \xi(t)(-\ln t).$$

This means that all points (t, x) of the set $\overline{D_1} \setminus \{(0, 0)\}$ satisfying the condition $t \in (0, t_{**}]$ belong to the set $D_2(\nu_*)$. Let $t^* = \min\{t_*, t_{**}\}$. According to the result presented above, the integral curve $J_* \colon (t, x^*(t))$ of Eq. (22) lies outside $\overline{D_1}$ for $t \in (t_-, t^*)$.

The statement is proved.

We define the operator $T \colon U \to U$ by setting $Tu = x_u$.

It is necessary to prove that $T: U \to U$ is a continuous operator. Let $u_i \in U$, $i \in \{1,2\}$, be arbitrary fixed functions and let $Tu_i = x_i$, $i \in \{1,2\}$. If $u_1 = u_2$, then we also have $x_1 = x_2$. Further, let

$$\|u_1 - u_2\|_B = h, \quad h > 0.$$

We investigate the behavior of the integral curves of the differential equation (22) with $u = u_1$. Further, by (22*), we denote the differential equation thus obtained. It is clear that $x_1: (0,\rho] \to \mathbb{R}$ is a solution of Eq. (22*). We set

$$\Phi_3 = \left\{(t,x): t \in (0,\rho], |x - x_2(t)| = h^\nu (t^m \xi(t))^{1-\nu}\right\},$$

$$D_3 = \left\{(t,x): t \in (0,\rho], |x - x_2(t)| < h^\nu (t^m \xi(t))^{1-\nu}\right\},$$

where ν is a constant satisfying the condition

$$0 < \nu < 1 - \frac{1}{m}. \tag{32}$$

Assume that the auxiliary function $A_3: D_0 \to [0,+\infty)$ is given by the equality

$$A_3(t,x) = (x - x_2(t))^2 (t^m \xi(t))^{-2(1-\nu)}.$$

By $a_3: D_0 \to \mathbb{R}$ we denote the derivative of this function in a sense of Eq. (22*). Since ρ is sufficiently small and ν satisfies inequalities (32), we conclude that, for $(t,x) \in \Phi_3$,

$$a_3(t,x) = 2\left(t^m \xi(t)\right)^{-2(1-\nu)} (\lambda(t))^{-1}$$
$$\times \left((a_{010} + o(1))(x - x_2(t))^2 + (x - x_2(t))o(1)h^\nu \left(t^m \xi(t)\right)^{1-\nu}\right), \quad t \to +0.$$

Here, we have used the estimates

$$\begin{aligned} |u_1(t) - u_2(t)| &= |u_1(t) - u_2(t)|^\nu |u_1(t) - u_2(t)|^{1-\nu} \\ &\le \|u_1 - u_2\|_B^\nu \left(|u_1(t) - S_m(t)| + |u_2(t) - S_m(t)|\right)^{1-\nu} \\ &\le h^\nu \left(2Mt^m \xi(t)\right)^{1-\nu}, \quad t \in (0,\rho], \end{aligned}$$

and

$$\begin{aligned} \left|u_1'(t) - u_2'(t)\right| &= \left|u_1'(t) - u_2'(t)\right|^\nu \left|u_1'(t) - u_2'(t)\right|^{1-\nu} \\ &\le \|u_1 - u_2\|_B^\nu \left(\left|u_1'(t) - S_m'(t)\right| + \left|u_2'(t) - S_m'(t)\right|\right)^{1-\nu} \\ &\le h^\nu (2qM\xi(t))^{1-\nu}, \quad t \in (0,\rho]. \end{aligned}$$

Since ρ is sufficiently small and

$$h^{\nu}(t^m\xi(t))^{1-\nu} = |x - x_2(t)|$$

for $(t, x) \in \Phi_3$, by virtue of (24), we obtain

$$\operatorname{sign} a_3(t, x) = \operatorname{sign} \alpha a_{010} \quad \text{for} \quad (t, x) \in \Phi_3.$$

Further, we successively consider two cases.

1. Let $\alpha a_{010} > 0$. Then $a_3(t, x) > 0$ for $(t, x) \in \Phi_3$. This implies that if we take an arbitrary point $(t_0, x_0) \in \Phi_3$ and denote the integral curve of Eq. (22*) passing through this point by $J_0\colon (t, x_0(t))$, then, for sufficiently small $\delta > 0$, we conclude that $(t, x_0(t)) \notin \overline{D_3}$ for $t \in (t_0, t_0 + \delta)$ (here, $t \le \rho$) and $(t, x_0(t)) \in D_3$ for $t \in (t_0 - \delta, t_0)$. This statement is proved by analogy with a similar statement for Φ_1 in the case where $\alpha a_{010} > 0$. Moreover,

$$x_1(\rho) = x_2(\rho) = x_G.$$

This enables us to conclude that if t decreases from $t = \rho$ to $t = 0$, then the integral curve $J_1\colon (t, x_1(t))$ of Eq. (22*) cannot have common points with Φ_3. Hence, this integral curve lies in D_3 for all $t \in (0, \rho]$. Therefore,

$$|x_1(t) - x_2(t)| \le h^{\nu}(t^m\xi(t))^{1-\nu}, \quad t \in (0, \rho]. \tag{33}$$

Further, we conclude that

$$\left|x_1'(t) - x_2'(t)\right| \le (|a_{010}| + o(1))|\lambda(t)|^{-1}h^{\nu}(t^m\xi(t))^{1-\nu}, \quad t \to +0.$$

By virtue of (24), this yields

$$\left|x_1'(t) - x_2'(t)\right| \le \frac{|a_{010} + o(1)|}{|\alpha| + o(1)}\, h^{\nu}(t^m\xi(t))^{1-\nu}t^{-m} = o(1)h^{\nu}t^{-m}, \quad t \to +0. \tag{34}$$

Since ρ is sufficiently small, then it follows from (33) and (34) that

$$|x_1(t) - x_2(t)| + \left|x_1'(t) - x_2'(t)\right| \le h^{\nu}t^{-m}, \quad t \in (0, \rho]. \tag{35}$$

2. Let $\alpha a_{010} < 0$. Then $a_3(t, x) < 0$ for $(t, x) \in \Phi_3$. Hence, if we take an arbitrary point $(t_0, x_0) \in \Phi_3$ and denote the integral curve of Eq. (22*) passing through this point by $J_0\colon (t, x_0(t))$, then, for sufficiently small $\delta > 0$, we obtain $(t, x_0(t)) \in D_3$ for $t \in (t_0, t_0 + \delta)$ (here, $t \le \rho$) and $(t, x_0(t)) \notin \overline{D_3}$ for $t \in (t_0 - \delta, t_0)$. This statement is proved by analogy with the corresponding assertion for Φ_1 in the case where $\alpha a_{010} < 0$. Moreover,

$$|x_1(t) - x_2(t)| \le |x_1(t) - S_m(t)| + |x_2(t) - S_m(t)| \le 2Mt^m\xi(t) < h^{\nu}(t^m\xi(t))^{1-\nu}$$

for $t \in (0, t(h)]$, where the constant $t(h) \in (0, \rho)$ is sufficiently small. Hence, the integral curve $J_1\colon (t, x_1(t))$ of Eq. (22*) lies in D_3 for $t \in (0, t(h)]$. This means that if t increases from $t = t(h)$ to $t = \rho$, then the integral curve $J_1\colon (t, x_1(t))$ of Eq. (22*) cannot have common points with Φ_3. Therefore, this integral curve lies in D_3 for all

$t \in (0, \rho]$. Hence, inequality (33) is true. Further, by analogy with the case $\alpha a_{010} > 0$, we arrive at estimates (34) and (35).

We now directly proceed to the proof of the continuity of the operator $T\colon U \to U$. Let $\varepsilon > 0$ be given. It is clear that there exists a sufficiently small $t_\varepsilon \in (0, \rho)$ such that

$$2Mt^m\xi(t) + 2qM\xi(t) \le \frac{\varepsilon}{2} \quad \text{for} \quad t \in (0, t_\varepsilon].$$

Therefore, if $t \in (0, t_\varepsilon]$, then

$$\begin{aligned} |x_1(t) + x_2(t)| + \left|x_1'(t) - x_2'(t)\right| &\le |x_1(t) - S_m(t)| + |x_2(t) - S_m(t)| \\ &\quad + \left|x_1'(t) - S_m'(t)\right| + \left|x_2'(t) - S_m'(t)\right| \\ &\le 2Mt^m\xi(t) + 2qM\xi(t) \\ &\le \frac{\varepsilon}{2}. \end{aligned}$$

If $t \in [t_\varepsilon, \rho]$, then, in view of (35), we get

$$|x_1(t) - x_2(t)| + \left|x_1'(t) - x_2'(t)\right| \le h^\nu t_\varepsilon^{-m}. \tag{36}$$

We set

$$\delta(\varepsilon) = \left(\frac{\varepsilon}{2}\, t_\varepsilon^m\right)^{\frac{1}{\nu}}.$$

If $h < \delta(\varepsilon)$, then it follows from (36) that

$$|x_1(t) - x_2(t)| + \left|x_1'(t) - x_2'(t)\right| \le \frac{\varepsilon}{2} \tag{37}$$

for $t \in [t_\varepsilon, \rho]$. At the same time, inequality (37) holds for $t \in (0, t_\varepsilon]$ and, in addition,

$$x_i(0) = 0, \quad x_i'(0) = c_1, \quad i \in \{1, 2\}. \tag{38}$$

Hence,

$$\max_{t \in [0,\rho]} \left(|x_1(t) - x_2(t)| + \left|x_1'(t) - x_2'(t)\right|\right) \le \frac{\varepsilon}{2} \quad \text{or} \quad \|x_1 - x_2\|_B \le \frac{\varepsilon}{2}.$$

Thus, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that the relation

$$\|u_1 - u_2\|_B = h < \delta(\varepsilon)$$

implies that

$$\|Tu_1 - Tu_2\|_B = \|x_1 - x_2\|_B \le \frac{\varepsilon}{2} < \varepsilon.$$

This reasoning depends neither on the choice of $\varepsilon > 0$, nor on the choice of $u_i \in U$, $i \in \{1, 2\}$. The continuity of the operator $T : U \to U$ is proved.

Therefore, the continuous operator $T : U \to U$ maps the closed bounded convex compact set U into itself. By virtue of the Schauder theorem, this operator has at least one fixed point in U, i.e., there exists at least one element $x_0 \in U$ such that

$$Tx_0 = x_0. \tag{39}$$

The function $x_0 : (0, \rho] \to \mathbb{R}$ is a solution of problem (1), (2) that belongs to the set $U(\rho, M)$. In addition, recall that, in the procedure of construction of the operator $T : U \to U$ for the case where $\alpha a_{010} > 0$, the point $G(\rho, x_G) \in H$ is chosen arbitrarily. Thus, in particular, it is possible to take the point $G(\rho, \beta)$ if the constant β satisfies inequality (13). In this case, the obtained solution $x_0 : (0, \rho] \to \mathbb{R}$ satisfies the condition $x_0(\rho) = \beta$.

Theorem 2 is proved.

The name Conditions (A) is used for the following set of conditions:

(i) the following inequality is true: $\alpha \neq 0$, where α is the constant given by relation (10);

(ii) $|\varphi(t, S_m(t), S_m'(t))| \le t^m \xi(t)$, $t \in (0, \tau)$, where $\xi : (0, \tau) \to (0, +\infty)$ is a continuously differentiable function satisfying conditions (11) [the second and third conditions in (11) imply that $0 \le L_2 \le 1$; moreover, if $L_2 < 1$, then definitely$L_1 = 0$ and, hence, the second condition in (11) is essential only in the case where $L_2 = 1$];

(iii) $|\varphi(t, x_1, y_1) - \varphi(t, x_2, y_2)| \le l_2 t^{m-1}|x_1 - x_2| + l_3 t^m |y_1 - y_2|$, $(t, x_i, y_i) \in D$, $i \in \{1, 2\}$, where l_2 and l_3 are constants and

$$l_2 + l_3 < \frac{|\alpha|}{2}.$$

Theorem 2. *Suppose that the Conditions (B) are satisfied. Then the following assertions are true:*

(a) if $\alpha a_{010} > 0$, then there exist ρ and M such that problem (1), (2) possesses an infinite set of solutions from the set $U(\rho, M)$; moreover, if the constant β satisfies condition (13), then there exists a unique solution $x_\beta \in U(\rho, M)$ of problem (1), (2) such that $x_\beta(\rho) = \beta$;

(b) if $\alpha a_{010} < 0$, then there exist ρ and M such that problem (1), (2) possesses a unique solution from the set $U(\rho, M)$.

Proof. We first choose constants ρ, M, and q. Assume that the constants q and M satisfy conditions (14) and (15), respectively. Here, we do not present the conditions specifying the choice of ρ. As in the proof of Theorem 1, we only note that ρ is sufficiently small. The validity of the subsequent reasoning based on the smallness of ρ is guaranteed by the choice of ρ; every time when the smallness of ρ is required, this is specially indicated in the proof.

Let B be a space of continuously differentiable functions $x : [0, \rho] \to \mathbb{R}$ with norm (16). By U we denote a subset of B each element $u : [0, \rho] \to \mathbb{R}$ of which satisfies inequalities (17) and, in addition, conditions (18) are satisfied. The set U is closed and bounded. We transform Eq. (1) to the form (21) and consider the differential equation (22) in which $u \in U$ is an arbitrary fixed function and $\lambda : \mathbb{R} \to \mathbb{R}$ is a function given by equality (23). Note that condition (24) is satisfied. Since ρ is sufficiently small, we have

$$(t, u(t), u'(t)) \in D, \quad t \in (0, \rho],$$

for all $u \in U$. We now consider the same sets D_0, Φ_1, D_1, and H and the auxiliary function $A_1\colon D_0 \to [0,+\infty)$ as in the proof of Theorem 1. By $a_1\colon D_0 \to \mathbb{R}$ we denote the derivative of the function $A_1\colon D_0 \to [0,+\infty)$ in a sense of Eq. (22). It is easy to see that

$$\operatorname{sign} a_1(t,x) = \operatorname{sign} \alpha a_{010} \quad \text{for} \quad (t,x) \in \Phi_1$$

because ρ is sufficiently small. We now successively consider two possible cases.

1. Let $\alpha a_{010} > 0$. As in the proof of Theorem 1, we show that each integral curve of Eq. (22) crossing H is defined for $t \in (0,\rho]$ and lies in D_1 for all $t \in (0,\rho]$. Let $G(\rho, x_G) \in H$ be an arbitrary fixed point. By $J_u\colon (t, x_u(t))$ we denote the integral curve of Eq. (22) passing through the point G. If conditions (28) are satisfied, then the function $x_u\colon [0,\rho] \to \mathbb{R}$ belongs to the set U. (To prove this statement, we use the same reasoning as in the proof of Theorem 1.) We now define an operator $T\colon U \to U$ by setting $Tu = x_u$. Here, it is worth noting that the point $G(\rho, x_G)$ remains fixed for any choice of the function $u \in U$ on the right-hand side of Eq. (22). Hence, $x_u(\rho) = x_G$ for any choice of $u \in U$.

2. Let $\alpha a_{010} < 0$. Then, as in the proof of Theorem 1, we can show that, among the integral curves of Eq. (22) crossing H, there exists one and only one integral curve (we denote it by $J_u\colon (t, x_u(t)))$ defined for $t \in (0,\rho]$ and lying in D_1 for all $t \in (0,\rho]$. In this case, we consider the same one-parameter families of sets $\Phi_2(\nu)$ and $D_2(\nu)$ (where ν is a parameter, $\nu \in (0,1]$), and $A_2\colon D_0 \to [0,+\infty)$ is an auxiliary function) as in the proof of Theorem 1. Assume that equalities (28) are true. Thus, as in the proof of Theorem 1, we conclude that $x_u \in U$. We define the operator $T\colon U \to U$ by the equality $Tu = x_u$.

We now prove that $T\colon U \to U$ is a contraction operator. Let $u_i \in U$, $i \in \{1,2\}$, be arbitrary fixed functions and let $Tu_i = x_i$, $i \in \{1,2\}$. If $u_1 = u_2$, then $x_1 = x_2$. Further, let

$$\|u_1 - u_2\|_B = h, \quad h > 0.$$

We study the behavior of the integral curves of the differential equation (22) with $u = u_1$. The differential equation thus obtained is denoted by (22*). It is clear that $x_1(0,\rho] \to \mathbb{R}$ is a solution of Eq. (22*). We set

$$\Phi_3 = \{(t,x)\colon t \in (0,\rho],\, |x - x_2(t)| = \eta t^m h\},$$

$$D_3 = \{(t,x)\colon t \in (0,\rho],\, |x - x_2(t)| < \eta t^m h\},$$

where η is a constant satisfying the condition

$$(l_2 + l_3)|a_{010}|^{-1} < \eta < (|\alpha| - (l_2 + l_3))|a_{010}|^{-1}.$$

We define the auxiliary function $A_3\colon D_0 \to [0,+\infty)$ by the equality

$$A_3(t,x) = (x - x_2(t))^2 t^{-2m}.$$

Moreover, by $a_3\colon D_0 \to \mathbb{R}$ we denote the derivative of this function in a sense of Eq. (22*). Since

$$|u_1(t) - u_2(t)| = \left| \int_0^t u_1'(s)\,ds - \int_0^t u_2'(s)\,ds \right| \le \left| \int_0^t \left|u_1'(s) - u_2'(s)\right|\,ds \right|$$

$$\leq \left| \int_0^t \max_{s\in[0,\rho]} \left(|u_1(s) - u_2(s)| + \left|u_1'(s) - u_2'(s)\right|\right) ds \right|$$

$$= \left| \int_0^t h\, ds \right| = ht, \quad t \in (0, \rho],$$

and, in addition,

$$\left|u_1'(t) - u_2'(t)\right| \leq \max_{t\in[0,\rho]} \left(|u_1(t) - u_2(t)| + \left|u_1'(t) - u_2'(t)\right|\right) = h, \quad t \in (0, \rho],$$

we conclude that

$$a_3(t, x) = 2t^{-2m}(\lambda(t))^{-1} \big((a_{010} + o(1))(x - x_2(t))^2$$

$$+(x - x_2(t))(l_2 + l_3 + o(1))t^m h\big), \quad t \to +0.$$

Since

$$t^m h = \frac{1}{\eta} |x - x_2(t)|$$

for $(t, x) \in \Phi_3$, ρ is sufficiently small, and

$$\frac{1}{\eta} (l_2 + l_3) < |a_{010}|,$$

by virtue of (24), we get

$$\operatorname{sign} a_3(t, x) = \operatorname{sign} \alpha a_{010} \quad \text{for} \quad (t, x) \in \Phi_3.$$

Further, we successively consider the following two cases:

1. Let $\alpha a_{010} > 0$. Then $a_3(t, x) > 0$ for $(t, x) \in \Phi_3$. If we take an arbitrary point $(t_0, x_0) \in \Phi_3$ and denote the integral curve of Eq. (22*) passing trough this point by J_0: $(t, x_0(t))$, then, for sufficiently small $\delta > 0$, we get $(t, x_0(t)) \notin \overline{D_3}$ for $t \in (t_0, t_0 + \delta)$ (here, $t \leq \rho$) and $(t, x_0(t)) \in D_3$ for $t \in (t_0 - \delta, t_0)$. This statement is proved by analogy with a similar statement for Φ_1 in the proof of Theorem 1 (in the case where $\alpha a_{010} > 0$). Moreover,

$$x_1(\rho) = x_2(\rho) = x_G.$$

Hence, if t decreases from $t = \rho$ to $t = 0$, then the integral curve J_1: $(t, x_1(t))$ of Eq. (22*) cannot have common points with Φ_3. Therefore, this integral curve lies in D_3 for all $t \in (0, \rho]$. This means that,

$$|x_1(t) - x_2(t)| \leq \eta t^m h, \quad t \in (0, \rho]. \tag{40}$$

Thus,

$$\left|x_1'(t) - x_2'(t)\right| \le (|a_{010}|\eta + l_2 + l_3 + o(1))|\lambda(t)|^{-1} t^m h, \quad t \to +0.$$

By virtue of (24), we obtain

$$\left|x_1'(t) - x_2'(t)\right| \le \left((|a_{010}|\eta + l_2 + l_3)|\alpha|^{-1} + o(1)\right) h, \quad t \to +0. \tag{41}$$

Denote

$$\omega = \frac{1}{2}\left(1 + (|a_{010}|\eta + l_2 + l_3)|\alpha|^{-1}\right). \tag{42}$$

Since, according to the assumption,

$$(|a_{010}|\eta + l_2 + l_3)|\alpha|^{-1} < 1,$$

we conclude that $0 < \omega < 1$. In view of the sufficient smallness of ρ, we derive the following inequality from (40)–(42):

$$|x_1(t) - x_2(t)| + \left|x_1'(t) - x_2'(t)\right| \le \omega h \tag{43}$$

for $t \in (0, \rho]$. It follows from (38) and (43) that

$$\|x_1 - x_2\|_B \le \omega h.$$

This yields

$$\|Tu_1 - Tu_2\|_B \le \omega \|u_1 - u_2\|_B, \quad \text{where} \quad 0 < \omega < 1. \tag{44}$$

2. Let $\alpha a_{010} < 0$. Then $a_3(t,x) < 0$ for $(t,x) \in \Phi_3$. Thus, if we choose any point $(t_0, x_0) \in \Phi_3$ and denote the integral curve of Eq. (22*) passing through this point by $J_0\colon (t, x_0(t))$, then, for sufficiently small $\delta > 0$, we obtain $(t, x_0(t)) \in D_3$ for $(t, t_0 + \delta)$ (here, $t \le \rho$) and $(t, x_0(t)) \notin \overline{D_3}$ for $t \in (t_0 - \delta, t_0)$. This assertion is proved by analogy with a similar statement for Φ_1 in the proof of Theorem 1 (in the case where $\alpha a_{010} < 0$). By virtue of (17), we get

$$|x_1(t) - x_2(t)| \le |x_1(t) - S_m(t)| + |x_2(t) - S_m(t)| \le 2Mt^m \xi(t) < \eta t^m h \tag{45}$$

provided that $t \in (0, t(h)]$, where the constant $t(h) \in (0, \rho)$ is sufficiently small. According to (45), the integral curve $J_1\colon (t, x_1(t))$ of Eq. (22*) lies in D_3 for $t \in (0, t(h)]$. If t increases from $t = t(h)$ to $t = \rho$, then, as shown above, the integral curve $J_1\colon (t, x_1(t))$ cannot have common points with Φ_3. Therefore, this integral curve lies in D_3 for all $t \in (0, \rho]$. Further, as in the case where $\alpha a_{010} > 0$, we successively obtain estimates (40)–(44).

The reasoning presented above is independent of the choice of the functions $u_i \in U$, $i \in \{1, 2\}$. Hence, $T\colon U \to U$ is a contraction operator.

Thus, the contraction operator $T\colon U \to U$ maps a closed and bounded set U into itself. By virtue of the Banach principle of contraction mappings, this operator has a unique fixed point in U, i.e., there exists a unique element $x_0 \in U$ for which equality (39) is true. It is obvious that the function $x_0\colon (0,\rho] \to \mathbb{R}$ is a unique solution of problem (1), (2) from the set $U(\rho, M)$. Furthermore, if we analyze the procedure of construction of the operator $T\colon U \to U$ in the case $\alpha a_{010} > 0$, then we conclude the point $G(\rho, x_G)$ of the set H is chosen arbitrarily. Hence, if β is any constant satisfying condition (13), then we can take the point $G(\rho, \beta)$ as the fixed point $G(\rho, x_G)$. Then the obtained (unique) solution $x\colon (0,\rho] \to \mathbb{R}$ of problem (1), (2) satisfies the condition $x_0(\rho) = \beta$.

Theorem 2 is proved.

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