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# SMOOTH CONTACT OF A SEMIINFINITE PUNCH WITH ROUNDED EDGE AND AN ELASTIC STRIP

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We consider a problem of contact of an elastic strip with a semiinfinite punch with rounded edge indented into one face of the strip, while the other face of the strip is fixed. Friction forces in the contact zone are neglected. By the Wiener–Hopf method, we obtain the exact analytic solution of the problem. The distributions of contact stresses, the stresses inside the strip and along its fixed face, and the normal displacements of points of a part of the load-free face of the strip are determined. We construct the isochores and determine the position of the point at which the maximum values of the principal shear stresses are attained.

Contact problems of the theory of elasticity for a strip in the case where a semiinfinite punch with rectilinear horizontal base is indented into a face of the strip, while the other face of the strip is under the conditions of smooth contact with the rigid base or is fixed were considered in [1, 2, 6]. The exact analytic solutions of these problems were found by the Wiener–Hopf method.

In what follows, we construct an analytic solution of the contact problem for an elastic strip in the case where the edge of a punch is rounded. The influence of rounding of the edge of a punch on the distribution of contact stresses in contact problems posed for an elastic half space was investigated in [8, 9].

## 1. Statement of the Problem

Consider the state of plane deformation in an elastic strip $-\infty < x < \infty$, $-h \le y \le h$ of width $2h$ with Poisson's ratio $\nu$ and shear modulus $G$ (Fig. 1). The bottom face $y = -h$, $-\infty < x < \infty$, of the strip is rigidly fixed and a semiinfinite punch with rectilinear base and rounded edge is indented into the top face of the strip $y = h$ within the interval $0 \le x < \infty$ under the action of a uniformly distributed normal load with intensity $p$. The remaining part $-\infty < x < 0$ of the face $y = h$ of the strip is free of loads. We neglect the friction forces acting in the contact zone $0 \le x < \infty$, $y = h$, of the punch and the strip.

The boundary conditions of the problem are as follows:

$$u_y\big|_{y=h} = f_0(x) H(\ell - x) - \delta, \quad 0 \le x < \infty,$$

$$\sigma_y\big|_{y=h} = 0, \quad -\infty < x < 0, \tag{1}$$

$$\tau_{yx}\big|_{y=h} = 0, \quad u_x\big|_{y=-h} = 0, \quad u_y\big|_{y=-h} = 0, \quad -\infty < x < \infty,$$

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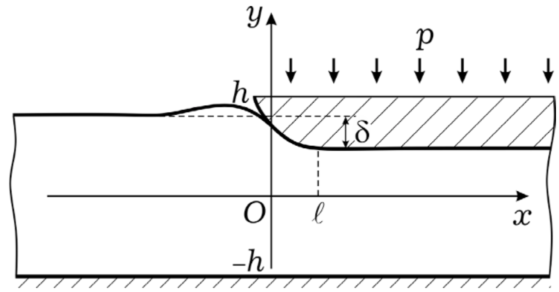


**Fig. 1**

where

$$y = f_0(x) \equiv \frac{1}{2R}(x-\ell)^2, \quad x \le \ell,$$

is the equation of the rounded edge of the punch, $R$ is the radius of curvature of the curvilinear part of the punch contour at the point $x = \ell$, $H(x)$ is a Heaviside function, and $\delta$ is the depression of the strip under the punch. In view of the conditions imposed at infinity,

$$\sigma_y = -p, \quad \varepsilon_x = 0, \quad \varepsilon_y = -\delta/(2h), \quad x \to \infty,$$

we express the depression $\delta$ in terms of pressure $p$ as follows:

$$\frac{\delta}{2h} = \frac{1-2\nu}{1-\nu}\frac{p}{2G}. \tag{2}$$

## 2. Integral Equation

We now introduce an unknown function of normal contact stresses

$$\sigma(x) = \frac{1}{2G}\sigma_y\Big|_{y=h}, \quad 0 < x < \infty, \tag{3}$$

and its Fourier transform $(\sigma(x) = 0,\ -\infty < x < 0)$

$$\tilde{\sigma}(\mu) = \frac{1}{2\pi}\int_0^\infty \sigma(r)\, e^{i\mu r}\, dr. \tag{4}$$

To deduce the integral equation, we consider the principal mixed boundary-value problem for the strip with the following conditions:

$$\frac{1}{2G}\sigma_y\Big|_{y=h} = \sigma(x), \quad \tau_{yx}\Big|_{y=h} = 0,$$
$$u_x\Big|_{y=-h} = 0, \quad u_y\Big|_{y=-h} = 0. \tag{5}$$

The general solution of the equilibrium equations for the strip has the form [5, 6]

$$u_x = \frac{\partial}{\partial x}\int_{-\infty}^{\infty} u(\mu, y)\, e^{-i\mu x}\, d\mu\,, \qquad u_y = \int_{-\infty}^{\infty} v(\mu, y)\, e^{-i\mu x}\, d\mu\,,$$

$$\frac{1}{2G}\sigma_x = \int_{-\infty}^{\infty} \sigma_1(\mu, y)\, e^{-i\mu x}\, d\mu\,, \qquad \frac{1}{2G}\sigma_y = \int_{-\infty}^{\infty} \sigma_2(\mu, y)\, e^{-i\mu x}\, d\mu\,,$$

$$\frac{1}{2G}\tau_{yx} = \frac{\partial}{\partial x}\int_{-\infty}^{\infty} \tau(\mu, y)\, e^{-i\mu x}\, d\mu\,,$$

$$\begin{aligned}
\mu^2 u(\mu, y) &= \mu A(\mu)\cosh\mu y + \mu B(\mu)\sinh\mu y \\
&\quad + C(\mu)\left[(3-4\nu)\cosh\mu y + \mu y\sinh\mu y\right] \\
&\quad - D(\mu)\left[(3-4\nu)\sinh\mu y + \mu y\cosh\mu y\right], \\
v(\mu, y) &= A(\mu)\sinh\mu y + B(\mu)\cosh\mu y + C(\mu) y\cosh\mu y - D(\mu) y\sinh\mu y, \\
\sigma_1(\mu, y) &= -\mu A(\mu)\cosh\mu y - \mu B(\mu)\sinh\mu y \\
&\quad - C(\mu)\left[(3-2\nu)\cosh\mu y + \mu y\sinh\mu y\right] \\
&\quad + D(\mu)\left[(3-2\nu)\sinh\mu y + \mu y\cosh\mu y\right], \\
\sigma_2(\mu, y) &= \mu A(\mu)\cosh\mu y + \mu B(\mu)\sinh\mu y \\
&\quad + C(\mu)\left[(1-2\nu)\cosh\mu y + \mu y\sinh\mu y\right] \\
&\quad - D(\mu)\left[(1-2\nu)\sinh\mu y + \mu y\cosh\mu y\right], \\
\mu\tau(\mu, y) &= \mu A(\mu)\sinh\mu y + \mu B(\mu)\cosh\mu y \\
&\quad + C(\mu)\left[2(1-\nu)\sinh\mu y + \mu y\cosh\mu y\right] \\
&\quad - D(\mu)\left[2(1-\nu)\cosh\mu y + \mu y\sinh\mu y\right],
\end{aligned} \tag{6}$$

where $A(\mu)$, $B(\mu)$, $C(\mu)$, and $D(\mu)$ are arbitrary functions.

Substituting solution (6) in the boundary conditions (5), we get the following system of linear algebraic equations:

$$\sigma_2(\mu, h) = \tilde{\sigma}(\mu)\,, \qquad \tau(\mu, h) = 0\,, \qquad u(\mu, -h) = 0\,, \qquad v(\mu, -h) = 0 \tag{7}$$

for the unknowns $A(\mu)$, $B(\mu)$, $C(\mu)$, and $D(\mu)$. Hence, we find

$$\begin{aligned}
A(\mu) &= \{2(1-\nu)(3-4\nu)\cosh 3\mu h + [(2\mu h)^2 + 2(1-\nu)(3-4\nu)]\cosh \mu h \\
&\quad + 2\mu h[(3-4\nu)(\cosh 2\mu h + 1) + 2(1-\nu)]\sinh \mu h\}\tilde{\sigma}(\mu)(\mu\Delta(2\mu h))^{-1}, \\
B(\mu) &= \{2(1-\nu)(3-4\nu)\sinh 3\mu h - [(2\mu h)^2 + 2(1-\nu)(3-4\nu)]\sinh \mu h \\
&\quad + 2\mu h[(3-4\nu)(\cosh 2\mu h - 1) - 2(1-\nu)]\cosh \mu h\}\tilde{\sigma}(\mu)(\mu\Delta(2\mu h))^{-1}, \\
C(\mu) &= -[(3-4\nu)\cosh 3\mu h + \cosh \mu h + 4\mu h \sinh \mu h]\tilde{\sigma}(\mu)(\Delta(2\mu h))^{-1}, \\
D(\mu) &= [(3-4\nu)\sinh 3\mu h - \sinh \mu h - 4\mu h \cosh \mu h]\tilde{\sigma}(\mu)(\Delta(2\mu h))^{-1}, \\
\Delta(2\mu h) &= (3-4\nu)(\cosh 4\mu h - 1) + 2(2\mu h)^2 + 8(1-\nu)^2 .
\end{aligned} \tag{8}$$

Relations (6) and (8) yield, in particular, the expressions for the normal displacements at the points of the top face of the strip

$$\begin{aligned}
u_y\Big|_{y=h} &= 2h\int_{-\infty}^{\infty} \mathcal{K}(2\mu h)\,\tilde{\sigma}(\mu)e^{-i\mu x}\,d\mu, \\
\mathcal{K}(2\mu h) &= \frac{\lambda(2\mu h)}{2\mu h\Delta(2\mu h)}, \\
\lambda(2\mu h) &= 2(1-\nu)[(3-4\nu)\sinh 4\mu h - 4\mu h].
\end{aligned} \tag{9}$$

Substituting the expression for $u_y\Big|_{y=h}$ from (9) in the first boundary condition in (1), we obtain the following equation:

$$2h\int_{-\infty}^{\infty} \mathcal{K}(2\mu h)\,\tilde{\sigma}(\mu)\,e^{-i\mu x}\,d\mu = f_0(x)H(\ell - x) - \delta, \quad 0 \le x < \infty. \tag{10}$$

In view of the behavior of the unknown function of contact stresses $\sigma(x)$ from (3) at infinity,

$$(\sigma(x) \sim -p/(2G),\ x \to \infty),$$

we can represent it as follows:

$$\sigma(x) = -\frac{p}{2G} + \sigma_*(x), \quad \sigma_*(\infty) = 0. \tag{11}$$

Then

$$\tilde{\sigma}(\mu) \;=\; -\frac{p}{4G}\left(\delta(\mu)+\frac{i}{\pi\mu}\right)+\frac{1}{2\pi}\int_0^\infty \sigma_*(r)\,e^{i\mu r}\,dr\,, \tag{12}$$

where $\delta(\mu)$ is the Dirac delta-function.

By the change of variables

$$x \;=\; 2h\xi, \qquad r \;=\; 2h\eta, \qquad \tau \;=\; 2\mu h, \qquad a \;=\; \ell/(2h), \tag{13}$$

we pass to the following new unknown function:

$$\varphi(\xi) \;=\; -\sqrt{2\pi}\,\frac{2h}{\delta}\,\sigma_*(2h\xi), \quad 0\le\xi<\infty. \tag{14}$$

As a result, we transform Eq. (10) into the integral equation

$$\int_0^\infty k(\xi-\eta)\,\varphi(\eta)\,d\eta \;=\; f(\xi), \quad 0\le\xi<\infty, \tag{15}$$

where

$$k(\xi-\eta) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{K}(\tau)\,e^{-i\tau(\xi-\eta)}\,d\tau,$$

$$f(\xi) \;=\; \sqrt{2\pi}\,\delta^{-1}\left(-f_0(2h\xi)H(a-\xi)+\frac{hp}{G}\sum_{k=1}^{\infty}\frac{\lambda(is_k)}{s_k^2\Delta'(is_k)}\,e^{-s_k\xi}\right);$$

here, $s_k$, $k=1,2,\ldots,$ are the roots of the equation $\Delta(is)=0$ from the half plane $\operatorname{Re} s>0$. The function $f(\xi)$ and the right-hand side of Eq. (15) have been transformed by using the residue theory.

## 3. Solution of the Integral Equation by the Wiener–Hopf Method [3]

We now extend the integral equation (15) to the entire real axis by assuming that $\varphi(\eta)=0$ for $\eta<0$ and apply the Fourier integral transformation to this equation. We now introduce the following unknown functions of the complex variable $z$:

$$\Phi^+(z) \;=\; \frac{1}{\sqrt{2\pi}}\int_0^\infty \varphi(\xi)\,e^{iz\xi}\,d\xi,$$

$$\Phi^-(z) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{iz\xi}\,d\xi\int_0^\infty k(\xi-\eta)\,\varphi(\eta)\,d\eta, \tag{16}$$

analytic in the half planes $\operatorname{Im} z>c^+$ and $\operatorname{Im} z<c^-$ $(c^+<0,\ c^->0)$, respectively.

By using the theorem on convolution for the Fourier integral transformation and representations (16), we reduce the integral equation (15) to the following functional equation:

$$\mathcal{K}(z)\Phi^+(z) - \Phi^-(z) = F^+(z), \qquad c^+ < \operatorname{Im} z < c^- . \tag{17}$$

The right-hand side of this equation

$$F^+(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(\xi) e^{iz\xi} d\xi$$

$$= \frac{1}{\delta}\left[ \frac{(2h)^2}{2R}\left( \frac{a^2}{iz} + \frac{2a}{(iz)^2} - \frac{2}{(iz)^3}(e^{iza} - 1) \right) - \frac{hp}{G}\sum_{k=1}^\infty \frac{\lambda(is_k)}{s_k^2\Delta'(is_k)}\frac{1}{iz - s_k} \right] \tag{18}$$

is an analytic function in the half plane $\operatorname{Im} z > c^+$.

We now represent the coefficient $\mathcal{K}(z)$ in Eq. (17) in the form of an infinite product

$$\mathcal{K}(z) = \mathcal{K}(0)\mathcal{K}^+(z)\mathcal{K}^-(z), \qquad \mathcal{K}(0) = \frac{1-2\nu}{1-\nu},$$
$$\mathcal{K}^+(z) \equiv \mathcal{K}^-(-z) = \prod_{k=1}^\infty \left(1 - \frac{iz}{\zeta_k}\right)\left(1 - \frac{iz}{s_k}\right)^{-1}, \tag{19}$$

where $\zeta_k$, $k = 1, 2, \ldots$, are the roots of the equation $\lambda(is) = 0$ from the half plane $\operatorname{Re} s > 0$.

We divide Eq. (17) by $\mathcal{K}^-(z)$ and represent the obtained right-hand side in the form

$$\frac{F^+(z)}{\mathcal{K}^-(z)} = f^+(z) - f^-(z), \qquad c^+ < \operatorname{Im} z < c^- , \tag{20}$$

where $f^+(z)$ and $f^-(z)$ are functions analytic in the half planes $\operatorname{Im} z > c^+$ and $\operatorname{Im} z < c^-$, respectively. We represent the function $f^+(z)$ in the form of a Cauchy-type integral as follows:

$$f^+(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F^+(\zeta)\, d\zeta}{\mathcal{K}^-(\zeta)(\zeta - z)} = f_1^+(z) + f_2^+(z), \quad \operatorname{Im} z > 0,$$

$$f_1^+(z) = \frac{1}{\delta}\frac{(2h)^2}{2R}\frac{\mathcal{K}(0)}{2\pi i}\int_{-\infty}^{\infty} \frac{\zeta\Delta(\zeta)\mathcal{K}^+(\zeta)}{\lambda(\zeta)(\zeta - z)}\left( \frac{a^2}{i\zeta} + \frac{2a}{(i\zeta)^2} - \frac{2}{(i\zeta)^3}(e^{i\zeta a} - 1) \right) d\zeta, \tag{21}$$

$$f_2^+(z) = -\frac{1}{\delta}\frac{hp}{G}\frac{\mathcal{K}(0)}{2\pi i}\int_{-\infty}^{\infty} \frac{\zeta\Delta(\zeta)\mathcal{K}^+(\zeta)}{\lambda(\zeta)(\zeta - z)}\sum_{k=1}^\infty \frac{\lambda(is_k)}{s_k^2\Delta'(is_k)}\frac{1}{i\zeta - s_k}\, d\zeta.$$

We find the integral for $f_1^+(z)$ via the residues of the integrand at its simple poles $\zeta = i\zeta_k$, $k = 1, 2, \ldots$, $\zeta = z$ from the half plane $\operatorname{Im}\zeta > 0$ in which $e^{i\zeta a} \to 0$ for $|\zeta| \to \infty$. Therefore,

$$f_1^+(z) = -\frac{1}{\delta}\frac{(2h)^2}{2R}\mathcal{K}(0)\left[\sum_{k=1}^{\infty}\frac{i\zeta_k\mathcal{K}^+(i\zeta_k)\Delta(i\zeta_k)}{\lambda'(i\zeta_k)(i\zeta_k - z)}\left(\frac{a^2}{\zeta_k} - \frac{2a}{\zeta_k^2} - \frac{2}{\zeta_k^3}(e^{-\zeta_k a} - 1)\right)\right.$$
$$\left. + \frac{z\Delta(z)\mathcal{K}^+(z)}{\lambda(z)}\left(-\frac{a^2}{iz} - \frac{2a}{(iz)^2} + \frac{2}{(iz)^3}(e^{iza} - 1)\right)\right]. \tag{22}$$

In expansion (22), we sum the components of the series that do not contain the coefficient $e^{-\zeta_k a}$ by passing from the series in terms of residues from the half plane $\operatorname{Im}\tau > c$, $0 < c < \zeta_1$, to the integral along the straight line $\operatorname{Im}\zeta = c$. We compute this integral via the residue at the single pole $\zeta = 0$ from the half plane $\operatorname{Im}\tau < c$. Hence, we get

$$\mathcal{K}(0)\sum_{k=1}^{\infty}\frac{i\zeta_k\mathcal{K}^+(i\zeta_k)\Delta(i\zeta_k)}{\lambda'(i\zeta_k)(i\zeta_k - z)}\left(\frac{a^2}{\zeta_k} - \frac{2a}{\zeta_k^2} + \frac{2}{\zeta_k^3}\right)$$
$$= -\frac{1}{\mathcal{K}^-(z)}\left(-\frac{a^2}{iz} - \frac{2a}{(iz)^2} - \frac{2}{(iz)^3}\right)$$
$$+ \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{1}{\mathcal{K}^-(\tau)(\tau - z)}\left(-\frac{a^2}{i\tau} - \frac{2a}{(i\tau)^2} - \frac{2}{(i\tau)^3}\right)d\tau$$
$$= -\frac{1}{\mathcal{K}(-z)}\left(-\frac{a^2}{iz} - \frac{2a}{(iz)^2} - \frac{2}{(iz)^3}\right) + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3}. \tag{23}$$

Here,

$$a_1 = i(a^2 + 2ab_1 + 2b_2), \qquad a_2 = 2(a + b_1), \qquad a_3 = -2i,$$

$$\frac{1}{\mathcal{K}^-(z)} = 1 + b_1 iz + b_2(iz)^2 + O((iz)^3), \qquad z \to 0,$$

$$b_1 = \sum_{k=1}^{\infty}\left(\frac{1}{s_k} - \frac{1}{\zeta_k}\right), \tag{24}$$

$$b_2 = \sum_{k=1}^{\infty}\left(\frac{1}{s_k} - \frac{1}{\zeta_k}\right)\left[\sum_{m=k+1}^{\infty}\left(\frac{1}{s_m} - \frac{1}{\zeta_m}\right) - \frac{1}{\zeta_k}\right]$$
$$= \frac{1}{2}\left(b_1^2 - \frac{\nu(1 - 4\nu)}{3(1 - \nu)(1 - 2\nu)}\right).$$

We transform the integral for $f_2^+(z)$ with the help of residues at the points $\zeta = -is_k$, $k = 1, 2, \ldots,$ from the half plane $\operatorname{Im}\zeta < 0$ into a series

$$f_2^+(z) = \frac{i}{\delta}\frac{hp}{G}\sum_{k=1}^{\infty}\frac{\lambda(is_k)}{s_k^2\Delta'(is_k)}\frac{1}{\mathcal{K}^-(-is_k)(z+is_k)} \tag{25}$$

and find the sum of this series by analogy with (23). We find

$$f_2^+(z) = \frac{i}{\delta}\frac{hp}{G}\mathcal{K}(0)\frac{1-\mathcal{K}^+(z)}{z}. \tag{26}$$

As a result, we obtain

$$f^+(z) = -\frac{1}{\delta}\frac{(2h)^2}{2R}\left(2\mathcal{K}(0)\sum_{k=1}^{\infty}\frac{\mathcal{K}^+(i\zeta_k)\Delta(i\zeta_k)}{i\zeta_k^2\lambda'(i\zeta_k)(i\zeta_k - z)}e^{-\zeta_k a}\right.$$

$$\left.+\frac{1}{\mathcal{K}^-(z)}\frac{2}{(iz)^3}e^{iza}+\frac{a_1}{z}+\frac{a_2}{z^2}+\frac{a_3}{z^3}\right)+\frac{i}{\delta}\frac{hp}{G}\mathcal{K}(0)\frac{1-\mathcal{K}^+(z)}{z}. \tag{27}$$

The operations performed according to relations (19) and (20) enable us to rewrite Eq. (17) in the form

$$\mathcal{K}(0)\mathcal{K}^+(z)\Phi^+(z) - f^+(z) = \frac{\Phi^-(z)}{\mathcal{K}^-(z)} - f^-(z), \qquad c^+ < \operatorname{Im} z < c^-. \tag{28}$$

We place the functions analytic in the half plane $\operatorname{Im} z > c^+$ on the left-hand side of Eq. (28) and the functions analytic in the half plane $\operatorname{Im} z < c^-$ on its right-hand side. In view of the fact that the half planes have the common band of analyticity $c^+ < \operatorname{Im} z < c^-$, the left- and right-hand sides of Eq. (28) serve as analytic extensions of each other to the entire complex plane and determine an entire function $P(z)$. In view of the estimates

$$\mathcal{K}^+(z) = O(z^{-1/2}), \quad \Phi^+(z) = o(1), \quad f^+(z) = O(z^{-1}), \quad |z| \to \infty,$$

we determine the asymptotic behavior of the function $P(z)$ as $|z| \to \infty$: $P(z) = o(z^{-1/2})$. Hence, we conclude that $P(z) \equiv 0$. Thus, we get the following solution of the functional equation (17):

$$\Phi^+(z) = \frac{f^+(z)}{\mathcal{K}(0)\mathcal{K}^+(z)}, \qquad \Phi^-(z) = f^-(z)\mathcal{K}^-(z). \tag{29}$$

According to the Watson lemma [7], it follows from the behavior of the function $\Phi^+(z) = O(z^{-1/2})$ as $|z| \to \infty$ and relations (3), (11), (14), and (16) that the contact stresses $\sigma_y\big|_{y=h}$ are unbounded at the end of the contact zone $(x \to +0)$. We demand that the stresses must be bounded at the point $x = 0$, $y = h$ and impose the condition

$$\lim_{|z|\to\infty} z f^{+}(z) = 0 .$$

In view of (27), this condition can be represented in the form

$$\mathcal{K}(0) - 2\frac{hG}{pR}\left[ 2\mathcal{K}(0)\sum_{k=1}^{\infty} \frac{\Delta(i\zeta_k)\mathcal{K}^{+}(i\zeta_k)}{\zeta_k^2\lambda'(i\zeta_k)}\, e^{-\zeta_k a} + a_1 \right] = 0 . \tag{30}$$

Equation (30) connects the relative size $a = \ell/(2h)$ of the curvilinear part of the punch base operating in contacts with the elastic strip with the cumulative parameter $hG/(pR)$ of the problem. Under condition (30), the behavior of contact stresses at the end of the contact zone can be described as follows:

$$\sigma_y\Big|_{y=h} = O\left(\sqrt{x}\right), \quad x \to +0 .$$

Applying the inverse Fourier transformation to the first equality of (16) and taking into account (29), we obtain the solution of the integral equation (15) in the form

$$\varphi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi^{+}(\tau) e^{-i\tau\xi}\, d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{f^{+}(\tau)}{\mathcal{K}(0) K^{+}(\tau)}\, e^{-i\tau\xi}\, d\tau . \tag{31}$$

## 4. Determination of Stresses and Displacements

We now find the contact stresses by using relations (3) and (31) with the help of relations (11) and (14). As a result of the transformation of the integral from (31) according to the residue theory, we get

$$\begin{aligned}
\frac{1}{p}\sigma_y\Bigg|_{y=h} = {} & -1 - \frac{1}{\mathcal{K}(0)}\frac{2hG}{pR}\left( \frac{2\nu(1-4\nu)}{3(1-\nu)(1-2\nu)} - (\xi - a)^2 \right) H\,(a - \xi) \\
& - \sum_{k=1}^{\infty} \frac{\zeta_k \Delta(i\zeta_k)}{\lambda'(i\zeta_k)} \mathcal{K}^{+}(i\zeta_k) e^{-\zeta_k \xi} \left\{ \frac{4hG}{pR}\left[ \frac{1}{\zeta_k^3 \mathcal{K}^{+}(i\zeta_k)} \right.\right. \\
& \times \operatorname{sgn}(\xi - a)\, e^{\zeta_k(\xi - |\xi - a|)} - \left( \frac{1}{\zeta_k^3} + \frac{a_2}{2\zeta_k^2} + \frac{a_1}{2\zeta_k} \right) \\
& \left.\left. - \mathcal{K}(0) \sum_{n=1}^{\infty} \frac{\Delta(i\zeta_n)\mathcal{K}^{+}(i\zeta_n)}{\zeta_n^2 \lambda'(i\zeta_n)(\zeta_n + \zeta_k)}\, e^{-\zeta_n a} \right] + \mathcal{K}(0)\frac{1}{\zeta_k} \right\},
\end{aligned} \tag{32}$$

$$\xi = \frac{x}{2h}, \quad 0 < x < \infty .$$

In view of expression (9) and equalities (12)–(14), and (16), the normal displacements of points of the load-free part of the top face of the strip can be transformed as follows:

$$u_y\Big|_{y=h} = -\frac{\delta}{2\pi}\int_{-\infty}^{\infty}\mathcal{K}(\tau)\Phi^+(\tau)e^{-i\tau\xi}\,d\tau - \frac{hp}{2G}\int_{-\infty}^{\infty}\mathcal{K}(\tau)\left[\delta(\tau)+\frac{i}{\pi\tau}\right]e^{-i\tau\xi}\,d\tau. \tag{33}$$

Substituting the solution $\Phi^+(\tau)$ from (29) in (33), after necessary transformations, we get

$$\frac{1}{\delta}u_y\Big|_{y=h} = \sum_{k=1}^{\infty}\frac{\lambda(is_k)}{s_k\Delta'(is_k)\mathcal{K}^+(is_k)}e^{s_k\xi}\left\{\frac{4h^2}{R\delta}\left[\frac{\Delta(i\zeta_n)\mathcal{K}^+(i\zeta_n)}{\zeta_n^2\lambda'(i\zeta_n)(s_k-\zeta_n)}e^{-\zeta_n a}\right.\right.$$

$$\left.\left.+\frac{1}{\mathcal{K}(0)}\left(\frac{1}{s_k^3}-\frac{a_2}{2s_k^2}+\frac{a_1}{2s_k}\right)\right]-\frac{1}{\mathcal{K}(0)s_k}\right\},$$

$$\xi = \frac{x}{2h}, \quad -\infty < x < 0\,. \tag{34}$$

By using representations (6) and (8), we write the stresses at any point of the strip in the form of the following expressions:

$$\frac{1}{2G}(\sigma_x+\sigma_y) = \int_{-\infty}^{\infty}\left\{[D(\mu)-C(\mu)]e^{\mu y}-[C(\mu)+D(\mu)]e^{-\mu y}\right\}e^{-i\mu x}\,d\mu\,,$$

$$\frac{1}{2G}(\sigma_y-\sigma_x+2i\tau_{yx}) = 2\int_{-\infty}^{\infty}\left\{\mu[A(\mu)+B(\mu)]e^{\mu y}\right.$$

$$\left.+[2(1-\nu)+\mu y][C(\mu)-D(\mu)]e^{\mu y}\right\}e^{-i\mu x}\,d\mu. \tag{35}$$

Hence, in view of equalities (8), (12)–(14), (16), and (29), we obtain

$$\frac{1}{p}(\sigma_x+\sigma_y) = -\frac{1}{2(1-\nu)}+\frac{1}{\pi}\int_{-\infty}^{\infty}M_1(\tau,\zeta)\,e^{-i\xi\tau}\,d\tau, \qquad \zeta = \frac{y}{2h}, \quad -h\le y\le h\,,$$

$$\frac{1}{p}(\sigma_y-\sigma_x+2i\tau_{yx}) = -\frac{1}{2}\mathcal{K}(0)+\frac{1}{\pi}\int_{-\infty}^{\infty}M_2(\tau,\zeta)\,e^{-i\xi\tau}\,d\tau\,, \tag{36}$$

where

$$M_j(\tau,\zeta) = \frac{\alpha_j(\tau,\zeta)}{\Delta(\tau)\mathcal{K}^+(\tau)}\left(\frac{1}{i\tau}-f_1^+(\tau)\right), \quad j = 1,2\,,$$

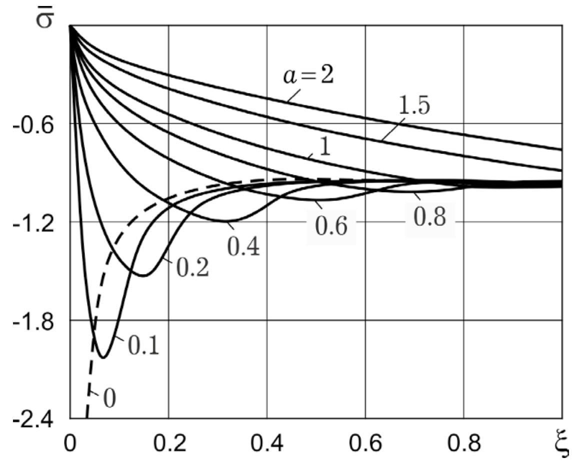


**Fig. 2**

$$\alpha_1(\tau,\zeta) = (3-4\nu)\cosh\left(\frac{3}{2}+\zeta\right)\tau + \cosh\left(\zeta-\frac{1}{2}\right)\tau - 2\tau\sinh\left(\zeta-\frac{1}{2}\right)\tau,$$

$$\alpha_2(\tau,\zeta) = (3-4\nu)\tau\left(\frac{1}{2}-\zeta\right)e^{(3/2+\zeta)\tau}$$

$$+\left[4(1-\nu)(1-2\nu)+\tau\left(\frac{1}{2}-\zeta\right)+2\tau^2\left(\frac{1}{2}+\zeta\right)\right]e^{(\zeta-1/2)\tau}.$$

Note that the integrals from (36) are exponentially convergent for $-h \le y < h$ and slowly convergent (with a power rate) for $y = h$. In this case, it is necessary to transform the integrals into series according to the residue theory, as in the case of contact stresses (32). The convergence of these series is exponential except the point $x = 0$.

## 5. Numerical Results

We performed calculations for Poisson's ratio $\nu = 1/3$ and different values of the relative size $a = \ell/(2h)$ of the curvilinear part of the punch base operating in contact with the elastic strip.

In Fig. 2, the distributions of the dimensionless contact stresses

$$\bar{\sigma} = \frac{1}{p}\sigma_y\Big|_{y=h}$$

given by relation (32) for different values of $a$ are displayed by the solid curves. As the parameter $a$ decreases from $a = 0.8$ to $a = 0.1$, the contact stresses approach their limit distribution for $a = 0$ (the dashed curve), which corresponds to a punch with rectilinear base without rounding [6]. As the parameter $a$ increases from $a = 1.0$ to $a = 2.0$, the contact stresses insignificantly differ from the contact stresses obtained in the case of a parabolic punch with finite contact zone $0 \le x \le 2\ell$ [4].

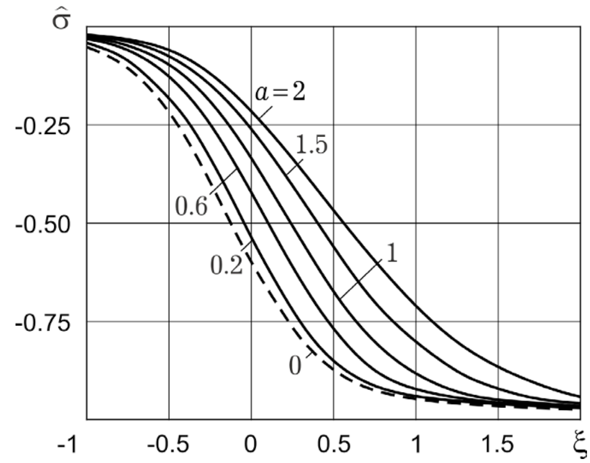


**Fig. 3**

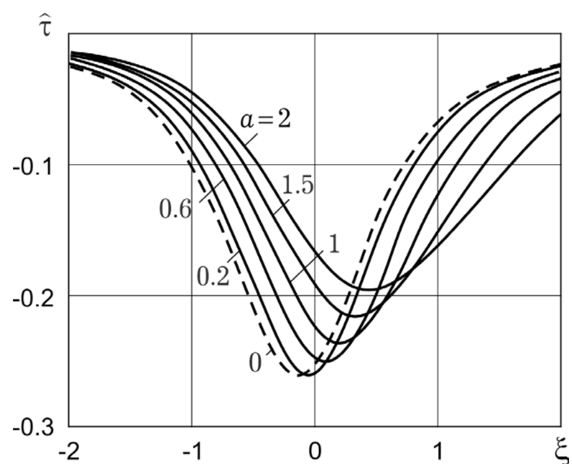


**Fig. 4**

In Figs. 3 and 4, we present the distributions of the dimensionless normal

$$\hat{\sigma} = \frac{1}{p}\sigma_y\Big|_{y=-h}$$

and tangential

$$\hat{\tau} = \frac{1}{p}\tau_{yx}\Big|_{y=-h}$$

stresses along the fixed face of the strip for the following values of the parameter:

$$a = 0,\ 0.2,\ 0.6,\ 1.0,\ 1.5,\ \text{and}\ \ 2.0.$$

In Fig. 5, we display the distributions of dimensionless tangential stresses inside the strip along the straight lines parallel to the faces of the strip for $a = 0.2$.

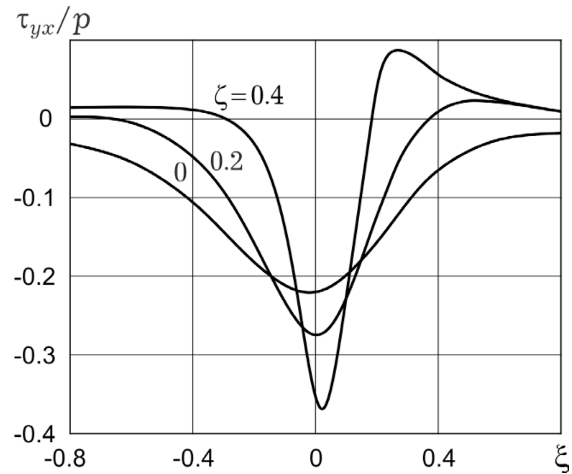


**Fig. 5**

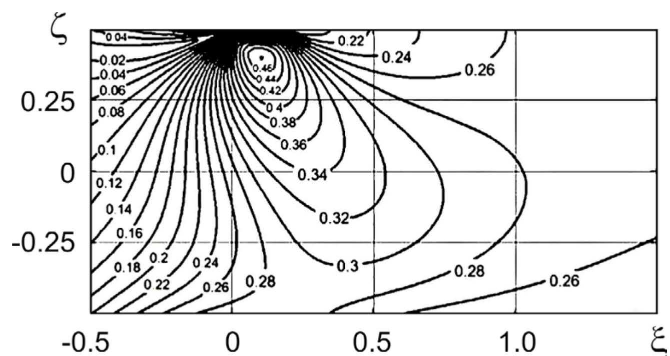


**Fig. 6**

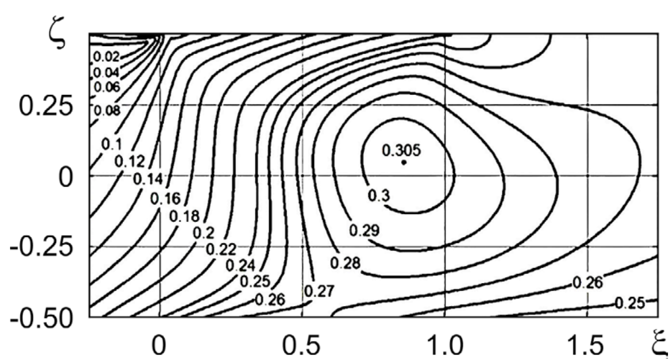


**Fig. 7**

In Figs. 6 and 7, we show the isochores, i.e., level lines of the principal shear stresses (related to the value of $p$):

$$\tau_{\max} = \frac{1}{2}\left| \sigma_y - \sigma_x + 2i\tau_{xy} \right|,$$

**Table 1**

| $\max\bar{\tau}_{\max}$ | 0.479 | 0.370 | 0.332 | 0.315 | 0.305 | 0.289 | 0.279 | 0.272 | 0.268 |
|---|---|---|---|---|---|---|---|---|---|
| $a$ | 0.2 | 0.4 | 0.6 | 0.8 | 1 | 1.5 | 2 | 2.5 | 3 |
| $\xi$ | 0.105 | 0.266 | 0.464 | 0.672 | 0.857 | 1.310 | 1.783 | 2.265 | 2.750 |
| $\zeta$ | 0.398 | 0.298 | 0.192 | 0.096 | 0.048 | 0.034 | 0.067 | 0.103 | 0.127 |
| $\alpha$ | 2.841 | 0.940 | 0.480 | 0.294 | 0.199 | 0.096 | 0.056 | 0.037 | 0.026 |

**Table 2**

| $\bar{u}_{\max}$ | 0.433 | 0.428 | 0.423 | 0.417 | 0.410 | 0.403 | 0.395 | 0.387 | 0.379 |
|---|---|---|---|---|---|---|---|---|---|
| $-\xi$ | 0.954 | 0.924 | 0.897 | 0.872 | 0.85 | 0.831 | 0.814 | 0.798 | 0.785 |
| $a$ | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |

for $a = 0.2$ and $a = 1$, respectively. If $a = 0.2$, then the value of

$$\bar{\tau}_{\max} = \frac{1}{p}\tau_{\max}$$

attains its maximum value $\max\bar{\tau}_{\max} = 0.479$ at the point $\xi = x/(2h) = 0.105$, $\zeta = y/(2h) = 0.398$. If $a = 1$, then the value $\max\bar{\tau}_{\max} = 0.305$ is attained at the point $\xi = 0.857$, $\zeta = 0.048$.

The values of $\max\bar{\tau}_{\max}$ for different $a$, together with the dimensionless coordinates $\xi, \zeta$ of the corresponding points and the values of the cumulative parameter $\alpha = hG/(pR)$, are presented in Table 1.

In finding the normal displacements at points of the free part of the top face of the strip according to relations (34), as in the case of a punch without rounding [6], it is established that, at a certain distance from the end of the contact zone, the deformed boundary of the strip elevates and forms a hill. In Table 2, we present the maximum relative values of displacements $\bar{u}_{\max} = 10\max u_y/\delta$ and corresponding dimensionless coordinates $\xi$ of the top face of the strip, which determine the height of the hill and the position of its vertex.

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