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STABILITY OF ONE-PHASE STATES IN VARIATIONAL PROBLEM OF THE THEORY OF ELASTICITY IN TWO-PHASE MEDIA. THE MULTI-DIMENSIONAL CASE

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We study the phenomenon of loss of stability of one-phase critical points of the energy functional of a two-phase elastic medium under perturbations of the temperature. Bibliography: 1 *title.*

1 Introduction

This section contains necessary preliminary information (cf. details in the survey [1]). In quadratic approximation, the strain energy density of an elastic medium is given by

$$F^{\pm}(M) = \langle A^{\pm}(e(M) - \zeta^{\pm}), e(M) - \zeta^{\pm} \rangle, \tag{1.1}$$

where $M \in R^{m\times m}$ is an $m\times m$ matrix, $m \geqslant 1$, $e(M) = 1/2(M+M^*)$, $\zeta^{\pm} \in R_s^{m\times m}$ are symmetric $m \times m$ matrices, $A^{\pm} : R_s^{m\times m} \to R_s^{m\times m}$ are linear mappings that are symmetric and positive definite with respect to the inner product $\langle \alpha, \beta \rangle = \operatorname{tr} \alpha\beta$, $\alpha, \beta \in R_s^{m\times m}$, i.e.

$$\langle A^{\pm}\xi, \zeta \rangle = \langle \xi, A^{\pm}\zeta \rangle, \quad \nu|\xi|^2 \leqslant \langle A^{\pm}\xi, \xi \rangle \leqslant \nu^{-1}|\xi|^2, \quad |\xi|^2 = \langle \xi, \xi \rangle, \tag{1.2}$$

for all $\xi, \zeta \in R_s^{m\times m}$ and some $\nu \in (0,1)$. Based on (1.1), we introduce the strain energy functional of a two-phase medium

$$I_0[u, \chi, t] = \int\limits_{\Omega} \{\chi(F^+(\nabla u) + t) + (1-\chi)F^-(\nabla u)\}\, dx, \tag{1.3}$$

where $u = u(x)$, $x \in \Omega \subset R^m$ is an m-dimensional vector-valued function, ∇u is the matrix of coefficients $(\nabla u)_{ij} = u^i_{x_j}$, $i, j = 1, \dots, m$, $t \in R^1$, and $\chi = \chi(x)$ is a characteristic function whose support is occupied by the phase labeled by +. In applications, $u(x)$ is a displacement field, $e(\nabla u)$ is the strain tensor, $\zeta^{\pm}$ is the residual strain tensor of the corresponding phase, and t is the temperature (constant in Ω) of the two-phase medium. Let the functional (1.3) be defined on pairs of functions

$$u \in \mathbb{X} = \overset{\circ}{W}{}^1_2(\Omega, R^m), \quad \chi \in \mathbb{Z}', \tag{1.4}$$

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where $\mathbb{Z}'$ is the collection of all measurable characteristic functions. Sometimes, we indicate the dependence of the functional (1.3) and sets (1.4) on the domain Ω. Throughout the paper, Ω is assumed to be bounded.

By the *equilibrium state* of a two-phase elastic medium with fixed t we mean the solution to the variational problem

$$I_0[\widehat{u}_t, \widehat{\chi}_t, t] = \inf_{u \in \mathbb{X}, \chi \in \mathbb{Z}'} I_0[u, \chi, t], \quad \widehat{u}_t \in \mathbb{X}, \quad \widehat{\chi}_t \in \mathbb{Z}'. \tag{1.5}$$

The equilibrium state is *one-phase* if

$$\widehat{\chi}_t = \chi^+ \equiv 1 \quad \text{or} \quad \widehat{\chi}_t = \chi^- \equiv 0 \tag{1.6}$$

and *two-phase* in the opposite case. It is easy to see that the variational problems

$$I_0[\widehat{u}^+, \chi^+, t] = \inf_{u \in \mathbb{X}} I_0[u, \chi^+, t], \quad I_0[\widehat{u}^-, \chi^-, t] = \inf_{u \in \mathbb{X}} I_0[u, \chi_-, t], \quad \widehat{u}^{\pm} \in \mathbb{X}, \tag{1.7}$$

are uniquely solvable and their solutions are given by

$$\widehat{u}^+ \equiv 0, \quad \widehat{u}^- \equiv 0. \tag{1.8}$$

Therefore, one-phase equilibrium states are realized only with zero displacement field.

It is proved that for the problem (1.5) there exist the phase transition temperatures $t_{\pm} \in R^1$ independent of Ω

$$t_- \leqslant t^* \leqslant t_+, \quad t^* = -[\langle A\zeta, \zeta \rangle] \equiv -(\langle A^+\zeta^+, \zeta^+ \rangle - \langle A^-\zeta^-, \zeta^- \rangle) \tag{1.9}$$

(both identities in (1.9) are realized simultaneously) that are characterized by

$$\begin{aligned}
&\text{in the case } t_- < t_+,\\
&\text{a single solution to the problem (1.5) with } t < t_- \text{ is a pair } \widehat{u}_t = \widehat{u}^+,\ \widehat{\chi}_t = \chi^+,\\
&\text{a single solution to the problem (1.5) with } t > t_+ \text{ is a pair } \widehat{u}_t = \widehat{u}^-,\ \widehat{\chi}_t = \chi^-,\\
&\text{for } t \in (t_-, t_+) \text{ the problem (1.5) has no one-phase equilibrium state}
\end{aligned} \tag{1.10}$$

and

$$\begin{aligned}
&\text{in the case } t_{\pm} = t^*, \text{ the first two assertions in (1.10) hold,}\\
&\text{for } t = t^* \text{ the unique solution to the problem (1.5) is the pair } \widehat{u}_{t^*} \equiv 0,\ \widehat{\chi}_{t^*},\\
&\text{where } \widehat{\chi}_{t^*} \text{ is an arbitrary element of } \mathbb{Z}'.
\end{aligned} \tag{1.11}$$

In the case (1.10), for $t \in (t_-, t_+)$ the problem (1.5) can have a solution or be unsolvable. It turns out that

$$[A\zeta] \equiv (A^+\zeta^+ - A^-\zeta^-) = 0 \text{ is a criterion for equality of } t_- \text{ and } t_+. \tag{1.12}$$

We set

$$|\Omega| i(t) = \inf_{u \in \mathbb{X}, \chi \in \mathbb{Z}'} I_0[u, \chi, t, \Omega], \quad |\Omega| i_{\min}(t) = \inf_{u \in \mathbb{X}, \chi = \{\chi^+, \chi^-\}} I_0[u, \chi, t, \Omega]. \tag{1.13}$$

The functions $i(t)$ and $i_{\min}(t)$ are continuous with respect to $t \in R^1$. By (1.8), the second infimum in (1.13) has the form

$$i_{\min}(t) = \begin{cases} I_0[\widehat{u}^+, \chi^+, t] = t + \langle A^+\zeta^+, \zeta^+\rangle, & t \leqslant t^*, \\ I_0[\widehat{u}^-, \chi^-, t] = \langle A^-\zeta^-, \zeta^-\rangle, & t \geqslant t^*. \end{cases} \tag{1.14}$$

It is obvious that $i_{\min}(t) \geqslant i(t)$. The equality $i_{\min}(t_0) = i(t_0)$ for some t_0 means that the problem (1.5) with $t = t_0$ possesses the equilibrium state $\widehat{u}^+$, χ^+ for $t_0 \leqslant t^*$ and $\widehat{u}^-$, χ^- for $t_0 \geqslant t^*$. Furthermore, $(t_-, t_+) = \mathscr{K}$, where

$$\mathscr{K} = \{t \in R^1 : i_{\min}(t) > i(t)\}, \tag{1.15}$$

and the pairs $\widehat{u}^\pm$, $\chi^\pm$ for all t are critical points of the energy functional (1.3).

2 The Main Results

Since $i_{\min}(t) = i(t)$ for $t \in R^1 \setminus \mathscr{K}$,

$$\begin{aligned} &\widehat{u}^+, \chi^+ \text{ is an equilibrium state only for } t \leqslant t_-, \\ &\widehat{u}^-, \chi^- \text{ is an equilibrium state only for } t \geqslant t_+ \end{aligned} \tag{2.1}$$

(cf. (1.6), (1.8)). For fixed t a pair $\widetilde{u} \in \mathbb{X}$, $\widetilde{\chi} \in \mathbb{Z}'$ is a W_2^1-*saddle point* of the energy functional if for any $\delta > 0$ there are $v_\pm^\delta \in \mathbb{X}$ and $\psi_\pm^\delta \in \mathbb{Z}'$ such that $\|v_\pm^\delta\|_{\mathbb{X}} < \delta$, $\|\widetilde{\chi} - \psi_\pm^\delta\|_{L_1} < \delta$ and

$$\begin{aligned} I_0[\widetilde{u} + v_+^\delta, \psi_+^\delta, t] > I_0[\widetilde{u}, \widetilde{\chi}, t], \\ I_0[\widetilde{u} + v_-^\delta, \psi_-^\delta, t] < I_0[\widetilde{u}, \widetilde{\chi}, t]. \end{aligned} \tag{2.2}$$

Theorem 2.1. *If for a given t some pair $\widehat{u}^\pm$, $\chi^\pm$ is not a solution to the problem* (1.5)*, then it is a W_2^1-saddle point of the energy functional* (1.3)*.*

The pair $\widehat{u}^+$, χ^+ is not an equilibrium state for the functional (1.3) only for $t > t_-$. Theorem 2.1 asserts that the stability of this pair is lost for $t > t_-$ under small energy perturbations of $\widehat{u}^+$ and small perturbations of χ^+ in $L_1(\Omega)$.

The pair $\widehat{u}^-$, χ^- is not an equilibrium state of the functional (1.3) only for $t < t_+$. Theorem 2.1 asserts that the stability of this pair is lost for $t < t_+$ under small energy perturbations of $\widehat{u}^-$ and small perturbations of χ^- in the space $L_1(\Omega)$.

The perturbation v_-^δ providing instability can be obtained for every $\delta > 0$ in a constructive way. It satisfies the inclusion $v_-^\delta \in W_\infty^1(\Omega, R^m)$, but $\|v_-^\delta\|_{W_\infty^1}$ does not tend to zero as $\delta \to 0$. Therefore, the perturbation used in the proof of Theorem 2.1 below is not small in the W_∞^1-norm.

We try to clarify whether the pairs $\widehat{u}^\pm$, $\chi^\pm$ are stable under small perturbations of $\widehat{u}^\pm$ in the $W_\infty^1(\Omega, R^m)$-norms and perturbations of $\chi^\pm$ in $L_1(\Omega)$. It turns out that such a relaxation of perturbations of $\widehat{u}^\pm$ essentially changes the stability character for the pairs $\widehat{u}^\pm$, $\chi^\pm$.

We adapt the definition of a saddle pont to the case of perturbations of $\widehat{u}^\pm$ in W_∞^1.

For fixed t a pair $\widetilde{u} \in \mathbb{X}$, $\widetilde{\chi} \in \mathbb{Z}'$ is a W_∞^1-*saddle point* of the energy functional if for any $\delta > 0$ there are functions $v_\pm^\delta \in \mathbb{X} \cap W_\infty^1(\Omega, R^m)$, $\psi_\pm^\delta \in \mathbb{Z}'$ such that $\|v_\pm^\delta\|_{W_\infty^1} < \delta$, $\|\widetilde{\chi} - \psi_\pm^\delta\|_{L_1} < \delta$ and (2.2) holds.

Theorem 2.2. *For $t < t^*$ the pair $\widehat{u}^+$, χ^+ is a local minimum of the energy functional under small perturbations of $\widehat{u}^+$ in $W^1_\infty(\Omega, R^m)$ and any perturbations of χ^+ in $L_1(\Omega)$, whereas for $t > t^*$ it is a W^1_∞-saddle point of this functional. For $t > t^*$ the pair $\widehat{u}^-$, χ^- is a local minimum of the energy functional under small perturbations of $\widehat{u}^-$ in $W^1_\infty(\Omega, R^m)$ and any perturbations of χ^- in $L_1(\Omega)$, whereas for $t < t^*$ it is a W^1_∞-saddle point of this functional.*

In the case $t_- < t_+$, $t = t^*$, the pairs $\widehat{u}^\pm$ $\chi^\pm$ are W^1_∞-saddle points of the energy functional.

In Theorems 2.1 and 2.2, the loss of stability of one-phase states $\widehat{u}^\pm$, $\chi^\pm$ is caused (depending on the values of the temperature t) by perturbation of only distributions of phases $\chi^\pm$ or by consistent perturbation of displacement fields $\widehat{u}^\pm$ and phase distribution $\chi^\pm$.

3 Proof of the Main Results

Proof of Theorem 2.1. Since $\widehat{u}^\pm$ is a unique solution to the problems (1.7), for any t the first inequality in (2.2) holds with $\widetilde{u} = \widehat{u}^\pm$, $\widetilde{\chi} = \psi^\delta_\pm = \chi^\pm$, $v^\delta_+ \in \mathbb{X}$, $v^\delta_+ \neq 0$. Since

$$I_0[\widehat{u}^+, \chi, t] = I_0[\widehat{u}^+, \chi^+, t] - (t - t^*) \int\limits_\Omega (\chi^+ - \chi)\, dx,$$

$$I_0[\widehat{u}^-, \chi, t] = I_0[\widehat{u}^-, \chi^-, t] - (t^* - t) \int\limits_\Omega (\chi - \chi^-)\, dx,$$

for every $t > t^*$ the second inequality in (2.2) holds with $\widetilde{u} = \widehat{u}^+$, $v^\delta_- = 0$, $\psi^\delta_- = \chi$, $\widetilde{\chi} = \chi^+$ for any $\chi \neq \chi^+$, whereas for each $t < t^*$ the same inequality holds with $\widetilde{u} = \widehat{u}^-$, $v^\delta_- = 0$, $\psi^\delta_- = \chi$, $\widetilde{\chi} = \chi^-$ for any $\chi \neq \chi^-$.

It remains to consider the case $t_- < t_+$ and for any sufficiently small $\delta > 0$ establish the existence of $v^\delta_- \in \mathbb{X}$, $\psi^\delta_- \in \mathbb{Z}'$ such that

$$\begin{aligned} &I_0[\widehat{u}^+ + v^\delta_-, \psi^\delta_-, t] < I_0[\widehat{u}^+, \chi^+, t], \quad \|v^\delta_-\|_{\mathbb{X}} < \delta, \quad \|\chi^+ - \psi^\delta_-\|_{L_1} < \delta, \quad t \in (t_-, t^*],\\ &I_0[\widehat{u}^- + v^\delta_-, \psi^\delta_-, t] < I_0[\widehat{u}^-, \chi^-, t], \quad \|v^\delta_-\|_{\mathbb{X}} < \delta, \quad \|\psi^\delta_- - \chi^-\|_{L_1} < \delta, \quad t \in [t^*, t_+). \end{aligned} \tag{3.1}$$

Taking into account (1.14) and the inequality $i(t) < i_{\min}(t)$ for $t \in (t_-, t_+)$, we find $u_>, u_< \in \mathbb{X}(\Omega)$, $\chi_>, \chi_< \in \mathbb{Z}'(\Omega)$ such that

$$\begin{aligned} &I_0[u_<, \chi_<, t, \Omega] < |\Omega| i_{\min}(t) = I_0[\widehat{u}^+, \chi^+, t, \Omega], \quad t \in (t_-, t^*],\\ &I_0[u_>, \chi_>, t, \Omega] < |\Omega| i_{\min}(t) = I_0[\widehat{u}^-, \chi^-, t, \Omega], \quad t \in [t^*, t_+). \end{aligned} \tag{3.2}$$

By the strict inequality in (3.2), the functions $u_>$, $u_<$ do not vanish. By the continuity of $I_0[., \chi, t]$ in the space $\mathbb{X}$, we can assume that these functions belong to the class $C^\infty_0(\Omega, R^m)$.

For any sufficiently small $\lambda > 0$ we choose $\xi = \xi(\lambda)$ such that the domain $\Omega_\lambda = \Omega_{\xi(\lambda),\lambda}$ constructed by the rule $\Omega_{\xi,\lambda} = \{x \in R^m : x = \lambda\widetilde{x} + \xi, \widetilde{x} \in \Omega\}$, $\lambda > 0$, $\xi \in R^m$, lies, together with its closure, in Ω. We consider the pairs $u_<$, $\chi_<$ of functions $u^\lambda_<$, $\chi^\lambda_<$ defined by the rule

$$u^\lambda_<(x) = \lambda u_<(\widetilde{x}), \quad \chi^\lambda_<(x) = \chi_<(\widetilde{x}), \quad \widetilde{x} \in \Omega, \quad x \in \Omega_\lambda. \tag{3.3}$$

The change of variables yields

$$\frac{1}{|\Omega_\lambda|} I_0[u^\lambda_<, \chi^\lambda_<, t, \Omega_\lambda] = \frac{1}{|\Omega|} I_0[u_<, \chi_<, t, \Omega]. \tag{3.4}$$

We denote by $\widetilde{u}^\lambda_<$ the extension of $u^\lambda_<$ by 0 and by $\widetilde{\chi}^\lambda_<$ the extension of $\chi^\lambda_<$ by 1 from Ω_λ to Ω. Then $\widetilde{u}^\lambda_< \in \mathbb{X}(\Omega)$ and $\widetilde{\chi}^\lambda_< \in \mathbb{Z}'(\Omega)$. Using (3.4) and the first inequality in (3.2), we have

$$I_0[\widetilde{u}^\lambda_<, \widetilde{\chi}^\lambda_<, t, \Omega] = I_0[u^\lambda_<, \chi^\lambda_<, t, \Omega_\lambda] + |\Omega \setminus \Omega_\lambda|(F^+(0) + t) = \frac{|\Omega_\lambda|}{|\Omega|} I_0[u_<, \chi_<, t, \Omega] + |\Omega \setminus \Omega_\lambda| i_{\min}(t)$$

$$< |\Omega_\lambda| i_{\min}(t) + |\Omega \setminus \Omega_\lambda| i_{\min}(t) = |\Omega| i_{\min}(t) = I_0[\widehat{u}^+, \chi^+, t, \Omega], \quad t \in (t_-, t^*]$$

(the condition $t \in (t_-, t^*]$ was used to change $F^+(0) + t$ with $i_{\min}(t)$ and $|\Omega| i_{\min}(t)$ with $I_0[u^+, \chi^+, t, \Omega]$). Using the obtained inequality $I_0[\widetilde{u}^\lambda_<, \widetilde{\chi}^\lambda_<, t] < I_0[\widehat{u}^+, \chi^+, t]$ and the relations

$$\begin{aligned} \|\widetilde{u}^\lambda_<\|^2_{\mathbb{X}} &= \int_\Omega |e(\nabla \widetilde{u}^\lambda_<)|^2\,dx = \int_{\Omega_\lambda} |e(\nabla u^\lambda_<)|^2\,dx \\ &= \frac{|\Omega_\lambda|}{|\Omega_\lambda|} \int_{\Omega_\lambda} |e(\nabla u^\lambda_<)|^2\,dx = \frac{|\Omega_\lambda|}{|\Omega|} \int_\Omega |e(\nabla u_<)|^2\,dx, \\ \int_\Omega |\chi^+ - \widetilde{\chi}^\lambda_<|\,dx &= \int_{\Omega_\lambda} |\chi^+ - \chi^\lambda_<|\,dx = \frac{|\Omega_\lambda|}{|\Omega_\lambda|} \int_{\Omega_\lambda} |\chi^+ - \chi^\lambda_<|\,dx = \frac{|\Omega_\lambda|}{|\Omega|} \int_\Omega |\chi^+ - \chi_<|\,dx, \end{aligned} \tag{3.5}$$

$|\Omega_\lambda||\Omega|^{-1} = \lambda^m$, we arrive at the first estimate in (3.1) with

$$v^\delta_- = \widetilde{u}^\lambda_< - \widehat{u}^+ = \widetilde{u}^\lambda_<, \quad \psi^\delta_- = \widetilde{\chi}^\lambda_<, \quad \delta = \lambda^{\frac{m}{2}} \|u_<\|_{\mathbb{X}(\Omega)} + \lambda^m \|\chi^+ - \chi_<\|_{L_1(\Omega)}. \tag{3.6}$$

The second estimate in (3.1) is proved in a similar way, but for $\widetilde{\chi}^\lambda_>$ one should take the extension of $\chi^\lambda_>$ by 0 from Ω_λ to Ω. □

By (3.6) and (3.3), for all δ the quantity $\|\nabla v^\delta_-\|_{L_\infty(\Omega)} = \|\nabla u_<\|_{L_\infty(\Omega)}$ is a positive constant independent of δ (recall that $u_< \not\equiv 0$). Therefore, the perturbation v^δ_- constructed in the proof of Theorem 2.1 is not small in the space $W^1_\infty(\Omega, R^m)$ as $\delta \to 0$.

Proof of Theorem 2.2. Since the functional is quadratic property (cf. (1.1)), we have

$$\begin{aligned} I_0[u, \chi, t] - I_0[\widehat{u}^+, \chi^+, t] &= \frac{1}{2} F^+_{M_{ij}M_{kl}} \int_\Omega e_{ij}(\nabla u) e_{kl}(\nabla u)\,dx \\ &+ \int_\Omega (1-\chi)((t^* - t) - (F^+ - F^-)_{M_{ij}}(0) e_{ij}(\nabla u))\,dx \\ &- \frac{1}{2} \int_\Omega (1-\chi)((F^+ - F^-)_{M_{ij}M_{kl}} e_{ij}(\nabla u) e_{kl}(\nabla u))\,dx, \\ I_0[u, \chi, t] - I_0[\widehat{u}^-, \chi^-, t] &= \frac{1}{2} F^-_{M_{ij}M_{kl}} \int_\Omega e_{ij}(\nabla u) e_{kl}(\nabla u)\,dx \\ &+ \int_\Omega \chi((t - t^*) + (F^+ - F^-)_{M_{ij}}(0) e_{ij}(\nabla u))\,dx \\ &+ \frac{1}{2} \int_\Omega \chi((F^+ - F^-)_{M_{ij}M_{kl}} e_{ij}(\nabla u) e_{kl}(\nabla u))\,dx. \end{aligned} \tag{3.7}$$

Then for all $u \in \mathbb{X} \cap W^1_\infty(\Omega, R^m)$, $\chi \in \mathbb{Z}'$ and some positive constant γ from the assumption (1.2) it follows that

$$I_0[u,\chi,t] - I_0[\widehat{u}^+,\chi^+,t] \geqslant \nu \int\limits_\Omega |e(\nabla u)|^2\,dx + \int\limits_\Omega (1-\chi)((t^*-t) - \gamma(1+\|\nabla u\|_{L_\infty})\|\nabla u\|_{L_\infty})\,dx, \tag{3.8}$$

$$I_0[u,\chi,t] - I_0[\widehat{u}^-,\chi^-,t] \geqslant \nu \int\limits_\Omega |e(\nabla u)|^2\,dx + \int\limits_\Omega \chi((t-t^*) - \gamma(1+\|\nabla u\|_{L_\infty})\|\nabla u\|_{L_\infty})\,dx$$

and for all $u \in \mathbb{X}$, $\chi \in \mathbb{Z}'$

$$\begin{aligned}
&I_0[u,\chi^+,t] - I_0[\widehat{u}^+,\chi^+,t] = \frac{F^+_{M_{ij}M_{kl}}}{2}\int\limits_\Omega e_{ij}e_{kl}\,dx,\\
&I_0[\widehat{u}^+,\chi,t] - I_0[\widehat{u}^+,\chi^+,t] = |\Omega|(1-Q)(t^*-t),\\
&I_0[u,\chi^-,t] - I_0[\widehat{u}^-,\chi^-,t] = \frac{F^-_{M_{ij}M_{kl}}}{2}\int\limits_\Omega e_{ij}e_{kl}\,dx,\\
&I_0[\widehat{u}^-,\chi,t] - I_0[\widehat{u}^-,\chi^-,t] = |\Omega|Q(t-t^*),\\
&e = e(\nabla u), \quad Q = \frac{1}{|\Omega|}\int\limits_\Omega \chi\,dx.
\end{aligned} \tag{3.9}$$

From (3.7) with $t = t^*$ and $e = e(\nabla u)$ we find

$$\begin{aligned}
I_0[u,\chi,t^*] - I_0[\widehat{u}^+,\chi^+,t^*] &= \int\limits_\Omega \{\chi\langle A^+e,e\rangle + (1-\chi)\langle A^-e,e\rangle + 2(\chi^+-\chi)\langle [A\zeta],e\rangle\}\,dx,\\
I_0[u,\chi,t^*] - I_0[\widehat{u}^-,\chi^-,t^*] &= \int\limits_\Omega \{\chi\langle A^+e,e\rangle + (1-\chi)\langle A^-e,e\rangle - 2(\chi-\chi^-)\langle [A\zeta],e\rangle\}\,dx.
\end{aligned} \tag{3.10}$$

We study the stability of the pair $\widehat{u}^+$, χ^+. Let $t < t^*$. Then (3.8) implies

$$I_0[u,\chi,t] > I_0[\widehat{u}^+,\chi^+,t]$$

for $\chi \in \mathbb{Z}'$, $u \in \mathbb{X} \cap W^1_\infty(0,l)$, $t^* - t > \gamma(1+\|\nabla u\|_{L_\infty})\|\nabla u\|_{L_\infty}$, $\|\widehat{u}^+ - u\|_{\mathbb{X}} + \|\chi^+ - \chi\|_{L_1} > 0$.

Let $t > t^*$. Then (3.9) implies

$$\begin{aligned}
&I_0[u,\chi^+,t] > I_0[\widehat{u}^+,\chi^+,t] \quad \forall\, u \in \mathbb{X}, \quad u \neq \widehat{u}^+,\\
&I_0[\widehat{u}^+,\chi,t] < I_0[\widehat{u}^+,\chi^+,t] \quad \forall\, \chi \in \mathbb{Z}', \quad \chi \neq \chi^+.
\end{aligned}$$

Let $t = t^*$. Then (3.10) implies

$$I_0[u,\chi^+,t^*] > I_0[\widehat{u}^+,\chi^+,t^*] \quad \forall\, u \in \mathbb{X},\ u \neq \widehat{u}^+.$$

In the case $t_- < t_+$, we have $[A\zeta] \neq 0$ in view of (1.12). Therefore, there exists a function $u_0 \in C_0^\infty(\Omega, R^m)$, $\|u_0\|_{W^1_\infty} = 1$, such that $\langle [A\zeta], e(\nabla u_0)\rangle \not\equiv 0$. Then, based on the equality

$$\int\limits_\Omega \langle [A\zeta], e(\nabla u_0)\rangle \, dx = 0,$$

we conclude that the set $E_- = \{x \in \Omega : \langle [A\zeta], e(\nabla u_0(x))\rangle < 0\}$ has positive measure. For every $\delta > 0$ we fix a function $\chi = \psi_-^\delta \in \mathbb{Z}'$ such that $\operatorname{supp}(\chi^+ - \psi_-^\delta) \subset E_-$ and $\|\chi^+ - \psi_-^\delta\|_{L_1} < \delta$. By the first identity in (3.10), for all sufficiently small $\epsilon > 0$ we have

$$I_0[\epsilon u_0, \psi_-^\delta, t^*] - I_0[\widehat{u}^+, \chi^+, t^*] = \epsilon^2 \int\limits_\Omega \{\psi_-^\delta \langle A^+ e, e\rangle + (1 - \psi_-^\delta)\langle A^- e, e\rangle\} \, dx$$

$$+ 2\epsilon \int\limits_\Omega (\chi^+ - \psi_-^\delta)\langle [A\zeta], e\rangle \, dx < 0, \quad e = e(\nabla u_0).$$

Thus, we obtain the second inequality in (2.2) for some $\epsilon = \epsilon(\delta) \in (0, \delta)$, $\widetilde{u} = \widehat{u}^+$, $\widetilde{\chi} = \chi^+$, $v_-^\delta = \epsilon(\delta) u_0$, and the above function ψ_-^δ. The pair $\widehat{u}^-$, χ^- is considered in a similar way. □

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