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EXACT SOLUTION OF THE NAVIER–STOKES EQUATION DESCRIBING NONISOTHERMAL LARGE-SCALE FLOWS IN A ROTATING LAYER OF LIQUID WITH FREE UPPER SURFACE

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Abstract. We present an analytic representation of an exact solution of the Navier–Stokes equations that describe flows of a rotating horizontal layer of a liquid with rigid and thermally isolated bottom and a free upper surface. On the upper surface, a constant tangential stress of an external force is given, and heat emission governed by the Newton law occurs. The temperature of the medium over the surface of the liquid is a linear function of horizontal coordinates. We find the solution of the boundary-value problem for ordinary differential equations for the velocity and temperature. and examine its form depending on the Taylor, Grashof, Reynolds, and Biot numbers. In rotating liquid, the motion is helical; account of the inhomogeneity of the temperature makes the helical motion more complicated.

***Keywords and phrases*:** horizontal convection, exact solution, nonisothermal flow.

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In modern models describing the ocean circulation, the value of tangential stresses of external forces (wind) is specified on the upper boundary of the layer. In this case, large-scale boundary flow of Ekman type is formed across the layer (see [7, 9, 11]). For such flows, there exist analytic representations of infinite horizontal layers of rotating liquid with free upper surface; these representations can be used for the study of nonlinear effects in Ekman layers (see [1]) and the construction of models of two-dimensional vortex motion of a liquid within the framework of the theory of shallow water (see [2–4, 8]).

In this paper, we discuss combined large-scale flows that appear in rotating horizontal layers of a liquid whose lower boundary is rigid and thermally insulated and the upper surface is free. On the upper surface, a constant tangential stress of an external force and a linear distribution of the temperature are given.

Consider an infinite horizontal layer of an incompressible liquid with rigid boundaries $z = \pm h$, which rotates with a constant angular velocity Ω_0 about the axis that coincides with the vertical coordinate axis Oz. Assume that the lower boundary $z = -h$ is rigid and thermally insulated,

$$\vec{v} = 0, \quad \frac{\partial T}{\partial z} = 0, \tag{1}$$

where $\vec{v}(t, x, y, z) = (v_x, v_y, v_z)$ is the velocity vector and $T(t, x, y, z)$ is the temperature. On the free upper surface $z = h$, the constant tangential stress of a certain external force is given, the condition of a "rigid cover" for the vertical component of the velocity is fulfilled (see [5, 12]), and the temperature depends linearly on the horizontal coordinate x,

$$\rho_0 \nu \frac{\partial v_{x,y}}{\partial z} = \tau_{x.y}, \quad v_z = 0, \quad \frac{\partial T}{\partial z} = -\gamma_A (T - Ax), \quad A = \text{const}, \tag{2}$$

where $\vec{\tau} = (\tau_x, \tau_y)$ is the vector of tangential stresses, ρ_0 is the average density of the liquid, ν is the viscosity, γ_A is the empirical coefficient of heat exchange (see [10]), and Ax is the temperature of the

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medium above the upper surface of the liquid. The closedness condition for the flow has the form

$$\int_{-h}^{h} v_x\,dz = 0, \quad \int_{-h}^{h} v_y\,dz = 0. \tag{3}$$

The study of flows is based on the convection equations in the Boussinesq approximation in the rotating Cartesian coordinate system (see [6, 9]). The ratio of the convective force that appears due to the inhomogeneity of the density in a centrifugal force and the convective force in the gravitational field is determined by the Froude number Fr (see [6]). We consider the case where $\mathrm{Fr} = \Omega_0^2 l/g \ll 1$, where l is the characteristic horizontal scale and g is the gravitational acceleration. In this case, the effect of gravity is significant and we can neglect the influence of the centrifugal force.

As the units of lengths x, y, and z, time t, velocities v_x, v_y, and v_z, temperature T, and pressure P, respectively, we choose h, h^2/ν, ν/h, Ah, and $\rho_0\nu^2/h^2$, where $\tau_0 = \sqrt{\tau_x^2 + \tau_y^2}$, and we obtain the equations in the dimensionless form:

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} - \sqrt{\mathrm{Ta}}\cdot v_y = -\frac{\partial P}{\partial x} + \Delta v_x, \tag{4}$$

$$\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z} + \sqrt{\mathrm{Ta}}\cdot v_x = -\frac{\partial P}{\partial y} + \Delta v_y, \tag{5}$$

$$\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z} = -\frac{\partial P}{\partial z} + \Delta v_z + \mathrm{Gr}\cdot T, \tag{6}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0, \tag{7}$$

$$\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z} = \frac{1}{\mathrm{Pr}}\Delta T. \tag{8}$$

Here

$$\mathrm{Ta} = \left(\frac{2\Omega_0 h^2}{\nu}\right)^2, \quad R = \frac{\tau_0 h^2}{\rho_0\nu^2}, \quad \mathrm{Gr} = \frac{g\beta A h^4}{\nu^2}, \quad \mathrm{Pr} = \frac{\nu}{\chi},$$

are the Taylor, Reynolds, Grashof, and Prandtl numbers, respectively, β is the thermal expansion coefficient (see [5]), χ is the thermal diffusivity, and

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the Laplace operator.

The boundary conditions for the dimensionless velocity and temperature with account of (1)–(3) take the following form:

(i) for $z = 1$

$$\frac{\partial v_x}{\partial z} = R\cos\alpha, \quad \frac{\partial v_y}{\partial z} = R\sin\alpha, \quad v_z = 0, \quad \frac{\partial T}{\partial z} = -\,\mathrm{Bi}(T - Ax); \tag{9}$$

(ii) for $z = -1$

$$v_x = v_y = v_z = 0, \quad \frac{\partial T}{\partial z} = 0, \tag{10}$$

where

$$\cos\alpha = \frac{\tau_x}{\sqrt{\tau_x^2 + \tau_y^2}}, \quad \sin\alpha = \frac{\tau_y}{\sqrt{\tau_x^2 + \tau_y^2}};$$

the angle α determines the direction of the vector of tangential stresses in the coordinate system chosen, and $\mathrm{Bi} = \gamma_A h$ is the Biot number.

Taking into account the boundary condition (9)–(10) and the incompressibility condition (7), we search for a solution of the system (4)–(8) in the following form:

$$v_x = u_0(z), \quad v_y = v_0(z), \quad v_z = 0, \quad T = x + \theta_0(z), \quad P = p_0(x, y, z). \tag{11}$$

Substituting the formulas (11) in the system (4)–(8), we obtain a system of equations for the velocity, temperature, and pressure. Introduce the complex-valued function of the velocity

$$M(z) = u_0(z) + iv_0(z),$$

where $i = \sqrt{-1}$, and introduce the notation

$$\lambda = \sqrt[4]{\frac{Ta}{4}} \cdot (1 + i).$$

Eliminating the pressure, we obtain the following boundary-value problem for the velocity:

$$M'''(z) - \lambda^2 M'(z) = \mathrm{Gr}, \quad M(-1) = 0, \quad M'(1) = R\exp(i\alpha), \quad \int_{-1}^{1} M(z)\,dz = 0. \tag{12}$$

The solution of the problem (12) has the form

$$M(z) = \frac{\mathrm{Gr}}{\lambda^2}\Big[f_1(z) - f_2(z) - z\Big] + R\exp(i\alpha)f_1(z), \tag{13}$$

where

$$f_1(z) = \frac{\sinh(\lambda(z+1)) - (\sinh\lambda + \sinh(\lambda z))\sinh\lambda/\lambda}{\lambda\cosh(2\lambda) - \sinh(2\lambda)/2},$$
$$f_2(z) = \frac{\lambda\cosh(\lambda(z-1)) - \sinh(2\lambda)/2}{\lambda\cosh(2\lambda) - \sinh(2\lambda)/2} \tag{14}$$
$$u_0(z) = \mathrm{Re}\, M(z), \qquad v_0(z) = \mathrm{Im}\, M(z).$$

The solution (13), (14) has the parametric form, the profile of the velocity depends on the Taylor number (i.e., on the intensity of rotation); the first term describes the advection and the second term describes the action of tangential stresses on the profile of the velocity of the flow. The temperature is also a solution of the boundary-value problem obtained after the substitution of (11) in (8) and account of the corresponding boundary condition (9) and (10),

$$\theta_0(z) = \frac{\mathrm{Pr}}{\sqrt{\mathrm{Ta}}}\left(\mathrm{Im}\, C_3 \cdot f_3(z) + f_4(z)\right),$$

where

$$f_3(z) = \frac{3 - 2z - z^2}{2}, \quad f_4(z) = v_0(z) - v_0(1) - v_0'(-1)\left(z - 1 - \frac{1}{\mathrm{Bi}}\right) - \left(\frac{R}{\mathrm{Bi}}\right)\sin\alpha,$$
$$C_3 = C_1\sinh\lambda, \quad C_1 = \frac{\mathrm{Gr}(-\sinh\lambda + \lambda\cosh\lambda)}{\lambda^2} + R\exp(i\alpha)\sinh\lambda.$$

If $\mathrm{Ta} = 0$ (i.e., rotation is absent), then the equations for the velocity and temperature becomes simpler:

$$u_0(z) = \mathrm{Gr}\,\frac{1 - 6z - 3z^2 + 4z^3}{24} + R\cos\alpha\frac{-1 + 2z + 3z^2}{8}, \quad v_0(z) = R\sin\alpha\frac{-1 + 2z + 3z^2}{8}, \tag{15}$$

$$\theta_0(z) = \mathrm{Pr}\,\mathrm{Gr}\,\frac{4z^5 - 5z^4 - 20z^3 + 10z^2 + 40z - 29}{96} + \mathrm{Pr}\,R\cos\alpha\frac{3z^4 + 4z^3 - 6z^2 - 12z + 11}{480}. \tag{16}$$

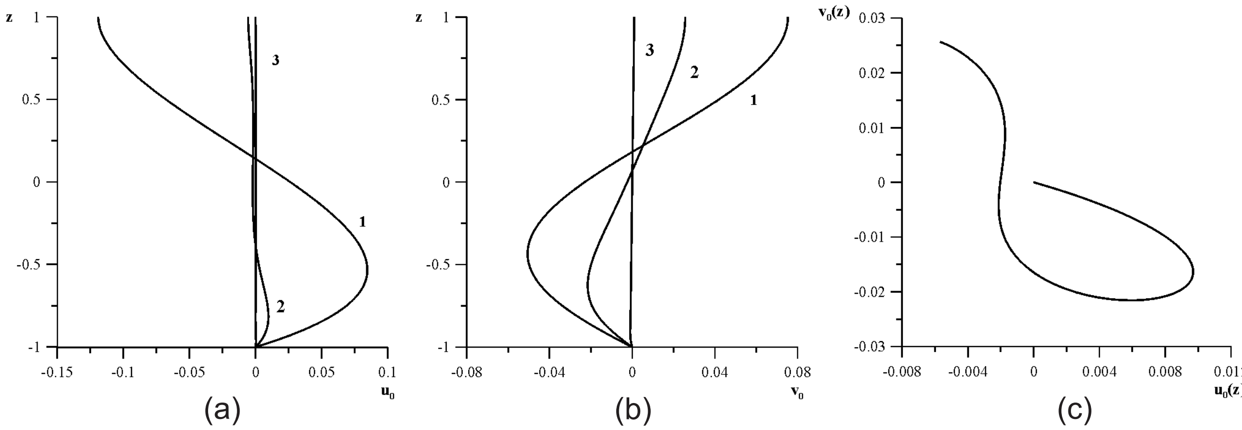

Fig. 1. Dependence of the components of the velocity $u_0(z)$ (a) and $v_0(z)$ (b) for $\alpha = 0$ and $R = 0$; the lines 1, 2, and 3 correspond to $\mathrm{Ta} = 10$, $\mathrm{Ta} = 10^3$, and $\mathrm{Ta} = 10^6$; (c) the hodograph of the velocity vector for the purely advection flow for $\mathrm{Ta} = 10^3$ and $\alpha = 0$.

In the isothermal case ($\mathrm{Gr} = 0$), the velocity (15) has parabolic profile. The velocity has the minimal value at $z = -1/3$ and the maximal value at $z = 1$; the direction of the velocity changes at $z = 1/3$. The temperature $\theta_0(z)$ is positive. In the case where the tangential stresses on the upper surface vanish ($R = 0$), then the velocity profile is cubic and the liquid moves in the direction opposite to the direction in the isothermal case and the temperature $\theta_0(z)$ is negative. Calculations made for $\alpha = 0$ show that if the Grashof number increases, then the velocity profile changes. If $\mathrm{Gr}/R \leq 1.01$, then in the upper part of the layer the liquid moves to the right and in the lower part to the left. If $\mathrm{Gr}/R > 1.01$, then three jets appear in the layer, whereas if $\mathrm{Gr}/R \geq 3$, there exist two jets. The temperature (16) has opposite signs for $\mathrm{Gr}/R > 5/3$ and is negative for $\mathrm{Gr}/R \geq 3$.

In the isothermal case, the flow that appears in the rotating layer changes depending on the Taylor number and the direction of the vector $\vec{\tau}$ on the free surface. For all values Ta, the profiles of the velocity $u_0(z)$ and $v_0(z)$ make a helical motion. The velocity attains the maximal values near the upper boundary or exactly at it. If the Taylor number increases, then the influence of tangential stresses on the velocity of the flow decreases, the maximum of the velocity also decreases, and the motion is localized near the free boundary. For $0 \leq \mathrm{Ta} \leq 35$, the modulus of the second component of the velocity first monotonically increases and then begins to decrease. Of special interest is the dependence of the angle α on the Taylor number Ta, when the module of the x-component of the velocity on the upper boundary for $z = 1$ attains the maximal value, whereas the second component of the velocity vanish. This dependence is determined by (11) according to the formula

$$\tan\alpha = -\frac{\operatorname{Im} f_1(1)}{\operatorname{Re} f_1(1)}.$$

As the Taylor number increases, the angle also increases from 0° for $\mathrm{Ta} = 0$; if $\mathrm{Ta} \gg 1$, then the angle tends to 45°.

For the nonisothermal case, for $R = 0$ the velocity of the purely advective flow also attains the maximal values near the upper boundary. For $0 \leq Ta \leq 24$, the modulus of the second component of the velocity first monotonically increases and then begins to decrease. The character of the motion is helical (see Fig. 1), but the direction of the helix is opposite.

In the general case, the flow is the combination of a wind flow and an advection flow and is a helical flow. Thus, advection causes a helical motion, which is superposed on the helical flow that appears due to the wind stress on the upper free surface.

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