DOI 10.1007/s10958-018-3776-2
Journal of Mathematical Sciences, Vol. 230, No. 5, May, 2018

OSCILLATION CRITERION FOR AUTONOMOUS DIFFERENTIAL EQUATIONS WITH BOUNDED AFTEREFFECT

V. V. Malygina UDC 517.929

Abstract. For autonomous functional-differential equations with delays, we obtain an oscillation criterion, which allows one to reduce the oscillation problem to the calculation of a unique root of a real-valued function determined by the coefficients of the original equation. The criterion is illustrated by examples of equations with concentrated and distributed aftereffect, for which convenient oscillation tests are obtained.

Keywords and phrases: differential equation with aftereffect, oscillation, concentrated and distributed delay.

AMS Subject Classification: 34K06, 34K11

Consider the functional-differential equation of the form

$$\dot{x}(t) + \mu \int_0^r x(t-s)dk(s) = 0, \quad t \geq 0, \tag{1}$$

where μ and r are real numbers, $r > 0$, k is a function of bounded variation satisfying the condition $k(0) = 0$, and the integral is meant in the Riemann–Stieltjes sense. As is known (see [1]), under these assumptions, there exists a unique solution of Eq. (1) with given initial conditions in the class of locally absolutely continuous functions.

A continuous function defined on the semi-axis is said to be *oscillating* if the sequence of its zeros is unbounded from the right. Equation (1) is said to be *oscillating* if all its solutions are oscillating functions.

To Eq. (1), we put in correspondence the characteristic function

$$F(\lambda) \equiv -\lambda + \mu \int_0^r e^{\lambda t}\, dk(t), \quad \lambda \in \mathbb{C}.$$

The following assertion states the relation between the oscillation property of Eq. (1) and the properties of its characteristic function.

Theorem 1 (see [4]). *Equation* (1) *is an oscillating equation if and only if the equation* $F(\lambda) = 0$ *has no real roots.*

Generally speaking, the function F has a complicated structure, and a direct study of the set of its real roots is a difficult problem. In the present paper, we obtain a criterion that is more efficient for analysis of the oscillation property of Eq. (1) than Theorem 1.

We exclude from consideration the trivial case where (1) is an ordinary differential equation. We assume that k is a nondecreasing on $[0, r]$ function. In this case, if $\mu = 0$, then $F(0) = 0$, and if $\mu < 0$, then $F(0)F(-\infty) < 0$, i.e., for $\mu \leq 0$ the characteristic function has real roots. Therefore, we assume that $\mu > 0$.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory, Vol. 132, Proceedings of International Symposium "Differential Equations–2016," Perm, 2016.

Introduce the notation

$$G(\zeta) \equiv \zeta \int_0^r se^{\zeta s} dk(s) - \int_0^r e^{\zeta s} dk(s),$$

where $\zeta \in \mathbb{R}$. It is easy to see that $G(\zeta) < 0$ for $\zeta \le 0$, whereas for $\zeta > 0$ we have $G'(\zeta) > 0$, i.e., G monotonically increases on the positive semi-axis. Since $G(\zeta) \to +\infty$ as $\zeta \to +\infty$, the equation $G(\zeta) = 0$ has a unique (moreover, positive) root on the real axis. We denote this root by ζ_*.

Theorem 2. *The function F is positive for all $\zeta \ge 0$ if and only if*

$$\mu \int_0^r e^{\zeta_* s}\, dk(s) > \zeta_*,$$

where ζ_* is the root of the equation $G(\zeta) = 0$.

The conditions of the theorem and the property ζ_* imply that

$$\int_0^r e^{\zeta_* s}\, dk(s) > 0.$$

Introduce the notation

$$\mu_* = \zeta_* \Big/ \int_0^r e^{\zeta_* s}\, dk(s)$$

and consider on the set $\mathbb{R}$ the function

$$F_*(\zeta) = -\zeta + \mu_* \int_0^r e^{\zeta s}\, dk(s)$$

and its derivatives

$$F_*'(\zeta) = -1 + \mu_* \int_0^r s\, e^{\zeta s} dk(s), \quad F_*''(\zeta) = \mu_* \int_0^r s^2\, e^{\zeta s} dk(s).$$

Since $F_*''(\zeta) > 0$ for all $\zeta \in \mathbb{R}$, we conclude that F_*' monotonically increases on $\mathbb{R}$ from -1 to $+\infty$; therefore, there exists a unique point $\zeta_0 \in \mathbb{R}$ at which $F_*'(\zeta_0) = 0$. On the other hand,

$$\int_0^r se^{\zeta_* s} dk(s) = \frac{1}{\zeta_*} \int_0^r e^{\zeta_* s} dk(s) = \frac{1}{\mu_*};$$

therefore,

$$F_*'(\zeta_*) = -1 + \frac{\mu_*}{\mu_*} = 0$$

and hence $\zeta_0 = \zeta_*$. Thus,

$$F_*(\zeta_0) = F_*'(\zeta_0) = 0$$

and, moreover, ζ_0 is a minimum point of the function F_*. Therefore, $F_*(\zeta) \ge 0$ for all $\zeta \ge 0$.

Proof of Theorem 2. Assume that the conditions of the theorem hold. Taking into account the notation introduced above, we can write the condition in the form $\mu > \mu_*$. Then for any $\zeta \in [0, \infty)$ we have

$$F(\zeta) = -\zeta + \mu \int_0^r e^{\zeta s}\, dk(s) = F_*(\zeta) + (\mu - \mu_*) \int_0^r e^{\zeta s}\, dk(s) > 0,$$

which was required. Conversely, if $\mu \le \mu_*$, then

$$F(\zeta_*) = F_*(\zeta_*) + (\mu - \mu_*) \int_0^r e^{\zeta s}\, dk(s) = (\mu - \mu_*) \int_0^r e^{\zeta s}\, dk(s) \le 0;$$

therefore, the function F is not positive on $[0, \infty)$. □

Combining Theorems 1 and 2, we obtain the following result.

Theorem 3. *Let k be a nondecreasing on $[0, r]$ function. Then the following assertions are equivalent*:

(1) *Eq.* (1) *is an oscillating equation*;

(2) *the characteristic function of Eq.* (1) *is positive on* $[0, \infty)$;

(3) *the following inequality is valid*:

$$\mu \int_0^r e^{\zeta_* s}\, dk(s) > \zeta_*,$$

where ζ_* is the root of the equation $G(\zeta) = 0$;

(4) *the following inequality is valid*:

$$\mu \int_0^r s e^{\zeta_* s}\, dk(s) > 1,$$

where ζ_* is the root of the equation $G(\zeta) = 0$.

Proof. Obviously, $F(\zeta) > 0$ for $\zeta < 0$; therefore, by Theorem 1, the assertions (1) and (2) are equivalent. Theorem 2 provides the equivalence of the assertions (2) and (3), and the definition of the root ζ_* — the equivalence of the assertions (3) and (4). □

Corollary 1. *If*

$$\mu \int_0^r s\, dk(s) > 1/e,$$

then Eq. (1) *is an oscillating equation.*

The function k, as any function of bounded variation, can be represented as the sum $k(t) = k_1(t) + k_2(t) + k_3(t)$, where $k_1(t)$ is a jump function, $k_2(t)$ is an absolutely continuous function, and $k_3(t)$ is a singular function. Each of these functions corresponds to its own type of functional-differential equations. We illustrate Theorem 3 by examples of equations with various types of aftereffect.

A. Equations with concentrated delays. Let $\mu = 1$,

$$k(t) = k_1(t) = \sum_{k=1}^n a_k \chi(t - r_k),$$

where $a_k > 0$, $0 < r_1 < r_2 < \cdots < r_n = r$, and χ is the Heaviside function. Equation (1) takes the form

$$\dot{x}(t) + \sum_{k=1}^n a_k x(t - r_k) = 0, \quad t \ge 0. \tag{2}$$

Applying Corollary 1, we obtain the well-known oscillation condition (see [2]).

Condition 1. *If* $a_k > 0$ *for any* $k = \overline{1,n}$ *and*

$$\sum_{k=1}^{n} a_k r_k > \frac{1}{e},$$

then Eq. (2) *is an oscillating equation.*

Setting $n = 1$ in Eq. (2), we obtain the equation

$$\dot{x}(t) + ax(t - r) = 0, \quad t \geq 0, \tag{3}$$

for which

$$G(\zeta) \equiv e^{\zeta r}(\zeta r - 1), \quad \zeta_* = \frac{1}{r}.$$

Applying assertions (3) or (4) of Theorem 3, we obtain for Eq. (3) the following oscillation criterion.

Condition 2 (see [3])**.** *Equation* (3) *is an oscillating equation if and only if* $ar > 1/e$.

Thus, in the case of a singe term, Condition 1 becomes a criterion. In particular, this implies that the constant $1/e$ in Condition 1 cannot be improved; moreover, the strict inequality cannot be replaced by nonstrict.

B. Equations with distributed delays. Let

$$k(t) = k_2(t) = \int_0^t p(s)ds,$$

where p is a summable on $[0, r]$ nonnegative function. Equation (1) takes the form

$$\dot{x}(t) + \mu \int_0^r p(s)x(t - s)ds = 0, \quad t \geq 0. \tag{4}$$

We clarify Corollary 1 for Eq. (4): in this case, the strict inequality can be replaced by nonstrict.

Condition 3. *If*

$$\mu \int_0^r p(s)s\,ds \geq \frac{1}{e},$$

then Eq. (4) *is an oscillating equation.*

Proof. The inequality $e^{\zeta_* s} > es\zeta_*$ valid for all $s \neq 1/\zeta_*$ and Condition (3) imply that $\mu > 0$ and

$$\int_0^r p(s)s\,ds > 0,$$

we have

$$\mu \int_0^r p(s)\,e^{\zeta_* s}ds > \mu e\zeta_* \int_0^r p(s)s\,ds \geq \frac{e\zeta_*}{e} = \zeta_*.$$

It remains to apply item (3) of Theorem 3. □

Apply Theorem 3 to an equation of the form (4) for a specific family of functions $p(t) = t^\alpha$, where $\alpha > -1$. Equation (4) takes the form

$$\dot{x}(t) + \mu \int_0^r s^\alpha x(t-s)\,ds = 0, \quad t \geq 0. \tag{5}$$

Introduce the notation

$$I(\xi) = \int_0^1 s^\alpha e^{\xi s} ds;$$

then

$$\int_0^r k(s)e^{\zeta s} ds = r^{\alpha+1} I(r\zeta), \quad \zeta \int_0^r k(s)se^{\zeta s}\,ds = r^{\alpha+1}(e^{\zeta r} - (\alpha+1)I(r\zeta)),$$

and the equation $G(\zeta) = 0$ is equivalent to the equation

$$(\alpha+2)I(\xi) = e^\xi, \tag{6}$$

where $\xi = r\zeta$. In this notation, the inequality from item (3) of Theorem 3 takes the form

$$\mu r^{\alpha+2} > (\alpha+2)\xi e^{-\xi}.$$

Thus, we obtain the following oscillation criterion for Eq.(5).

Condition 4. *Equation* (5) *is an oscillating equation if and only if*

$$\mu r^{\alpha+2} > (\alpha+2)\xi_\alpha e^{-\xi_\alpha},$$

where ξ_α is the root of Eq. (6).

We indicate a simple particular case of Eq. (5) for $\alpha = 0$.

Condition 5. *The equation*

$$\dot{x}(t) + \mu \int_0^r x(t-s)\,ds = 0$$

is an oscillating equation if and only if

$$\mu r^2 > 2\xi_0 e^{-\xi_0},$$

where ξ_0 is the positive root of the equation

$$1 - \frac{\xi}{2} = e^{-\xi}.$$

A numerical calculation yields

$$\xi_0 \approx 1.594, \quad 2\xi_0 e^{-\xi_0} \approx 0.648.$$

The roots of Eq. (6) for other values of $\alpha > -1$ can also be easily found:

$$\zeta_{-1/2} \approx 1.914, \quad \zeta_1 \approx 1.361, \quad \zeta_2 \approx 1{,}262, \quad \zeta_3 \approx 1{,}206, \quad \zeta_{1/2} \approx 1{,}447, \quad \zeta_{3/2} \approx 1{,}303.$$

To each root, its own oscillation conditions corresponds.

We note the following interesting fact: Condition 4 allows one to prove the exactness of the constant $1/e$ in Condition 3. To prove this, we first obtain certain properties of the family ξ_α.

Lemma 1. *For all $\alpha > -1$, the inequality*

$$\xi_\alpha > 1$$

holds. Moreover,

$$\lim_{\alpha\to\infty} \xi_\alpha = 1.$$

Proof. From the definition of ξ_α we have

$$(\alpha+2)\int_0^1 s^\alpha e^{\xi_\alpha(s-1)}ds = 1.$$

Integrating by parts, we obtain

$$(\alpha+2)\int_0^1 s^{\alpha+1} e^{\xi_\alpha(s-1)}ds = \frac{1}{\xi_\alpha}.$$

Subtracting the second relation from the first, we have

$$1 - \frac{1}{\xi_\alpha} = (\alpha+2)\int_0^1 s^\alpha(1-s)e^{\xi_\alpha(s-1)}ds, \tag{7}$$

which immediately implies that for any $\alpha > -1$ the estimate $\xi_\alpha > 1$ is valid. Taking this inequality into account, we obtain from (7)

$$0 < 1 - \frac{1}{\xi_\alpha} \le (\alpha+2)\int_0^1 (s^\alpha - s^{\alpha+1})ds = \frac{1}{\alpha+1}$$

and hence

$$\lim_{\alpha\to\infty} \xi_\alpha = 1.$$

The lemma is proved. □

We show that in Condition (3) the constant $1/e$ cannot be improved. Setting $r = 1$ in Eq. (5) and applying Condition 3, we obtain the following sufficient condition of oscillation:

$$\mu r^{\alpha+2} \ge \frac{\alpha+2}{e}.$$

Applying Condition (4), we obtain the following oscillation criterion:

$$\mu r^{\alpha+2} > (\alpha+2)\xi_\alpha e^{-\xi_\alpha}.$$

By Lemma 1, $\xi_\alpha > 1$; therefore,

$$\xi_\alpha e^{-\xi_\alpha} < \frac{1}{e},$$

but

$$\xi_\alpha e^{-\xi_\alpha} \to \frac{1}{e};$$

therefore, it is impossible to decrease the constant $1/e$ in Condition 3.

There are very few examples of functional-differential equations with singular components. We mention the paper [5] in which an equation of the form (1) was examined, where $r = 1$ and k is the Cantor function. For this equation, the exact domain of positivity of the fundamental solution in the form $\mu \le \mu_0$ was found in [5], where μ_0 is determined by a root of a certain auxiliary transcendent equation; its approximate value is $\mu_0 \approx 0.618$.

Since the positivity of the fundamental solution is equivalent to the presence of real roots of the characteristic function, Theorem 1 implies that the domain of oscillation for this equation has the form $\mu > \mu_0$.

Acknowledgment. This work was done within the framework of the state task of the Ministry of Education and Science of the Russian Federation (the base part of the task No. 2014/152, project No. 1890).

REFERENCES

1. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, *Introduction to the Theory of Linear Functional Differential Equations*, World Federation Publ., Atlanta (1995).
2. I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, The Clarendon Press, Oxford Univ. Press, New York (1991).
3. A. D. Myshkis, "On solutions of first-order linear homogeneous differential equations of stable type with delayed argument," *Mat. Sb.*, **28**, No. 3, 641–658 (1951).
4. T. L. Sabatulina, "On oscillating solutions of autonomous differential equations with aftereffect," *Vestn. Perm. Univ.*, **3 (34)**, 25–31 (2016).
5. T. Sabatulina and V. Malygina, "On positiveness of the fundamental solution for a linear autonomous differential equation with distributed delay," *Electron. J. Qual. Theory Differ. Equ.*, **61**, 1–16 (2014).

V. V. Malygina
Perm National Research Polytechnic University, Perm, Russia
E-mail: mavera@list.ru