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ON SOME PROPERTIES OF ENDOMORPHISM RINGS OF ABELIAN GROUPS

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Abstract. Some equivalent conditions under which a group can be (fully) transitive, endotransitive, or weakly transitive are presented.

The term "(full) transitivity" was introduced by I. Kaplansky in [24] in studying modules over the complete discrete valuation rings. Fully transitive torsion free Abelian groups were studied first by P. A. Krylov in [25] (he called these groups transitive). The definition of an arbitrary (fully) transitive Abelian group was introduced by Yu. B. Dobrusin in [6]. Describing (fully) transitive groups is still an open problem although studies connected with these objects are steadily being carried out. For example, (fully) transitive torsion groups were considered in [1, 4, 5, 11, 15, 16, 23, 31]; torsion free groups, in [3, 17, 26, 27]; mixed groups, in [9, 18–20, 30]; (fully) transitive modules, in [10, 22]; and weakly transitive torsion free groups, in [14, 29]. Endotransitive groups introduced in [7] for torsion free groups were also studied in [2, 8, 21].

This paper demonstrates some connections between these concepts and presents some equivalent conditions for a group to be (fully) transitive, endotransitive, or weakly transitive.

In [4], Corner considers the following concept: let Φ be a subring with unity of the ring $\mathrm{E}(G)$, and let H be a Φ-invariant subgroup of a reduced p-group G; then, he says that Φ acts (fully) transitively on H if the existence of an (element $\varphi \in \Phi$) invertible element $\varphi \in \Phi$ such that $\varphi(x) = y$ follows for any $x, y \in H$ such that $(U_G(x) \leq U_G(y))$ $U_G(x) = U_G(y)$. Roughly speaking, we say that the subgroup H is (fully) transitive over Φ. Thus, G is a (fully) transitive group in the sense of Kaplansky if and only if $\mathrm{E}(G)$ acts (fully) transitively on G. Theorem 6 describes (fully) transitive action of the ring $\mathrm{E}(G)$ on an arbitrary reduced group G.

The following Problem 41.1 was posed in [28]: "Is the class of transitive [strongly homogeneous] torsion free groups with respect to taking direct summands?" We recall that homogeneous transitive torsion free groups are called strongly homogeneous. Theorem 8 yields necessary and sufficient conditions under which a direct summand of an arbitrary transitive group is a transitive group.

In this work, the word "group" means an Abelian group. All standard definitions and designations can be found in [12, 13]. If G is a group, then the group (ring) of all its automorphisms (endomorphisms) is denoted as $\mathrm{Aut}(G)$ $(\mathrm{E}(G))$; $H(a)_A$ denotes the height matrix of an element a in a subgroup A of the group G; $H_p(a)_A$ denotes a row of the height matrix $H(a)_A$ corresponding to a prime number p; $T_p(G)$ denotes the p-component of the periodic subgroup $T(G)$ of the group G.

We recall that a group G is called (fully) transitive if the existence of $\varphi \in \mathrm{Aut}(G)$ $(\varphi \in \mathrm{E}(G))$ carrying an element a into element b follows for any pair of elements $a, b \in G$ such that $H(a) = H(b)$ $(H(a) \leq H(b))$.

Let us consider fully invariant subgroups of the group G associated with a nonzero element $a \in G$:

$$\mathrm{bfc}(a) = \{b \in G \mid H(a) \leq H(b)\},$$
$$\mathrm{lfc}(a) = \{b \in G \mid \exists \varphi \in \mathrm{E}(G),\ \varphi(a) = b\}.$$

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We call them the large and small fully invariant subgroups of the group G containing the element a, respectively.

Remark 1. For any nonzero element $a \in G$, there exists an epimorphism $\psi_a\colon \mathrm{E}^+(G) \to \mathrm{lfc}(a)$ acting by the rule $\psi_a(\varphi) = \varphi(a)$ for any $\varphi \in \mathrm{E}^+(G)$.

Further, we consider some equivalent conditions of full transitivity of the group G.

Proposition 1. *For a reduced group G, the following conditions are equivalent*:

(1) *G is a fully transitive group*;
(2) $\mathrm{bfc}(a) = \mathrm{lfc}(a)$ *for any nonzero element $a \in G$*;
(3) *for any nonzero element $a \in G$, there exists an epimorphism $\psi_a\colon \mathrm{E}^+(G) \to \mathrm{bfc}(a)$ acting by the rule $\psi_a(\varphi) = \varphi(a)$ for any $\varphi \in \mathrm{E}^+(G)$.*

Proof. (1) $\Longrightarrow$ (2). Let a be an arbitrary element of the group G. Since $\mathrm{lfc}(a) \subseteq \mathrm{bfc}(a)$, let $c \in \mathrm{bfc}(a)$. Then $H(a) \le H(c)$. Since G is a fully transitive group, there exists $\varphi \in \mathrm{E}(G)$ such that $\varphi(a) = c$. Therefore, $c \in \mathrm{lfc}(a)$ and $\mathrm{bfc}(a) = \mathrm{lfc}(a)$.

(2) $\Longrightarrow$ (1). Consider arbitrary elements $a, b \in G$ such that $H(a) \le H(b)$. Since $b \in \mathrm{bfc}(a)$ and $\mathrm{bfc}(a) = \mathrm{lfc}(a)$, there exists $\varphi \in \mathrm{E}(G)$ such that $\varphi(a) = b$.

(2) $\Longrightarrow$ (3). Follows from the condition and Remark 1.

(3) $\Longrightarrow$ (2). Let a be an arbitrary element of the group G. Since $\mathrm{lfc}(a) \subseteq \mathrm{bfc}(a)$, let c be an arbitrary element of the subgroup $\mathrm{bfc}(a)$. Then, for the element c, there exists $\eta \in \mathrm{E}^+(G)$ such that $c = \psi_a(\eta) = \eta(a)$. Therefore, $c \in \mathrm{lfc}(a)$ and $\mathrm{bfc}(a) = \mathrm{lfc}(a)$. □

By analogy with fully invariant subgroups of the group G that are connected with a nonzero element $a \in G$, we consider invariant subsets of the group G

$$\mathrm{bc}(a) = \{b \in G \mid H(b) = H(a)\},$$
$$\mathrm{lc}(a) = \{b \in G \mid \exists \varphi \in \mathrm{Aut}(G),\ \varphi(a) = b\},$$

which we call large and small invariant subsets of the group G containing the element a. Here, by an invariant subset of the group G we mean a subset closed with respect to the action of automorphisms of the group G.

Remark 2. For an arbitrary nonzero element $a \in G$, we have the following connection between fully invariant subgroups and invariant subsets defined above:

$$\mathrm{lc}(a) \subseteq \mathrm{lfc}(a) \subseteq \mathrm{bfc}(a),$$
$$\mathrm{lc}(a) \subseteq \mathrm{bc}(a) \subseteq \mathrm{bfc}(a).$$

Below, the term "weakly transitive group" introduced for torsion free groups in [14] is defined for an arbitrary Abelian group.

Definition 1. A group G is called weakly transitive if for arbitrary elements $x, y \in G$ the existence of endomorphisms $\varphi, \psi \in \mathrm{E}(G)$ such that $\varphi(x) = y$, $\psi(y) = x$ implies the existence of $\alpha \in \mathrm{Aut}\, G$ such that $\alpha(x) = y$.

For a characterization of weakly transitive groups, we need the following set defined for any nonzero element $a \in G$:

$$\mathrm{weak}(a) = \{b \in G \mid \exists \varphi,\ \psi \in \mathrm{E}(G),\ \varphi(a) = b,\ \psi(b) = a\}.$$

Remark 3. It is easy to verify that $\mathrm{lc}(a) \subseteq \mathrm{weak}(a) \subseteq \mathrm{bc}(a)$ and $\mathrm{lc}(a) \subseteq \mathrm{weak}(a) \subseteq \mathrm{lfc}(a)$ for any nonzero element $a \in G$.

Lemma 2. *A group G is weakly transitive if and only if* $\mathrm{weak}(a) = \mathrm{lc}(a)$ *for any nonzero element $a \in G$.*

Proof. Necessity. Let G be a weakly transitive group. Then, for any nonzero element $a \in G$ and for any $x \in \mathrm{weak}(a)$, there exist $\varphi, \psi \in \mathrm{E}(G)$ such that $\varphi(x) = a$ and $\psi(a) = x$. Since G is a weakly transitive group, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(a) = x$. Therefore, $x \in \mathrm{lc}(a)$. The reverse inclusion follows from Remark 3.

Sufficiency. Consider arbitrary $x, y \in G$ and $\varphi, \psi \in \mathrm{E}(G)$ such that $\varphi(x) = y$ and $\psi(y) = x$. Then $y \in \mathrm{weak}(x) = \mathrm{lc}(x)$. Therefore, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(x) = y$. □

Remark 4. For each nonzero element $a \in G$, there exists an epimorphism $\psi_a \colon \mathrm{Aut}(G) \to \mathrm{lc}(a)$ acting by the rule $\psi_a(\varphi) = \varphi(a)$ for any $\varphi \in \mathrm{Aut}(G)$.

Since any transitive group is weakly transitive, it is interesting to find an additional condition under which a weakly transitive group is transitive. The following statement proposes such a condition.

Proposition 3. *For a group G, the following conditions are equivalent*:

(1) *G is a transitive group*;
(2) $\mathrm{bc}(a) = \mathrm{lc}(a)$ *for any nonzero element $a \in G$*;
(3) *for any nonzero element $a \in G$, there exists an epimorphism $\psi_a \colon$* $\mathrm{Aut}(G) \to \mathrm{bc}(a)$ *acting by the rule $\psi_a(\varphi) = \varphi(a)$ for any $\varphi \in$* $\mathrm{Aut}(G)$;
(4) *G is a weakly transitive group and* $\mathrm{weak}(a) = \mathrm{bc}(a)$ *for any nonzero element $a \in G$.*

Proof. (1) $\Longrightarrow$ (2). Let a be an arbitrary nonzero element of the group G and $c \in \mathrm{bc}(a)$. Then $H(c) = H(a)$. Therefore, there exist $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(a) = c$. Then $c \in \mathrm{lc}(a)$ and $\mathrm{bc}(a) \subseteq \mathrm{lc}(a)$. The reverse inclusion follows from Remark 2.

(2) $\Longrightarrow$ (1). Consider arbitrary nonzero elements $a, b \in G$ such that $H(a) = H(b)$. Then $b \in \mathrm{bc}(a)$. Since $\mathrm{bc}(a) = \mathrm{lc}(a)$, there exists $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(a) = b$.

(2) $\Longrightarrow$ (3). Follows from the condition and Remark 4.

(3) $\Longrightarrow$ (2). Let a be an arbitrary nonzero element of the group G. Since $\mathrm{lc}(a) \subseteq \mathrm{bc}(a)$, let $x \in \mathrm{bc}(a)$. Since $\psi_a \colon \mathrm{Aut}(G) \to \mathrm{bc}(a)$ is an epimorphism, there exists $\varphi \in \mathrm{Aut}(G)$ such that $x = \psi_a(\varphi) = \varphi(a)$. Therefore, $x \in \mathrm{lc}(a)$ and $\mathrm{bc}(a) = \mathrm{lc}(a)$.

The equivalence of conditions (2) and (4) follows from Remark 3 and Lemma 2. □

Definition 2. A group G is called endotransitive if the existence of $\varphi \in \mathrm{E}(G)$ such that $\varphi(x) = y$ follows for arbitrary elements $x, y \in G$ such that $H(x) = H(y)$.

In [28], Problem 44 was formulated: "Are there weakly transitive torsion free groups (here, by the term "weak transitivity," we mean the term "endotransitivity") that are neither transitive nor fully transitive?" Remark 4 and the following lemma bring hope that such groups can exist.

Lemma 4. *A reduced group G is endotransitive if and only if* $\mathrm{bc}(a) \subseteq \mathrm{lfc}(a)$.

Proof. Necessity. Consider arbitrary $0 \neq a \in G$ and $x \in \mathrm{bc}(a)$. Then $H(a) = H(x)$. Since G is an endotransitive group, there exists $\varphi \in \mathrm{E}(G)$ such that $\varphi(a) = x$. Therefore, $x \in \mathrm{lfc}(a)$.

Sufficiency. Consider arbitrary elements $a, b \in G$ such that $H(a) = H(b)$. Then $b \in \mathrm{bc}(a) \subseteq \mathrm{lfc}(a)$. Therefore, there exists an endomorphism $\varphi \in \mathrm{E}(G)$ such that $\varphi(a) = b$. Thus, G is an endotransitive group. □

Remark 5. Propositions 1 and 3, Remark 4, and Lemma 4 imply a well-known result that if a group is (fully) transitive, it is endotransitive.

For the sake of completeness, we recall the following lemma proved by Corner in [4].

Lemma 5 ([4]). *A reduced p-group G is* (*fully*) *transitive if and only if* $\mathrm{E}(G)$ *acts* (*fully*) *transitively on $p^{\omega}G$.*

Let us extend the concept introduced by Corner for p-groups to arbitrary reduced Abelian groups. Here, in contrast to Corner, we suppose that the height matrix $H(a)$ is taken in a subgroup A of the group G for any $a \in A$.

Definition 3. Let Φ be a subring with unity of the ring $\mathrm{E}(G)$ and A be a Φ-invariant subgroup of a reduced Abelian group G. We say that Φ acts (fully) transitively on A or the subgroup A is (fully) transitive on Φ if the existence of an (element $\varphi \in \Phi$) invertible element $\varphi \in \Phi$ such that $\varphi(x) = y$ follows for any $x, y \in A$.

Theorem 6. *A reduced group G is (fully) transitive if and only if $\mathrm{E}(G)$ acts (fully) transitively on $p^\sigma G$ for any ordinal number σ and arbitrary prime number p.*

Proof. We prove the theorem for the case of full transitivity; the transitive case is proved similarly.

Necessity. Let p be an arbitrary prime number; if G is a p-divisible group, then $p^\sigma G = G$, i.e., $p^\sigma G$ is a fully transitive group over $\mathrm{E}(G)$ follows for any ordinal number σ.

Let $pG \neq G$. We prove the theorem by induction on σ. If $\sigma = 0$, then $\mathrm{E}(G)$ acts fully transitively on G.

Let the statement of the theorem be fulfilled for any δ such that $0 \leq \delta < \sigma$. Let us show that $\mathrm{E}(G)$ acts fully transitively on $p^\sigma G$. Let $a, b \in p^\sigma G$ and $H(a)_{p^\sigma G} \leq H(b)_{p^\sigma G}$. Then $H_p(a)_{p^\sigma G} \leq H_p(b)_{p^\sigma G}$, where $H_p(a)_{p^\sigma G} = (\alpha_0, \alpha_1, \ldots, \alpha_k, \ldots)$ and $H_p(b)_{p^\sigma G} = (\beta_0, \beta_1, \ldots, \beta_m, \ldots)$.

Let σ be an isolated ordinal number. Then $p^\sigma G = p(p^{\sigma-1}G)$ and, therefore, there exist elements $c_1, c_2 \in p^{\sigma-1}G$ such that $a = pc_1$, $b = pc_2$. Then $H_p(c_1)_{p^{\sigma-1}G} = (\mu, \alpha_0, \alpha_1, \ldots, \alpha_k, \ldots)$ and $H_p(c_2)_{p^{\sigma-1}G} = (\nu, \beta_0, \beta_1, \ldots, \beta_m, \ldots)$; here, $H_q(c_1)_{p^{\sigma-1}G} = H_q(a)_{p^\sigma G}$ and $H_q(c_2)_{p^{\sigma-1}G} = H_q(b)_{p^\sigma G}$ for any prime number q, where $q \neq p$. If $\mu \leq \nu$, then $H(c_1)_{p^{\sigma-1}G} \leq H(c_2)_{p^{\sigma-1}G}$. Since, by the induction hypothesis, the subgroup $p^{\sigma-1}G$ is fully transitive over $\mathrm{E}(G)$, there exists $\varphi \in \mathrm{E}(G)$ such that $\varphi(c_1) = c_2$. Then $p\varphi(c_1) = pc_2$ and $\varphi(a) = b$.

Let $\nu < \mu$. Suppose that there is a jump between ν and β_0 (otherwise, $\mu \leq \nu$). Then the νth Ulm–Kaplansky invariant of the group $T_p(p^{\sigma-1}G)$ is different from zero, i.e., there exists $d \in p^{\sigma-1}G$ such that $o(d) = p$ and $H_p(d)_{p^{\sigma-1}G} = (\nu, \infty, \ldots)$. Consider an element $c_1 + d \in p^{\sigma-1}G$. Since $H_p(c_1+d)_{p^{\sigma-1}G} = (\nu, \alpha_0, \alpha_1, \ldots, \alpha_k, \ldots) \leq H_p(c_2)_{p^{\sigma-1}G}$ and $H_q(c_1+d)_{p^{\sigma-1}G} = H_q(c_1)_{p^{\sigma-1}G}$ for any prime q, $q \neq p$, we have $H(c_1+d)_{p^{\sigma-1}G} \leq H(c_2)_{p^{\sigma-1}G}$. Since, by the induction hypothesis, the subgroup $p^{\sigma-1}G$ is fully transitive over $\mathrm{E}(G)$, there exists $\varphi \in \mathrm{E}(G)$ such that $\varphi(c_1+d) = c_2$. Then $p\varphi(c_1+d) = pc_2$ and $\varphi(a) = b$.

Let σ be the limit ordinal number, i.e., $p^\sigma G = \bigcap_{\delta<\sigma} p^\delta G$. Therefore, $a, b \in p^\delta G$ for any $\delta < \sigma$. Then, be definition of the generalized height, $h_p^*(a)_{p^\delta G} = h_p^*(a)_{p^\sigma G}$ for any $\delta < \sigma$ and, therefore, $h_p^*(p^k a)_{p^\delta G} = h_p^*(p^k a)_{p^\sigma G}$ for any $\delta < \sigma$ and natural k, i.e., $H_p(a)_{p^\delta G} = H_p(a)_{p^\sigma G}$ for any $\delta < \sigma$. Since $H_q(a)_{p^\delta G} = H_q(a)_{p^\sigma G}$ for any $\delta < \sigma$ and prime q, where $q \neq p$, we have $H(a)_{p^\delta G} = H(a)_{p^\sigma G}$ for any $\delta < \sigma$. Similar reasonings show that $H(b)_{p^\delta G} = H(b)_{p^\sigma G}$ for any $\delta < \sigma$. Then $H(a)_{p^\delta G} \leq H(b)_{p^\delta G}$. By the induction hypothesis, the subgroup $p^\delta G$ is fully transitive over $\mathrm{E}(G)$ for any $\delta < \sigma$, i.e., there exists $\varphi \in \mathrm{E}(G)$ such that $\varphi(a) = b$.

Sufficiency. Let $\mathrm{E}(G)$ acts fully transitively on the subgroup $p^\sigma G$ for any ordinal number σ and for any prime number p. Then, in particular, $\mathrm{E}(G)$ acts fully transitively on $p^0 G = G$, i.e., G is a fully transitive group. □

Corollary 7. *For a reduced p-group G, the following conditions are equivalent*:

(1) *G is a (fully) transitive group*;
(2) $\mathrm{E}(G)$ *acts (fully) transitively on $p^\omega G$ (in the sense of Corner)*;
(3) $\mathrm{E}(G)$ *acts (fully) transitively on $p^\sigma G$ for any ordinal number σ (in the sense of Definition 3).*

Proof. Equivalences of conditions (1) and (2), (1) and (3) are obtained from Lemma 5 and Theorem 6, respectively. □

Let us introduce the following concept.

Definition 4. Let $G = A \oplus B$. We say that automorphisms of the group A are induced by automorphisms of the group G if, for any $a \in A$ and for any $x \in \mathrm{bc}(a)$, the existence of $\varphi \in \mathrm{Aut}\, G$ such that $\varphi\rho(a) = \rho(x)$

implies the existence of $\psi \in \operatorname{Aut} A$ such that $\pi\varphi\rho(a) = \psi(a)$, where $\rho\colon A \to G$ and $\pi\colon G \to A$ are the canonical embedding and projection, respectively.

Theorem 8. *A direct summand A of a transitive group G is a transitive group if and only if automorphisms of the group A are induced by automorphisms of the group G.*

Proof. Necessity. Let $G = A \oplus B$, where G and A are transitive groups. Let $\rho\colon A \to G$ and $\pi\colon G \to A$ be the canonical embedding and projection, respectively. Then, for any nonzero element $a \in A$ and for any $x \in \mathrm{bc}(a)$, the transitivity of the group A implies the existence of $\psi \in \operatorname{Aut} A$ such that $\psi(a) = x$.

Consider an element ρx. Since $H(\rho x) = H(\rho a)$, the transitivity of the group G implies the existence of $\varphi \in \operatorname{Aut} G$ such that $\varphi\rho a = \rho x$. Therefore, $\pi\varphi\rho a = x = \psi(a)$.

Sufficiency. For the transitivity of the group A, according to Proposition 3, it is sufficient to show that $\mathrm{bc}(a) = \mathrm{lc}(a)$ follows for any nonzero element $a \in A$. For an arbitrary element $x \in \mathrm{bc}(a)$, we have $H(x) = H(a)$. Since $H(\rho x) = H(\rho a)$, the existence of $\varphi \in \operatorname{Aut} G$ such that $\varphi\rho(a) = \rho(x)$ follows from the transitivity of the group G. Since automorphisms of the group G induce automorphisms of the group A, there exists $\psi \in \operatorname{Aut} A$ such that $x = \pi\varphi\rho(a) = \psi(a)$. Therefore, $\mathrm{bc}(a) = \mathrm{lc}(a)$. □

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