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NORMAL DETERMINABILITY OF TORSION-FREE ABELIAN GROUPS BY THEIR HOLOMORPHS

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Abstract. We investigate torsion-free Abelian groups that are decomposable into direct sums or direct products of homogeneous groups normally defined by their holomorphs. Properties of normal Abelian subgroups of holomorphs of torsion-free Abelian groups are also studied.

Let G be an Abelian group, and $\Gamma(G)$ be its holomorph, i.e., a semiprime extension of the group G by its automorphism group $\mathrm{Aut}(G)$. The group operation in the group $\mathrm{Aut}(G)$ is written multiplicatively; group operations in G and $\Gamma(G)$, additively. The group $\Gamma(G)$ can be considered as the set of all ordered pairs (g,φ), where $g \in G$ and $\varphi \in \mathrm{Aut}(G)$. The group operation in $\Gamma(G)$ is defined by the following rule: $(g,\varphi)+(h,\psi)=(g+\varphi h,\varphi\psi)$ for any $(g,\varphi),(h,\psi)\in\Gamma(G)$. The neutral element in $\Gamma(G)$ is the element $(0,\varepsilon)$ (ε is the identical automorphism); the element $(-\varphi^{-1}g,\varphi^{-1})$ is the inverse element for the element (g,φ). Elements (g,ε) form in the holomorph $\Gamma(G)$ a normal subgroup isomorphic to the group G; elements $(0,\varphi)$ form a subgroup isomorphic to the group $\mathrm{Aut}(G)$. We identify these subgroups with the groups G and $\mathrm{Aut}(G)$, respectively. It is clear that $G\cap\mathrm{Aut}(G)=\{(0,\varepsilon)\}$. Instead of the notation (g,ε) and $(0,\varphi)$ for elements of the group $\Gamma(G)$ we simply write g and φ, respectively.

Note that if G is an Abelian group, then it is the maximal Abelian subgroup of its holomorph $\Gamma(G)$. Indeed, if we assume the existence of an Abelian subgroup G_1 of the holomorph $\Gamma(G)$ such that $G\subset G_1$ and $G\neq G_1$, then G_1 contains an element (g,σ) not belonging to G, and, therefore, $\sigma\neq\varepsilon$. Then $(-g,\varepsilon)+(g,\sigma)=(0,\sigma)\in G_1$. By virtue of commutativity of G_1, we have $(a,\varepsilon)+(0,\sigma)=(0,\sigma)+(a,\varepsilon)$ for any element $a\in G$, i.e., $\sigma a=a$ and, therefore, $\sigma=\varepsilon$. We have obtained a contradiction. Thus, G is the maximal Abelian group of its holomorph.

Note also that if H is a normal Abelian subgroup of the group $\Gamma(G)$ and H_1 and Φ_1 are sets of first and second components of elements of the group H, respectively, then H_1 is a fully invariant subgroup of the group G [13] and Φ_1 is a normal subgroup in $\mathrm{Aut}(G)$.

In this paper, we consider questions connected with normal definiteness of torsion-free Abelian groups by their holomorphs.

Two groups are said to be *holomorphically isomorphic* if their holomorphs are isomorphic. A group A *is defined by its holomorph* in a certain class of groups $\Re$ if any group B belonging to this class and holomorphically isomorphic to the group A is isomorphic to the group A. There are known examples of nonisomorphic finite noncommutative groups with isomorphic holomorphs [11]. In [13], W. Mills demonstrated that any finitely generated Abelian group is defined by its holomorph in the class of all finitely generated Abelian groups. Some interesting results about properties of holomorphs of Abelian groups and about definiteness of Abelian groups by their holomorphs were obtained by I. Kh. Bekker [1–6].

The notion of almost holomorphic isomorphism is a generalization of the notion of holomorphic isomorphism. Groups A and B are said to be *almost holomorphically isomorphic* if each of them is isomorphic to a normal subgroup of the holomorph of the other group. It is clear that if two groups are holomorphically isomorphic, they are almost holomorphically isomorphic. The converse, generally speaking, is not true. Almost holomorphically isomorphic finitely generated Abelian groups were studied by W. Mills in [13]. Almost holomorphically isomorphic Abelian p-groups were studied in [6, 10]. Note

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that if the isomorphism of two groups follows from almost holomorphic isomorphism of the groups in a certain class $\Re$, then any group from the class $\Re$ is defined by its holomorph in this class.

We say that a group G *is normally defined in the class* $\Re$ *by its holomorph* if for any group H of this class the isomorphism of the groups G and H follows from almost holomorphic isomorphism of the groups G and H.

In studying holomorphs of Abelian groups, an important part is played by normal Abelian subgroups of such holomorphs. the following results are valid.

Lemma 1 ([12]). *If S is a normal Abelian group in $\Gamma(G)$, $(a,\sigma) \in S$, $g \in G$, then*

$$\sigma a - a \in S, \ (2a, \varepsilon) \in S, \ (0, \sigma^2) \in S; \tag{1}$$

$$\sigma g - g \in S; \tag{2}$$

$$\sigma(\sigma g - g) = \sigma g - g; \tag{3}$$

$$\sigma^n g = g + n(\sigma g - g); \tag{4}$$

$$n(a,\sigma) = \left(na + \frac{n(n-1)}{2}(\sigma a - a), \sigma^n \right); \tag{5}$$

$$2(\sigma a - a) = 0. \tag{6}$$

Lemma 2. *Let S be a normal Abelian subgroup of the holomorph $\Gamma(G)$ of a torsion-free Abelian group G and S_1 be the set of first components of elements of the group S.*

(1) *S is a torsion-free group.*

(2) *If $S \neq 0$, then $S_1 \neq 0$.*

Proof. (1) Let $(a,\sigma) \in S$ and $n(a,\sigma) = (0,\varepsilon)$ for some natural number n. We have (formula (5)) that

$$n(a,\sigma) = \left(na + \frac{n(n-1)}{2}(\sigma a - a), \sigma^n \right).$$

Since G is a torsion-free group, it follows from equality (6) that $\sigma a - a = 0$, and formula (5) takes the form

$$n(a,\sigma) = (na, \sigma^n). \tag{7}$$

Thus, $(na, \sigma^n) = (0, \varepsilon)$. It follows that $na = 0$ and $\sigma^n = \varepsilon$. Therefore, $a = 0$ because G is a torsion-free group. We demonstrate that $\sigma = \varepsilon$. By formula (4), $\sigma^n g = g + n(\sigma g - g)$ for any $g \in G$. Taking into account that $\sigma^n = \varepsilon$, we obtain that $g = g + n(\sigma g - g)$, i.e., $n(\sigma g - g) = 0$. Since G is a torsion-free group, $\sigma g - g = 0$ and, therefore, $\sigma = \varepsilon$.

(2) Let $(a,\sigma) \in S$ and $(a,\sigma) \neq (0,\varepsilon)$. If $\sigma \neq \varepsilon$, then there exists an element $g \in G$ such that $\sigma g - g \neq 0$. Applying formula (2), we obtain $\sigma g - g \in S$. Therefore, $\sigma g - g \in S_1$ and $S_1 \neq 0$. If $\sigma = \varepsilon$, then $a \neq 0$ and S_1 is also different from zero. □

Let us consider connections between types of elements of almost holomorphically isomorphic torsion-free Abelian group.

We recall some notations and terms from the theory of torsion-free Abelian groups.

Let A be an Abelian group, $a \in A$. The largest nonnegative number k for which the equation $p^k x = a$ has a solution is called *p-height* of the element a in the group A (designation: $h_p^A(a)$ or $h_p(a)$). If the equation $p^k x = a$ have a solution at any k, then a is called an *element of infinite p-height*, i.e., $h_p(a) = \infty$.

Let $\mathbf{X}$ be the set of all sequences of the form $v = (k_1, k_2, \ldots, k_i, \ldots)$, where k_i is a nonnegative integer or the symbol ∞ ($i \in \mathbb{N}$). Such sequences are called *characteristics*.

In the set $\mathbf{X}$, a partial order is introduced in a natural way; namely, $\left(k_1^{(1)}, k_2^{(1)}, \ldots, k_n^{(1)}, \ldots\right) \leq \left(k_1^{(2)}, k_2^{(2)}, \ldots, k_n^{(2)}, \ldots\right)$ if and only if for every $i \in \mathbb{N}$ the condition $k_i^{(1)} \leq k_i^{(2)}$ is satisfied. With respect to this partial order, the set $\mathbf{X}$ is a full lattice.

Let $\mathbf{P} = (p_1, p_2, \ldots, p_n, \ldots)$ be the set of all prime numbers enumerated in the ascending order. If A is a torsion-free Abelian group and $a \in A$, then the *characteristic* $\chi_A(a)$ (or $\chi(a)$) of the element a in the group A is a characteristic $\chi = (k_1, k_2, \ldots, k_n, \ldots)$ in which each k_i is the p_i-height $h^A_{p_i}(a)$ of the element a in the group A [8]. Note that, according to the definition, the characteristic of a nonzero element is the sequence $(\infty, \ldots, \infty, \ldots)$.

Two characteristics $v_1 = \bigl(k_1^{(1)}, k_2^{(1)}, \ldots, k_n^{(1)}, \ldots\bigr)$ and $v_2 = \bigl(k_1^{(2)}, k_2^{(2)}, \ldots, k_n^{(2)}, \ldots\bigr)$ are considered as *equivalent* if and only if the set $M = \bigl\{n \in \mathbb{N} \mid k_n^{(1)} \neq k_n^{(2)}\bigr\}$ is finite; here, if $k_n^{(1)} \neq k_n^{(2)}$, then $k_n^{(1)} \neq \infty$ and $k_n^{(2)} \neq \infty$.

The class of equivalence in the set of characteristics is called the *type*. If the characteristic of an element a of a torsion-free Abelian group A belongs to the type $\mathbf{t}$, then the element a is said to have the type $\mathbf{t}$ (which is written as follows: $\mathbf{t}_A(a) = \mathbf{t}$ or $\mathbf{t}(a) = \mathbf{t}$ if it is clear which of the groups A is meant).

A type $\mathbf{t}$ is called *p_n-divisible* ($p_n \in \mathbf{P}$) if for each characteristic $\chi = (k_1, k_2, \ldots, k_n, \ldots)$ belonging to the type $\mathbf{t}$ we have $k_n = \infty$.

We consider the set of types as a partially ordered set with respect to the natural ordering (i.e., $\mathbf{t}_1 \leq \mathbf{t}_2$ if and only if there exist characteristics v_1 and v_2 belonging to types $\mathbf{t}_1$ and $\mathbf{t}_2$, respectively, such that $v_1 \leq v_2$). The partially ordered set of all types is a full lattice.

A torsion-free Abelian group all nonzero elements of which have the same type $\mathbf{t}$ is called *homogeneous* [8]. To emphasize that all nonzero elements of a homogeneous group A have a fixed type $\mathbf{t}$, we say that A is a *homogeneous group of type* $\mathbf{t}$ and write it as follows: $\mathbf{t}(A) = \mathbf{t}$. It is evident that any torsion-free rank 1 group is homogeneous.

For a torsion-free Abelian group A, $T(A)$ denotes the set of all types of elements of the group A.

Lemma 3. *Let S be a normal Abelian subgroup of the holomorph $\Gamma(G)$ of a torsion-free Abelian group G. Then, for any type $\mathbf{t} \in T(S)$, there exists a type $\mathbf{t}' \in T(G)$ such that $\mathbf{t}' \geq \mathbf{t}$.*

Proof. Let the type $\mathbf{t}$ belong to the type set of the group S. Then there exists a nonzero element $(a, \sigma) \in S$ such that its characteristic belongs to the type $\mathbf{t}$ ($\chi\bigl((a, \sigma)\bigr) \in \mathbf{t}$). This characteristic has the form $\chi\bigl((a, \sigma)\bigr) = (k_1, k_2, \ldots, k_n, \ldots)$.

Let $a \neq 0$. We denote its type as $\mathbf{t}'$. If $k_n < \infty$, then there exists an element $(x_n, \eta_n) \in S$ such that $p_n^{k_n}(x_n, \eta_n) = (a, \sigma)$. Then, taking into account formula (7), we have that $(p_n^{k_n} x_n, \eta_n^{p_n^{k_n}}) = (a, \sigma)$. We obtain that $p_n^{k_n} x_n = a$. Therefore, the equation $a = p_n^{k_n} x_n$ is solvable in the group G. Therefore, $h_{p_n}^{(G)}(a) \geq k_n$. If $k_n = \infty$, then for any natural number m there exists an element $(y_m, \xi_m) \in S$ such that the equation $p_n^m(y_m, \xi_m) = (a, \sigma)$ is solvable in S; therefore, the equation $p_n^m y_m = a$ is solvable in G. This means that $h_{p_n}^{(G)}(a) = \infty$.

Thus, $\chi_{(G)}(a) \geq \chi_{(S)}\bigl((a, \sigma)\bigr)$, and, therefore $\mathbf{t}(a) = \mathbf{t}' \geq \mathbf{t}$.

Let $a = 0$. Then $\sigma \neq \varepsilon$. If $k_n < \infty$, then there exists an element $(0, \eta_n) \in S$ such that $p_n^{k_n}(0, \eta_n) = (0, \sigma)$ or $\eta_n^{p_n^{k_n}} = \sigma$. Since $\sigma \neq \varepsilon$, there exists an element $g \in G$ such that $\sigma g \neq g$. According to formula (4), we have that $\sigma g = \eta_n^{p_n^{k_n}} g = g + p_n^{k_n}(\eta_n g - g)$. It follows that $\sigma g - g = p_n^{k_n}(\eta_n g - g)$. The equation $\sigma g - g = p_n^{k_n} x$ is solvable in G. Therefore, $h_{p_n}^{(G)}(\sigma g - g) \geq k_n$.

If $k_n = \infty$, then $h_{p_n}^{(G)}(\sigma g - g) = \infty$. Thus, $\chi_{(G)}(\sigma g - g) \geq \chi_{(S)}\bigl((0, \sigma)\bigr)$. Therefore, $\mathbf{t}(\sigma g - g) = \mathbf{t}' \geq \mathbf{t}$. □

Let S_1 be the set of all first components of elements of the subgroup S from Lemma 3. We have $\sigma g - g \in S_1$ for each element $g \in G$ if $(a, \sigma) \in S$. Then, from the proof of Lemma 3, we obtain the following statement.

Proposition 4. *Let S be a normal Abelian subgroup of the holomorph $\Gamma(G)$ of a torsion-free Abelian group G and S_1 be the set of first components of elements of the group S. Then, for any type $\mathbf{t} \in T(S)$, there exists a type $\mathbf{t}' \in T(S_1)$ such that $\mathbf{t}' \geq \mathbf{t}$.*

Theorem 5. *Let G and H be almost holomorphically isomorphic torsion-free Abelian groups, G be a homogeneous group, and the group H have the following property: for any elements $b_1, b_2 \in H$ such that $\mathbf{t}(b_1)$ is comparable with $\mathbf{t}(b_2)$, we have that $\mathbf{t}(b_1) = \mathbf{t}(b_2)$. Then H is a homogeneous group and $\mathbf{t}(G) = \mathbf{t}(H)$.*

Proof. Let the type of the homogeneous group G be equal to $\mathbf{t}$, and let $\mathbf{t}_1 \in T(H)$. The groups G and H are almost holomorphically isomorphic; therefore, $H \cong G'$, where G' is a normal subgroup of the holomorph $\Gamma(G)$ and $G \cong H'$, where H' is a normal subgroup of the holomorph $\Gamma(H)$. It follows from the almost holomorphic isomorphism of the groups G and H that $\mathbf{t}_1 \in T(G')$. Then, from Lemma 3, we obtain that the type $\mathbf{t}$ satisfies the condition $\mathbf{t} \geq \mathbf{t}_1$. Since $G \cong H'$, we have that $\mathbf{t} \in T(H')$. According to Lemma 2, for the $\mathbf{t}$, there exists a type $\mathbf{t}_2 \in T(H)$ such that $\mathbf{t}_2 \geq \mathbf{t}$. We obtain that $\mathbf{t}_2 \geq \mathbf{t} \geq \mathbf{t}_1$. Since $\mathbf{t}_1, \mathbf{t}_2 \in T(H)$, there exist elements $b_1, b_2 \in H$ such that $\mathbf{t}(b_1) = \mathbf{t}_1$ and $\mathbf{t}(b_2) = \mathbf{t}_2$. We have that $\mathbf{t}(b_2) \geq \mathbf{t}(b_1)$, i.e., the types of the elements b_1 and b_2 are comparable. Taking into account the condition of theorem on types of elements of the group H, we obtain that $\mathbf{t}(b_1) = \mathbf{t}(b_2)$, i.e., $\mathbf{t}_1 = \mathbf{t}_2$ and, therefore, $\mathbf{t}_1 = \mathbf{t}$. By virtue of the arbitrariness of the choice of the type $\mathbf{t}_1$, we obtain that H is a homogeneous group and its type is $\mathbf{t}$. □

Corollary 6. *If G and H are homogeneous almost holomorphically isomorphic groups, then $\mathbf{t}(G) = \mathbf{t}(H)$.*

Theorem 7. *Let $G = \bigoplus_{\mathbf{t} \in T_1} G_{\mathbf{t}}$ and $H = \bigoplus_{\bar{\mathbf{t}} \in T_2} H_{\bar{\mathbf{t}}}$, where $G_{\mathbf{t}}$ and $H_{\bar{\mathbf{t}}}$ are homogeneous groups of types $\mathbf{t}$ and $\bar{\mathbf{t}}$, respectively, and T_1 and T_2 are sets consisting of pairwise independent types. If G and H are almost holomorphically isomorphic groups, then $T_1 = T_2$.*

Proof. Groups G and H are almost holomorphically isomorphic, i.e., $G \cong H'$ and $H \cong G'$, where G' and H' are normal Abelian subgroups of the holomorphs $\Gamma(G)$ and $\Gamma(H)$, respectively.

Let $\mathbf{t}_0 \in T_1$. It follows from the almost holomorphic isomorphism of the groups G and H that $\mathbf{t}_0 \in T(H')$. By Lemma 3, there exists a type $\bar{\mathbf{t}}_0 \in T(H)$ such that $\bar{\mathbf{t}}_0 \geq \mathbf{t}_0$.

Since types in T_2 are pairwise incomparable, $\bar{\mathbf{t}}_0 \leq \bar{\mathbf{t}}_i$ for all $i = \overline{1, k}$.

We have that $\bar{\mathbf{t}}_1 > \bar{\mathbf{t}}_0 \geq \mathbf{t}_0$. It follows from the almost holomorphic isomorphism of the groups G and H that $\bar{\mathbf{t}}_1 \in T(G')$. By Lemma 3, there exists a type $\mathbf{t}_1 \in T(G)$ such that $\mathbf{t}_1 \geq \bar{\mathbf{t}}_1$.

Two cases are possible.

(1) Let $\mathbf{t}_1 \in T_1$. Then $\mathbf{t}_1 \geq \bar{\mathbf{t}}_1 > \bar{\mathbf{t}}_0 \geq \mathbf{t}_0$. Hence $\mathbf{t}_1 > \mathbf{t}_0$. We obtain that the types $\mathbf{t}_0$ and $\mathbf{t}_1$ are comparable. This contradicts the condition of the theorem.

(2) Let $\mathbf{t}_1 \in T(G) \setminus T_1$. Then $\mathbf{t}_1 = \inf\{\mathbf{t}_2, \mathbf{t}_3, \ldots, \mathbf{t}_m\}$, where $\mathbf{t}_j \in T_1$, $j = \overline{2, m}$. By analogy to what was proved earlier, we obtain that $\mathbf{t}_j > \mathbf{t}_1$ for all $j = \overline{2, m}$. We have that $\mathbf{t}_2 > \mathbf{t}_1 \geq \bar{\mathbf{t}}_1 > \bar{\mathbf{t}}_0 \geq \mathbf{t}_0$. The types $\mathbf{t}_0$ and $\mathbf{t}_2$ belong to T_1 and are comparable with each other, a contradiction. Therefore, $\bar{\mathbf{t}}_0 \in T_2$ and $\bar{\mathbf{t}}_0 \geq \mathbf{t}_0$.

One can similarly prove that, for the type $\bar{\mathbf{t}}_0 \in T_2$, there exists a type $\mathbf{t}' \in T_1$ such that $\mathbf{t}' \geq \bar{\mathbf{t}}_0$.

Thus, $\mathbf{t}' \geq \bar{\mathbf{t}}_0 \geq \mathbf{t}_0$. Since the types in T_1 are not pairwise comparable, $\mathbf{t}' = \mathbf{t}_0$. This means that $\bar{\mathbf{t}}_0 = \mathbf{t}_0$ and the inclusion $T_1 \subset T_2$ is valid.

The inverse inclusion $T_2 \subset T_1$ can be proved similarly. Therefore, $T_1 = T_2$. □

Note that a statement similar to Theorem 7 holds for direct products of homogeneous groups; namely, the following result is valid.

Theorem 7′. *Let $G = \prod_{\mathbf{t} \in T_1} G_{\mathbf{t}}$, $H = \prod_{\bar{\mathbf{t}} \in T_2} H_{\bar{\mathbf{t}}}$, where $G_{\mathbf{t}}$ and $H_{\bar{\mathbf{t}}}$ are homogeneous groups of types $\mathbf{t}$ and $\bar{\mathbf{t}}$, respectively, and T_1 and T_2 are sets consisting of pairwise incomparable types. If G and H are almost holomorphically isomorphic groups, then $T_1 = T_2$.*

Now let us consider divisible subgroups of almost holomorphically isomorphic torsion-free Abelian groups.

Theorem 8. *If two Abelian groups are almost holomorphically isomorphic and one of them is torsion free, then the divisible subgroups of these groups are isomorphic.*

Proof. Let G be a torsion-free Abelian group and H be an Abelian group almost holomorphically isomorphic to it. Then $G \cong H'$ and $H \cong G'$, where G' and H' are normal subgroups of the groups $\Gamma(G)$ and $\Gamma(H)$, respectively. By Lemma 2, the group H is also torsion free. We show that if one of the groups G or H is a nonreduced group, the other group is also nonreduced.

Let $D(G)$, $D(H)$, $D(G')$, and $D(H')$ denote the divisible subgroups of the groups G, H, G', and H', respectively; let $\bar{G}_1$, $\bar{\Phi}$, $\bar{H}_1$, and $\bar{\Psi}$ be the sets of all first and second components of the groups $D(G')$ and $D(H')$, respectively.

Let G be a nonreduced group; then H' is also nonreduced and $D(H') \neq 0$. Then there exist $(h, \psi) \in D(H')$, $(h, \psi) \neq (0, \varepsilon)$. It follows from divisibility of the group $D(H')$ that, for any natural number n, there exists an element $(a_n, \theta_n) \in D(H')$ such that $n(a_n, \theta_n) = (h, \psi)$. Since H is a torsion-free group, we have by formula (7) that $n(a_n, \theta_n) = (na_n, \theta_n^n)$, i.e., $h = na_n$ and $\psi = \theta_n^n$.

If $h \neq 0$, then the group H is nonreduced.

Let $h = 0$. Then $\psi \neq \varepsilon$ and $\theta_n^n = \psi \neq \varepsilon$. BY Lemma 1, for any element $a \in H$, we have that $\psi a - a \in H'$ and there exists an element $h_1 \in H$ such that $\psi h_1 - h_1 \neq 0$. From formula (4) we obtain that $\theta_n^n h_1 = h_1 + n(\theta_n h_1 - h_1)$ or $\psi h_1 - h_1 = n(\theta_n h_1 - h_1)$. For a nonzero element $\psi h_1 - h_1$ of the torsion-free group H, we obtained that the equation $\psi h_1 - h_1 = nx$ is solvable in this group for any natural number n. This means that the group H is nonreduced. Let $(g, \sigma) \in D(G')$ and $(g, \sigma) \neq (0, \varepsilon)$. Then, for any natural number n, there exists an element $(b_n, \omega_n) \in D(G')$ such that $n(b_n, \omega_n) = (g, \sigma)$, and it follows that $nb_n = g$ and $\omega_n^n = \sigma$. Therefore, $g \in D(G)$ and $\bar{G}_1$ is a divisible subgroup of the group $D(G)$.

Similarly, one can prove that $\bar{H}_1$ is a divisible subgroup of the group $D(H)$.

Let us show that $D(G')$ is decomposable. Let us consider an automorphism η of the group G, acting as follows: $\eta g = 2g$ if $g \in D(G)$ and $\eta g = g$ if $g \in R(G)$ ($G = D(G) \oplus R(G)$). We have $-(0, \eta) + (2g, \varepsilon) + (0, \eta) = (g, \varepsilon)$; however, $(2g, \varepsilon) \in D(G')$. Therefore, $(g, \varepsilon) \in D(G')$. Then $(0, \sigma) \in D(G')$. We obtain that $D(G') = \bar{G}_1 \oplus \bar{\Phi}$.

Similarly, $D(H') = \bar{H}_1 \oplus \bar{\Psi}$.

$D(G')$ and $D(H')$ are nonzero normal Abelian subgroups of the groups $\Gamma(G)$ and $\Gamma(H)$, respectively. Then $\bar{G}_1 \neq 0$ and $\bar{H}_1 \neq 0$ (Lemma 2). Since $\bar{G}_1$ and $\bar{H}_1$ are invariant subgroups of the groups G and H, $\bar{G}_1 = D(G)$ and $\bar{H}_1 = D(H)$.

Taking into account the almost holomorphic isomorphism of the groups G and H, we have that $D(G) \cong D(H') = \bar{H}_1 \oplus \bar{\Psi} = D(H) \oplus \bar{\Psi}$, and, therefore, $r\big(D(G)\big) \geq r\big(D(H)\big)$. At the same time, $D(H) \cong D(G') = \bar{G}_1 \oplus \bar{\Phi} = D(G) \oplus \bar{\Phi}$, and hence $r\big(D(H)\big) \geq r\big(D(G)\big)$. Thus, $r\big(D(G)\big) = r\big(D(H)\big)$ and, therefore, $D(G) \cong D(H)$. □

Theorem 9. *The holomorph of a divisible torsion-free Abelian group G has no nonzero normal Abelian subgroups different from G.*

Proof. Let G' be a nonzero normal Abelian subgroup of the holomorph $\Gamma(G)$ of a divisible torsion-free group G. By $D(G')$, we denote the divisible subgroup of the group G'. Let $(a, \sigma) \in G'$ and $(a, \sigma) \neq (0, \varepsilon)$.

If $\sigma \neq \varepsilon$, then there exists an element $g \in G$ such that $\sigma g - g \neq 0$. For any natural number n, one can find an element g_n such that $ng_n = g$ and therefore, $n(\sigma g_n - g_n) = \sigma g - g$. Since $\sigma g_n - g_n \in G'$ and $\sigma g - g \in G'$, we have that $\sigma g - g \in D(G')$.

If $\sigma = \varepsilon$, then for each $n \in \mathbb{N}$ we consider such an automorphism η_n of the group G that

$$\eta_n h = \frac{h}{n}$$

for any element $h \in G$. Then $-(0, \eta_n) + (a, \varepsilon) + (0, \eta_n) = (a_n, \varepsilon)$, where a_n is determined from the equality $na_n = a$. We have that $(a_n, \varepsilon) \in G'$; therefore, $a \in D(G')$.

We have obtained that the subgroup $D(G') \neq 0$.

It was established in the proof of Theorem 8 that $D(G') = D(G) \oplus \bar{\Phi}$.

Since G is a divisible group, $D(G) = G$ and, therefore, $D(G') = G \oplus \bar{\Phi}$. This means that $G \subset G'$. Taking into account that G is the maximal Abelian normal subgroup of its holomorph, we obtain that $G = G'$. □

The following result follows from Theorem 9.

Theorem 10. *Any divisible torsion free Abelian group is normally determined by its holomorph in the class of all Abelian groups.*

Hereinafter, taking into account the obtained results, we consider only reduced groups when studying normal determinability of torsion-free Abelian groups.

Note that the statement similar to Theorem 9 for Abelian p-groups is not valid. The holomorph of a divisible p-group G can contain nonzero normal Abelian subgroups different from G. For example, subgroups of the form $G[p^k]$, where k is an arbitrary natural number refer to such subgroups.

We turn to considering almost holomorphically isomorphic completely decomposable torsion-free Abelian groups.

Let $G = \bigoplus_{i\in I} G_i$ be a completely decomposable torsion-free Abelian group ($r(G_i) = 1$ for any $i \in I$), $G = \bigoplus_{\mathbf{t}\in T} G_{\mathbf{t}}$ be the canonical decomposition of the group G, where $G_{\mathbf{t}} = \bigoplus_{i\in I(\mathbf{t})} G_i$, $I(\mathbf{t}) = \{i \in I \mid \mathbf{t}(G_i) = \mathbf{t}\}$.

Lemma 11. *If S is an invariant subgroup of the group G and $S \cap G_{\mathbf{t}'} \neq 0$ for some $\mathbf{t}' \in T$, then $S \cap G_j \neq 0$ for each $j \in I(\mathbf{t}')$.*

Proof. Let $S \cap G_{\mathbf{t}'} = A$ and a be a nonzero element of the group A. We have $G = \langle a\rangle_* \oplus G'$ [8, p. 137], $G = G_j \oplus G''$, $j \in I(\mathbf{t}')$. It follows from the theorem about the isomorphism of direct decompositions of completely decomposable groups [8, Proposition 86.1] that $G' \cong G''$. Let ω be an automorphism of the group G mapping $\langle a\rangle_*$ on G_j and G' on G''. We obtain $\omega(a) \in G_j$. Therefore, $S \cap G_j \neq 0$. □

The following remark about normal subgroups of holomorphs of torsion-free Abelian groups is used below.

Remark. If S is a normal Abelian subgroup of the holomorph $\Gamma(G)$ of a torsion-free group G, and S_1 and Φ_1 are the sets of first and second components of elements of the group S, respectively, then $\langle 2S_1, \Phi_1^2\rangle \subset S$ (Lemma 1). Since S_1 and Φ_1 are torsion-free Abelian groups, $2S_1 \cong S$ and $\Phi_1^2 \cong \Phi_1$. Thus, $\langle 2S_1, \Phi_1^2\rangle = 2S_1 \oplus \Phi_1^2 \cong S_1 \oplus \Phi_1$, i.e., the group $S_1 \oplus \Phi_1$ can be isomorphically embedded into the group S.

Theorem 12. *A completely decomposable homogeneous group is normally determined by its holomorph in the class of completely decomposable homogeneous groups.*

Proof. Let $G = \bigoplus_{i\in i} G_i$ be a completely decomposable homogeneous group and H be a completely decomposable homogeneous group almost holomorphically isomorphic to G. We show that $G \cong H$. We have $G \cong H'$, $H \cong G'$, where G' and H' are normal Abelian subgroups of holomorphs $\Gamma(G)$ and $\Gamma(H)$, respectively. By Corollary 6, $T(G) = T(H)$. Let G_1, H_1, Φ_1, and Ψ_1 be sets of first and second components of the groups G' and H', respectively. By Lemma 11, $r(G_1) = r(G)$ and $r(H_1) = r(H)$. Since $G_1 \oplus \Phi_1$ can be isomorphically embedded into the group G' and $G' \cong H$,

$$r(H) = r(G') = r(G_1) + r(\Phi_1) \geq r(G_1) = r(G).$$

Similarly, we obtain that

$$r(G) = r(H') = r(H_1) + r(\Psi_1) \geq r(H_1) = r(H).$$

This means that $r(G) = r(H)$ and, therefore, $G \cong H$. Thus, the group G is normally determined by its holomorph in the class of completely decomposable homogeneous groups. □

Lemma 13. *Let* $G = \bigoplus_{\mathbf{t} \in T} G_{\mathbf{t}}$*, where* $G_{\mathbf{t}}$ *is a homogeneous Abelian group of type* $\mathbf{t}$*, and* T *be a set of pairwise incomparable types. Then,* $\mathbf{t}(g) \not> \mathbf{t}$ *for any nonzero element* $g \in G$ *and any type* $\mathbf{t} \in T$*.*

Proof. Suppose the contrary. Let $0 \neq g_0 \in G$, $\mathbf{t}(g_0) > \mathbf{t}_0$ for some $\mathbf{t}_0 \in T$. Let us write g_0 in the form $g_0 = g_{\mathbf{t}_1} + g_{\mathbf{t}_2} + \cdots + g_{\mathbf{t}_n}$, where $g_{\mathbf{t}_i} \neq 0$ are elements from different components $G_{\mathbf{t}_i}$, and let $\chi(g_{\mathbf{t}_i}) = \chi_i \in \mathbf{t}_i$, $i = \overline{1, n}$, $\mathbf{t}_i \in T$. Then $\chi_i(g_0) = \inf_{\mathbf{X}} \{\chi_1, \chi_2, \ldots, \chi_n\}$ and, therefore, $\mathbf{t}_0 < \mathbf{t}(g_0) < \mathbf{t}_1$. Thus, we obtain $\mathbf{t}_1 > \mathbf{t}_0$, where $\mathbf{t}_0, \mathbf{t}_1 \in T$, which is impossible. □

A statement similar to Lemma 13 is also valid for the direct summand of homogeneous groups with pairwise incomparable types; namely, the following result takes place.

Lemma 13′. *Let* $G = \prod_{\mathbf{t} \in T} G_{\mathbf{t}}$*, where* $G_{\mathbf{t}}$ *is a homogeneous Abelian group of type* $\mathbf{t}$ *and* T *is a set of pairwise incomparable types. Then,* $\mathbf{t}(g) \not> \mathbf{t}$ *for any nonzero element* $g \in G$ *and any type* $\mathbf{t} \in T$*.*

Let $\mathfrak{M}$ denote the class of completely decomposable torsion-free Abelian groups with pairwise types of direct summands of their canonical decompositions.

Theorem 14. *Any group from the class* $\mathfrak{M}$ *is normally determined in this class by its holomorph.*

Proof. Let $G = \bigoplus_{\mathbf{t} \in T} G_{\mathbf{t}}$ be a group from the class $\mathfrak{M}$ and $H = \bigoplus_{\mathbf{t}' \in T'} H_{\mathbf{t}'}$ be an arbitrary group belonging to this class and almost holomorphically isomorphic to G. Let $G_{\mathbf{t}}$ and $H_{\mathbf{t}'}$ denote homogeneous completely decomposable direct summands of types $\mathbf{t}$ and $\mathbf{t}'$ of the groups G and H, respectively. T and T' are some sets consisting of pairwise incomparable types. According to Theorem 7, $T = T'$.

Let $\mathbf{t}_0 \in T$. We have $G \cong H'$ and $H \cong G'$, where G' and H' are normal Abelian subgroups of the holomorphs $\Gamma(G)$ and $\Gamma(H)$, respectively. Let H_1 and Ψ_1 denote the sets of the first and second components of elements of the group H'. In the group H_1, one can find an element $h_1 \neq 0$ such that $\mathbf{t}_{H_1}(h_1) \geq \mathbf{t}_0$ (Proposition 4). If $\mathbf{t}_{H_1}(h_1) > \mathbf{t}_0$, then $\mathbf{t}_H(h_1) > \mathbf{t}_0$, which is impossible by Lemma 13. Therefore, $\mathbf{t}_{H_1}(h_1) = \mathbf{t}_0$, i.e., $\mathbf{t} \in T(H_1)$.

Let us show that if $h \in H_1$ and $\mathbf{t}_{H_1}(h) = \mathbf{t}_0$, then $h \in H_{\mathbf{t}_0}$. Indeed, let $h \in H_1$ and $\mathbf{t}_{H_1}(h) = \mathbf{t}_0$. Then $\mathbf{t}_H(h) = \mathbf{t}_0$ by Lemma 13. If $h \notin H_{\mathbf{t}_0}$, then there exists a type $\mathbf{t}_1 \in T$, $\mathbf{t}_1 \neq \mathbf{t}_0$, such that

$$h = h_{\mathbf{t}_1} + h',$$

where

$$0 \neq h_{\mathbf{t}_1} \in H_{\mathbf{t}_1}, \quad h' \in \bigoplus_{\substack{\mathbf{t}' \in T, \\ \mathbf{t}' \neq \mathbf{t}_1}} H_{\mathbf{t}'}.$$

Therefore, $\mathbf{t}_0 = \mathbf{t}_H(h) < \mathbf{t}_H(h_{\mathbf{t}_1}) = \mathbf{t}_1$, which is impossible because $\mathbf{t}_1, \mathbf{t}_0 \in T$.

Thus, it has been established that the group H_1 contains an element h_1 such that $\mathbf{t}_{H_1}(h_1) = \mathbf{t}_0$ and $h_1 \in H_{\mathbf{t}_0}$. Let $h_1' \in H_1 \cap H_{\mathbf{t}_0}$, $h_1' \neq 0$. We have $H_{\mathbf{t}_0} = \langle h_1 \rangle_* \oplus C$, $H_{\mathbf{t}_0} = \langle h_1' \rangle_* \oplus C_1$. There exists an automorphism η of the group H which maps $\langle h_1 \rangle_*$ on $\langle h_1' \rangle_*$. Since H_1 is an invariant subgroup of the group H, $\mathbf{t}_{H_1}(h_1) = \mathbf{t}_{H_1}(\eta h_1)$. However, $s(\eta h_1) = m h_1'$ for some integers s and m; therefore, $\mathbf{t}_{H_1}(h_1') = \mathbf{t}_{H_1}(\eta h_1) = \mathbf{t}_{H_1}(h_1) = \mathbf{t}_0$. This means that $H_1 \cap H_{\mathbf{t}_0}$ is a homogeneous group and $\mathbf{t}(H_1 \cap H_{\mathbf{t}_0}) = \mathbf{t}_0$. By virtue of Lemma 11, $r(H_1 \cap H_{\mathbf{t}_0}) = r(H_{\mathbf{t}_0})$.

By Lemma 13, since $G \cong H'$, the group H' has no elements whose types are larger than $\mathbf{t}_0$. The group $H_1 \oplus \Psi_1$ can be isomorphically embedded into H'. In this embedding, the image of any element from $H_1 \cap H_{\mathbf{t}_0}$ has the type $\mathbf{t}_0$ in the group H'. Thus, $r(G_{\mathbf{t}_0}) \geq r(H_1 \cap H_{\mathbf{t}_0}) = r(H_{\mathbf{t}_0})$. Similarly, we obtain $r(H_{\mathbf{t}_0}) \geq r(G_{\mathbf{t}_0})$. By virtue of the arbitrariness of the type $\mathbf{t}_0$ from T, we have $G \cong H$. □

Theorem 14 involved completely decomposable groups from the class $\mathfrak{M}$. The result of this theorem cannot be extended to arbitrary completely decomposable groups.

Lemma 15. *Let $G = A \oplus B$, where A is an invariant subgroup of the group G,*

$$S(A) = \{(a,\sigma) \in \Gamma(G) \mid a \in A,\ (\forall\, \bar a \in A)\ \sigma\bar a = \bar a,\ (\exists\, \eta \in \mathrm{Hom}(B,A))\ (\forall\ b \in B)\ \sigma b = b + \eta b\}.$$

Then $S(A)$ is a normal Abelian subgroup of the holomorph of the group G and $S = A \oplus C$, where $C \cong \mathrm{Hom}(B,A)$.

Proof. Let us show that $S(A)$ is a subgroup of the holomorph of the group $\Gamma(G)$. Let $(a,\sigma),(a_1,\sigma_1) \in A$. Then

$$(a,\sigma) - (a_1,\sigma_1) = (a - \sigma\sigma_1^{-1}a_1, \sigma\sigma_1^{-1}) = (a - a_1, \sigma\sigma_1^{-1}).$$

For any element $g \in A$, we have that $\sigma\sigma_1^{-1}g = g$. Let $b \in B$ and $\sigma b = b + \eta b$, $\sigma_1 b = b + \eta_1 b$, where $\eta, \eta_1 \in \mathrm{Hom}(B,A)$. We have that

$$\sigma_1^{-1}b = b - \sigma_1^{-1}\eta_1 b = b - \eta_1 b.$$

Thus,

$$\sigma\sigma_1^{-1}b = \sigma(b - \eta_1 b) = b + \eta b - \eta_1 b = b + (\eta - \eta_1)b.$$

This means that $(a,\sigma) - (a_1,\sigma_1) \in S(A)$ and $S(A)$ is a subgroup of the group $\Gamma(G)$.

If (a_1,σ_1) and (a_2,σ_2) are elements of the group $S(A)$, then $\sigma_1\sigma_2 = \sigma_2\sigma_1$. We have that $(a_1,\sigma_1) + (a_2,\sigma_2) = (a_1 + a_2, \sigma_1\sigma_2)$ and $(a_2,\sigma_2) + (a_1,\sigma_1) = (a_2 + a_1, \sigma_2\sigma_1)$. This means that $S(A)$ is an Abelian subgroup of the group $\Gamma(G)$.

Let us show that $S(A)$ is a normal subgroup of the group $\Gamma(G)$. Let (g,φ) be an arbitrary element from $\Gamma(G)$. Consider the sum

$$-(g,\varphi) + (a,\sigma) + (g,\varphi) = (-\varphi^{-1}g + \varphi^{-1}a + \varphi^{-1}\sigma g, \varphi^{-1}\sigma\varphi) = (\bar a, \bar\sigma).$$

The element g, as an element of the direct sum, can be written in the form $a_0 + b_0$, where $a_0 \in A$, $b_0 \in B$. We obtain that

$$\varphi^{-1}\sigma g = \varphi^{-1}a_0 + \varphi^{-1}b_0 + \varphi^{-1}\eta b_0,$$

where $\sigma b_0 = b_0 + \eta b_0$, $\eta \in \mathrm{Hom}(B,A)$. Since A is invariant in G, we have that $\varphi^{-1}\eta b_0 \in A$ and $\varphi^{-1}a \in A$. Thus,

$$-\varphi^{-1}g + \varphi^{-1}a + \varphi^{-1}\sigma g = -\varphi^{-1}a_0 - \varphi^{-1}b_0 + \varphi^{-1}a + \varphi^{-1}a_0 + \varphi^{-1}b_0 + \varphi^{-1}\eta b_0 = \varphi^{-1}(a + \eta b_0) = \bar a \in A.$$

Let $\varphi b = b' + a'$, $b' \in B$, $a' \in A$. Then $\sigma\varphi b = \sigma b' + \sigma a' = b' + \lambda b' + a'$, where $\sigma b' = b' + \lambda b'$, $\lambda \in \mathrm{Hom}(B,A)$. We have that

$$\varphi^{-1}\sigma\varphi b = \varphi^{-1}b' + \varphi^{-1}\lambda b' + \varphi^{-1}a';$$

however, $\varphi^{-1}b' = b - \varphi^{-1}a'$, hence

$$\varphi^{-1}\sigma\varphi b = b + \varphi^{-1}\lambda b' = b + \varphi^{-1}\lambda\pi\varphi|Bb,$$

where π is a projection of G on B and $\varphi|B$ is a restriction of the automorphism φ on B. It is evident that $\varphi^{-1}\lambda\pi\varphi|B \in \mathrm{Hom}(B,A)$. Since $\sigma a = a$ for all $a \in A$, we have that $\varphi^{-1}\sigma\varphi$ acts identically on elements of A. Therefore, it has all the properties of the second components of elements from $S(A)$. Thus, $S(A)$ is a normal Abelian subgroup of the holomorph $\Gamma(G)$. Let any element $(a,\sigma) \in S(A)$ be in correspondence with a pair (a,η), where η is a homomorphism of the group B into the group A induced by the automorphism σ. The obtained correspondence is just the isomorphism $S(A) \cong A \oplus \mathrm{Hom}(B,A)$. □

Theorem 16. *There exists a completely decomposable torsion-free Abelian group that is not normally determined by its holomorph in the class of all completely decomposable torsion-free groups.*

Proof. Let π be a set of prime numbers. We say that the type $\mathbf{t}$ is *π-divisible* if it is p-divisible if and only if the prime number p belongs to π.

For any two types $\mathbf{t}_1$ and $\mathbf{t}_2$, where $\mathbf{t}_1 \geq \mathbf{t}_2$, we define their *difference* $\mathbf{t}_1 - \mathbf{t}_2$ as a type containing the characteristic $\chi_1 - \chi_2$ ($\chi_1 \in \mathbf{t}_1$, $\chi_2 \in \mathbf{t}_2$, $\chi_1 \geq \chi_2$). The difference of characteristics is defined componentwise, where, obviously, ∞ minus anything is ∞.

Let $G = \bigoplus_{i=1}^{n} G_{\mathbf{t}_i}$, where $G_{\mathbf{t}_i}$ is a homogeneous completely decomposable finite rank group of the type $\mathbf{t}_i$, all $\mathbf{t}_i$ are π-divisible types (π is a set of prime numbers), $\mathbf{t}_1 > \mathbf{t}_2 > \cdots > \mathbf{t}_n$, $r(G_{\mathbf{t}_1}) = 1$, and, for some $k \in \{2, \ldots, n\}$, $\mathbf{t}_1 - \mathbf{t}_k \neq \mathbf{t}_i$ for any $i \in \{2, \ldots, n\}$.

Consider the group $H = \bigoplus_{i=1}^{n} H_{\tau_i}$, where H_{τ_i} is a homogeneous completely decomposable of the type τ_i, where $\tau_i = \mathbf{t}_1 - \mathbf{t}_i$ for all $i = 1, 2, \ldots, n$, $\tau_1 = \mathbf{t}_1$, and $r(G_{\mathbf{t}_i}) = r(H_{\tau_i})$ $(i = 1, 2, \ldots, n)$. $G_{\mathbf{t}_1}$ is an invariant subgroup of the group G. We use this subgroup to construct a normal Abelian subgroup $S(G_{\mathbf{t}_1})$ of the holomorph $\Gamma(G)$ as was shown in Lemma 15. We have that

$$S(G_{\mathbf{t}_1}) \cong G_{\mathbf{t}_1} \oplus \left(\bigoplus_{i=2}^{n} \operatorname{Hom} G_{\mathbf{t}_i}, G_{\mathbf{t}_1} \right).$$

It is well known that if C_1 and C_2 are torsion-free rank 1 groups, then $\operatorname{Hom}(C_1, C_2) \neq 0$ if and only if $\mathbf{t}(C_1) \leq \mathbf{t}(C_2)$ and, in this case, $\operatorname{Hom}(C_1, C_2)$ is a torsion-free rank 1 group and $\mathbf{t}\big(\operatorname{Hom}(C_1, C_2)\big) = \mathbf{t}(C_2) - \mathbf{t}(C_1)$. Therefore,

$$S(G_{\mathbf{t}_1}) \cong G_{\mathbf{t}_1} \oplus \bigoplus_{i=2}^{n} H_{\tau_i} \cong \bigoplus_{i=1}^{n} H_{\tau_i} \cong H.$$

Similarly, consider a normal Abelian subgroup $S(H_{\tau_1})$ in the holomorph $\Gamma(H)$. We have that

$$S(H_{\tau_1}) \cong H_{\tau_1} \oplus \operatorname{Hom}\left(\bigoplus_{i=2}^{n} H_{\tau_i}, H_{\tau_1} \right) \cong H_{\tau_1} \oplus \bigoplus_{i=2}^{n} G_{\mathbf{t}_i} \cong \bigoplus_{i=1}^{n} G_{\mathbf{t}_i} \cong G.$$

Thus, the groups G and H are almost holomorphically isomorphic but $G \ncong H$ because $\tau_k \neq \mathbf{t}_i$ for any $i \in \{2, \ldots, n\}$.

Therefore, a group G cannot be normally determined by its holomorph in the class of completely decomposable groups. $\square$

Let us now consider direct products of homogeneous groups.

When considering direct products $A = \prod_{\mathbf{t} \in T} A_{\mathbf{t}}$, we identify the group $A_{\mathbf{t}}$ with the isomorphic subgroup $\rho_{\mathbf{t}} \pi_{\mathbf{t}} A$ of the group A. Any element $a \in A_{\mathbf{t}}$ is identified with the element $\rho_{\mathbf{t}} \pi_{\mathbf{t}} a$, where $\pi_{\mathbf{t}}$ is a projection of the group A onto the group $A_{\mathbf{t}}$, and $\rho_{\mathbf{t}}$ is a coordinate embedding of the group A into the group $A_{\mathbf{t}}$.

Proposition 17. *Let $G = \prod_{\mathbf{t} \in T} G_{\mathbf{t}}$, where $G_{\mathbf{t}}$ is a homogeneous group of the type $\mathbf{t}$ and T be a set of pairwise incomparable types. If for some torsion-free Abelian group H there exists an isomorphic mapping μ of the group G on a normal subgroup H' of the holomorph $\Gamma(H)$, then $\mu G_{\mathbf{t}}$ is a normal subgroup of the holomorph $\Gamma(H)$ for any type $\mathbf{t} \in T$.*

Proof. Let $\mathbf{t}_0$ be an arbitrary type from T and $S = \mu G_{\mathbf{t}_0}$. It is evident that S is a subgroup of the group H' and consists exactly of all elements of the group H' having the type $\mathbf{t}_0$. We prove that S is a normal subgroup of the holomorph $\Gamma(H)$.

Let $(s, \omega) \in S$, $(b, \sigma) \in \Gamma(H)$ and $\chi\big((s, \omega)\big) = (k_1, k_2, \ldots, k_n, \ldots)$, where $\chi\big((s, \omega)\big) \in \mathbf{t}_0$. For any natural number n for which $k_n < \infty$, there exists an element $(s_n, \omega_n) \in H'$ such that $p_n^{k_n}(s_n, \omega_n) = (s, \omega)$. Then

$$p_n^{k_n}\big(-(b, \sigma) + (s_n, \omega_n) + (b, \sigma)\big) = -(b, \sigma) + p_n^{k_n}(s_n, \omega_n) + (b, \sigma) = -(b, \sigma) + (s, \omega) + (b, \sigma) \in H'.$$

Therefore, the p_n-height of the element $-(b, \sigma) + (s, \omega) + (b, \sigma)$ in the group H' is not less than k_n. It is clear that if $k_m = \infty$ for some natural number m, then the p_m-height of the element $-(b, \sigma) + (s, \omega) + (b, \sigma)$ in the group H' is also equal to ∞. This means that $\mathbf{t}\big(-(b, \sigma) + (s, \omega) + (b, \sigma)\big) \geq \mathbf{t}_0$. Applying Lemma 2, we obtain that $\mathbf{t}\big(-(b, \sigma) + (s, \omega) + (b, \sigma)\big) = \mathbf{t}_0$. Therefore, $-(b, \sigma) + (s, \omega) + (b, \sigma) \in S$ and S is a normal subgroup $\Gamma(H)$. $\square$

We recall that a torsion-free Abelian group G is called *transitive* if for any two elements $a, b \in G$ such that $\chi(a) = \chi(b)$ there exists an automorphism $\varphi \in \mathrm{Aut}(G)$ such that $b = \varphi a$.

Theorem 18. *Let $G = \prod_{\mathbf{t} \in T_1} G_{\mathbf{t}}$, $H = \prod_{\mathbf{t} \in T_2} H_{\mathbf{t}}$, where $G_{\mathbf{t}}$ ($\mathbf{t} \in T_1$) and $H_{\mathbf{t}}$ ($\mathbf{t} \in T_2$) are transitive homogeneous groups and the sets T_1 and T_2 consist of pairwise incomparable types. If the groups G and H are almost holomorphically isomorphic, then $T_1 = T_2$ and the rank $r(G_{\mathbf{t}})$ of the group $G_{\mathbf{t}}$ is equal to the rank $r(H_{\mathbf{t}})$ of the group $H_{\mathbf{t}}$ for any type $\mathbf{t} \in T_1$.*

Proof. Since the groups G and H are almost holomorphically isomorphic, $G \cong H'$, $H \cong G'$, where G' and H' are normal Abelian subgroups of the holomorphs $\Gamma(G)$ and $\Gamma(H)$, respectively. Let μ denote the isomorphic mapping of the group G on the group H'.

Since the sets T_1 and T_2 consist of pairwise incomparable types, we obtain, according to Theorem 2, that the sets T_1 and T_2 coincide. Thus, one can write $G = \prod_{\mathbf{t} \in T} G_{\mathbf{t}}$, $H = \prod_{\mathbf{t} \in T} H_{\mathbf{t}}$, where $T_1 = T_2 = T$.

Let $\mathbf{t}_0 \in T$ and let $\mu G_{\mathbf{t}_0} = S$. By H_1 and Ψ, we denote the sets of first and second components of the group H', respectively; S_1 and Φ denote the sets of first and second components of the group S, respectively. It is evident that S is a subgroup of the group H'. Applying Proposition 17, we obtain that S is a normal Abelian subgroup of the holomorph $\Gamma(H)$. This means that $S_1 \neq 0$ (Lemma 2).

Let us show that the type of any element $s \in S_1$ in the group H equals $\mathbf{t}_0$. Since $s \in S_1$, there exists an element $(s, \omega) \in S$. The type of an element is preserved under isomorphisms; therefore, $\mathbf{t}_{H'}\big((s, \omega)\big) = \mathbf{t}_0$. Let $\chi\big((s, \omega)\big) = (k_1, k_2, \ldots, k_n, \ldots)$.

Let $k_n < \infty$. Then there exists an element $(x, \delta) \in H'$ such that $p_n^{k_n}(x, \delta) = (s, \omega)$. By formula (7), we have that $(s, \omega) = (p_n^{k_n} x, \delta^{p_n^{k_n}})$. Then $s = p_n^{k_n} x$. It follows that the p_n-height of the element s in the group H is not less than k_n. If $k_n = \infty$, then we obtain that the p_n-height of the element s in the group H is ∞.

Thus, $\chi(s) \geq \chi(s, \omega)$ and, therefore, $\mathbf{t}(s) \geq \mathbf{t}_0$. Since, according to Lemma 2, $\mathbf{t}(s) \ngtr \mathbf{t}_0$, we obtain that $\mathbf{t}(s) = \mathbf{t}_0$.

We prove that S_1 is a subgroup of the group $H_{\mathbf{t}_0}$. Suppose the contrary. Let there be an element $s \in S_1$ such that $s \notin H_{\mathbf{t}_0}$, i.e., $\pi_{\mathbf{t}_j} s \neq 0$ for some type $\mathbf{t}_j \neq \mathbf{t}_0$ ($\mathbf{t}_j \in T$), where $\pi_{\mathbf{t}_j}$ is a projection of the group H on the group $H_{\mathbf{t}_j}$. We have $\mathbf{t}(s) \leq \mathbf{t}_j$. Since $\mathbf{t}(s) = \mathbf{t}_0$, we obtain that $\mathbf{t}_0 \leq \mathbf{t}_j$. We have obtained a contradiction with the incomparability of types in T. Therefore, S_1 is a subgroup of the group $H_{\mathbf{t}_0}$.

Since S is a normal Abelian subgroup of the holomorph $\Gamma(H)$, we can apply formula (1) $2S_1 \subset S$. Therefore,

$$r(S_1) = r(2S_1) \leq r(S). \tag{8}$$

Let us prove that the subgroup $2S_1$ is mapped into the subgroup S_1 under any automorphism $\lambda \in \mathrm{Aut}(H_{\mathbf{t}_0})$. For the automorphism $\lambda \in \mathrm{Aut}(H_{\mathbf{t}_0})$ we construct an automorphism $\lambda' \in \mathrm{Aut}(H)$ in the following way: for any element $b \in H$, we put $\pi_{\mathbf{t}_0} \lambda' b = \lambda \pi_{\mathbf{t}_0} b$; $\pi_{\mathbf{t}_j} \lambda' b = \pi_{\mathbf{t}_j} b$ if $\mathbf{t}_j \neq \mathbf{t}_0$.

Consider an element $(s, \omega) \in S$. Let $\lambda'(2s) = u$. Then $\mathbf{t}(u) = \mathbf{t}(2s) = \mathbf{t}(s) = \mathbf{t}_0$. Taking into account that S_1 is a subgroup of the group $H_{\mathbf{t}_0}$, we obtain $\lambda'(2s) = \lambda(2s)$ and, therefore, $u = \lambda(2s)$.

Since H_1 is an invariant subgroup of the group H and $2s \in H_1$, $u \in H_1$.

H' is a normal Abelian subgroup of the holomorph $\Gamma(H)$ and, therefore, $2s \in H'$. We have that

$$(0, \lambda') + (2s, \varepsilon) + (0, \lambda')^{-1} = (\lambda'(2s), \varepsilon) = (u, \varepsilon).$$

It follows that $u \in H'$.

Since $H' = \mu G$, there exists an element $g \in G$ such that $\mu g = u$. Since isomorphisms preserve types, $\mathbf{t}(g) = \mathbf{t}_0$.

We have $g \in G_{\mathbf{t}_0}$. This means that $u \in S$; therefore, $u \in S_1$. This proves that $\lambda(2s) \in S_1$ for any element $s \in S_1$ and any automorphism $\lambda \in \mathrm{Aut}(H_{\mathbf{t}_0})$.

Let $\{a_i\}_{i \in I}$ be a maximal linearly independent system of elements in $H_{\mathbf{t}_0}$. Since $2S_1 \neq 0$, there exists an element $x \in 2S_1$, $x \neq 0$. The group $H_{\mathbf{t}_0}$ is homogeneous; therefore, for any $i \in I$ we have $\mathbf{t}(a_i) = \mathbf{t}_0$. Since $\mathbf{t}(x) = \mathbf{t}_0$, there exist numbers $m_i, n_i \in \mathbb{Z}$ such that $\chi(m_i a_i) = \chi(n_i x)$.

The group $H_{\mathbf{t}_0}$ is transitive; therefore, there exists an automorphism $\varphi \in \mathrm{Aut}(H_{\mathbf{t}_0})$ such that $\varphi(n_i x) = m_i a_i$. Since $\varphi(2S_1) \subset S_1$ by virtue of what was proved above, we obtain that $m_i a_i \in S_1$ for any $i \in I$.

The system $\{m_i a_i\}_{i\in I}$ is a linearly independent system of elements in the group S_1. Therefore, $r(S_1) \geq r(H_{\mathbf{t}_0})$. But S_1 is a subgroup of the group $H_{\mathbf{t}_0}$, this means that $r(S_1) \leq r(H_{\mathbf{t}_0})$. It follows from these inequalities that $r(S_1) = r(H_{\mathbf{t}_0})$.

Since $S = \mu G_{\mathbf{t}_0}$, we have that $r(G_{\mathbf{t}_0}) = r(S)$. Applying inequality (8), we obtain that $r(G_{\mathbf{t}_0}) = r(S) \geq r(S_1) = r(H_{\mathbf{t}_0})$. This means that $r(G_{\mathbf{t}_0}) \geq r(H_{\mathbf{t}_0})$.

Similarly, one can prove that

$$r(H_{\mathbf{t}_0}) \geq r(G_{\mathbf{t}_0}). \tag{9}$$

Comparing inequalities (8) and (9), we obtain that $r(G_{\mathbf{t}_0}) = r(H_{\mathbf{t}_0})$. □

Let $\mathfrak{A}$ denote a group class consisting of all direct products of the form $G = \prod_{\mathbf{t}\in T} G_{\mathbf{t}}$, where $G_{\mathbf{t}}$ are homogeneous completely decomposable groups and the set T consists of pairwise incomparable types.

Theorem 19. *Any group form the class $\mathfrak{A}$ is normally determined by its holomorph in this class.*

Proof. Let A be an arbitrary homogeneous completely decomposable group, a_1 and a_2 be nonzero elements of the group A, and $\chi(a_1) = \chi(a_2)$. Let $\langle a_1\rangle_*$ and $\langle a_2\rangle_*$ denote pure subgroups generated by elements a_1 and a_2, respectively. The subgroups $\langle a_1\rangle_*$ and $\langle a_2\rangle_*$ are of rank 1 and the same type. This means that $\langle a_1\rangle_* \cong \langle a_2\rangle_*$. Since A is a homogeneous separable group, we have that each of the groups $\langle a_1\rangle_*$ and $\langle a_2\rangle_*$ are direct summands in it [8, Proposition 87.2], i.e., $A = \langle a_1\rangle_* \oplus A_1$ and $A = \langle a_2\rangle_* \oplus A_2$. The groups A_1 and A_2 are completely decomposable groups [8, Theorem 86.7]; at the same time, they are homogeneous groups of the same type and of the same rank. Therefore, $A_1 \cong A_2$.

It is clear that there exists an automorphism φ of the group A such that $\varphi(a_1) = a_2$.

Let $G \in \mathfrak{A}$. Let us demonstrate that any group $H \in \mathfrak{A}$ almost holomorphically isomorphic to the group G is isomorphic to G.

Let $G = \prod_{\mathbf{t}\in T_1} G_{\mathbf{t}}$ and $H = \prod_{\mathbf{t}\in T_2} H_{\mathbf{t}}$, where $G_{\mathbf{t}}$ ($\mathbf{t} \in T_1$) and $H_{\mathbf{t}}$ ($\mathbf{t} \in T_2$) are homogeneous completely decomposable groups and the sets T_1 and T_2 consist of pairwise incomparable types. Since any group $G_{\mathbf{t}}$ ($\mathbf{t} \in T_1$) and any group $H_{\mathbf{t}}$ ($\mathbf{t} \in T_2$) are homogeneous completely decomposable groups, these groups are transitive. Therefore, by Theorem 18, $T_1 = T_2$ and $r(G_{\mathbf{t}}) = r(H_{\mathbf{t}})$ for any type $\mathbf{t} \in T_1$. Since $G_{\mathbf{t}}$ and $H_{\mathbf{t}}$ are homogeneous completely decomposable groups of the same rank, we have that $G_{\mathbf{t}} \cong H_{\mathbf{t}}$ for any type $\mathbf{t} \in T_1$. This means that $G \cong H$. □

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